

# Near-wall modelling in Eulerian-Eulerian simulations

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## Abstract

The near-wall region in turbulent Eulerian-Eulerian (E-E) simulations has hitherto received little to no attention. A standard approach to modelling this region is through the employment of single-phase wall-functions in the fluid-phase and it is unclear whether such an approach is capable of capturing the turbulent fluid-particle interaction in the near-wall region. In order to both investigate and alleviate E-E models reliance on single-phase wall-functions we propose an E-E elliptic relaxation model to account for the near-wall non-homogeneity which arises in wall-bounded flows. The proposed model is derived within an E-E framework and enables the full resolution of the boundary layer and arbitrary wall sensitivity. The model is then compared against the conventional  $k_f - \varepsilon_f$  turbulence model with standard single-phase wall-functions. Additionally, the modelling is compared against a low-Re number turbulence model. The elliptic relaxation model is implemented within the open-source CFD toolbox OpenFOAM, applied to a vertical downward-facing channel and validated against the benchmark experimental data of Kulick et al.

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[20]. Model results show marked improvements over the conventional turbulence model across mean flow and turbulence statistics predictions. The use of conventional single-phase wall functions were shown to negatively impede on the prediction of the velocity covariance coupling term and as a result the particle fluctuation energy. Moreover, this also lead to an underestimation of the near-wall volume fraction accumulation. Finally, the elliptic relaxation model, E-E model and accompanying validation cases are made open-source.

*Keywords:* v2-f, Boundary-layer, E-E, Near-wall

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1 **1. Introduction**

2 Many researchers have investigated turbulent wall-bounded fluid-particle flow  
3 through experimentation [4, 11, 15, 20, 23, 49]. Owing to their turbulent nature such  
4 flows exhibit complex physical behaviour giving rise to turbulence modulation which  
5 can be caused by: particle-particle, fluid-particle and/or particle-wall interactions.  
6 Thus, research has been ongoing to understand and model these phenomena; most  
7 of which are common in engineering processes, e.g. pneumatic conveyance and coal  
8 particle combustion. The aforementioned experimental studies provide invaluable  
9 physical insights and validation data for the development of predictive models. One  
10 notable study is that of Kulick et al. [20] which has received considerable attention  
11 from researchers developing, predominantly Euler-Lagrange (E-L), models [19, 25, 42,  
12 47, 51, 52, 54]. This study is particularly attractive as there are several particle classes  
13 giving rise to various particle-fluid and particle-wall interactions which contribute to  
14 turbulence modulation.

15 Having identified the aspects of physical behaviour which are significant in these  
16 flows, researchers can investigate them separately in a reductionist approach. We  
17 now highlight some studies that contribute to the understanding of particle behaviour  
18 within the case of Kulick et al. [20]; starting with the so-called feedback-force of the  
19 particle phase on the fluid turbulence in the flow. Vreman [51] recently examined the  
20 effect of the mean feedback-force and how it is exacerbated by wall roughness. An  
21 increase in wall roughness enhances turbulence attenuation i.e. a reduction in fluid-  
22 phase velocity fluctuations. This explains the over prediction of the mean particle  
23 velocities seen in previous studies Kubik and Kleiser [19], Wang and Squires [52],

24 Yamamoto et al. [54] as smooth walls were simulated.

25 Another phenomenon that has been investigated is turbophoresis, which refers to  
26 the tendency of particles in the flow to migrate towards regions of lower turbulence.  
27 The turbophoresis force is responsible for particles drifting from regions of high tur-  
28 bulence intensity to low turbulence intensity [27, 35], which often results in particles  
29 accumulating in the near-wall region characterised by low-speed streaks [32, 34].  
30 This accumulation in the near-wall region is referred to as deposition and has been  
31 researched numerically by [24, 27, 28, 30]. One of the first models for particle deposi-  
32 tion by Young and Leeming [55] showed that the turbophoretic velocity depends on  
33 the gradient of wall-normal fluctuating velocities and provided one of the first physi-  
34 cal basis for explaining the turbophoresis force. Strömngren et al. [47] investigated the  
35 effect of the turbophoresis force within an Eulerian-Eulerian (E-E) framework and  
36 found that even for small volume fractions,  $\alpha_p = 2 \times 10^{-4}$ , two-way coupling effects  
37 are non-negligible and the near-wall region may require special attention. This is due  
38 to the accumulation of particles in the near-wall region i.e viscous sub layer, leading  
39 to higher volume fractions in which two-way coupling effects become more relevant  
40 [14].

41 In turbulent single-phase simulations the near-wall region is typically modelled.  
42 Wall functions are applied to turbulence quantities,  $\varepsilon_f$  &  $\nu_{ft}$  with a zero gradient  
43 condition given to  $k_f$ , in order to avoid the computational overhead of detailed reso-  
44 lution of the flow in the near-wall region. Such wall functions are based on the law-  
45 of-the-wall, which is that the dimensionless velocity,  $u^+$  varies through some function  
46 expressed generically as,  $u^+ = f_{log}(y^+)$ . The function  $f_{log}$  is logarithmic representing

47 the outer log-law region of the turbulent boundary layer. This corresponds to the  
48 constant-stress layer in which the turbulent shear stress is proportional to the fric-  
49 tion velocity [45]. In turbulent quantity terms this means that the production and  
50 dissipation of turbulent kinetic energy are equal.

51 The law-of-the-wall is assumed to be universal and is found through dimensional  
52 reasoning, this then leads to a description of the near-wall region through dimen-  
53 sionless variables i.e. velocity and wall-normal coordinates. The dimensionless wall-  
54 normal coordinate is defined as  $y^+ = yu_\tau/\nu_f$  and the log-law is applicable in the  
55 range of  $30 < y^+ < 300$ , this then gives a universal relation that can be applied to  
56 turbulent wall-bounded flows. This criterion places a requirement on the first com-  
57 putational cell i.e. the distance of the cell centre must be further than  $y^+ > 30$ . As  
58 can already be deduced, the calculation of  $y^+$  depends on the friction velocity, which  
59 is not known *a priori*. Hence, this quantity is estimated prior to calculation using  
60 standard skin friction relations and informs mesh generation. This approach then  
61 sacrifices near-wall resolution for a computationally cheaper simulation. An impor-  
62 tant assumption about the nature of the law-of-the-wall has been made throughout  
63 i.e. its universal nature. This is in fact not true as it has been shown experimen-  
64 tally that the boundary layer is affected by adverse pressure gradients and geometric  
65 changes [16].

66 The viability of single-phase wall functions applied to multiphase simulations has  
67 attracted some interest from researchers. A theoretical study by Rizk and Elghobashi  
68 [40] showed that increasing volume fraction can adversely effect the mean profile pre-  
69 diction. It was found that with increasing particle volume fraction the log layer broke

70 down resulting in an overestimation. Interestingly, a similar relationship between an  
71 increasing mass loading and a reduction in the mean log-layer momentum was re-  
72 cently found experimentally by Saber et al. [44]. Benyahia et al. [2] included the  
73 effect of the particle phase directly into the wall function. An additional term that  
74 contains the drag and velocity fluctuation covariance is introduced in the log-law re-  
75 lation. This formulation allows the presence of the particles to influence the velocity  
76 profile, although when extended to more complex geometries the short-comings of  
77 single-phase wall functions remain.

78 Attempts to circumvent the reliance on single-phase wall functions have been  
79 made by several authors [3, 6, 40, 57] in which a low-Re number turbulence model  
80 is used. This allows the transport equations to be integrated up to the wall. This  
81 approach has proven fruitful for numerous authors as without the use of wall func-  
82 tions, the presence of the particles within the boundary layer can exert their influence  
83 [40, 57]. The low-Re turbulence model uses a damping function and a near wall cor-  
84 rection of Kolmogorov scaling [31]. The damping of the viscosity can be somewhat  
85 arbitrary and validated on relatively simple flow leading to a range of different mod-  
86 els [5, 21, 41, 46] with an extensive summary found in Patel et al. [31]. The damping  
87 functions used in Patel et al. [31] are often non-linear and can lead to numerical  
88 stiffness further complicating their application.

89 Durbin [8] proposes another way of accounting for wall-induced non-homogeneity.  
90 The quantity  $\overline{v_f^2}$ , which represents the turbulence-stress normal to streamlines, is in-  
91 troduced. This quantity is derived from the exact Reynolds-stress transport equation  
92 and contains a source term that accounts for the redistribution of turbulence kinetic

93 energy. This inclusion explicitly accounts for the wall-induced non-homogeneity and  
94 enables the wall-normal component to be dampened. The energy redistribution is  
95 governed by an *elliptic relaxation* equation  $f$ , that is free of geometric dependence  
96 or arbitrary fitting. The  $\overline{v_f^2} - f$  elliptic relaxation model has been validated across  
97 various challenging single-phase flows [1, 7, 26, 29, 48] highlighting the benefit of  
98 such a modelling technique.

99 There are two closely linked issues with the current E-E modelling approaches:  
100 the modelling of the near-wall region, through single-phase wall functions, and the  
101 subsequent consequences of such an approach i.e. the prediction of turbulence mod-  
102 ulation and turbophoresis. In this work we seek to investigate this by carrying out  
103 a side-by-side comparison of a conventional E-E simulation method with a newly-  
104 derived elliptic relaxation model in which the near-wall region has been resolved.  
105 The main aim then is to reveal the consequences of modelling the near-wall region  
106 whilst proposing new modelling to circumvent these consequences.

107 We begin at a recently proposed E-E model, namely the Reynolds-Averaged Two-  
108 Fluid Model of Fox [13]. This approach has proven particularly fruitful in modelling  
109 high  $Re$  number flows due to the inclusion of particle inertia induced energy sep-  
110 aration (see Février et al. [12], Fox [13]) and has lead to a high level of validation  
111 [38, 39]. The elliptic relaxation model of Durbin [8] is derived within the RA-TFM  
112 framework and applied to the vertical downward facing channel of Kulick et al. [20].  
113 The elliptic relaxation model alleviates the use of wall functions and/or the use of  
114 ad-hoc damping functions and their geometric dependency. To ascertain the con-  
115 sequences of a conventional E-E simulation, the RA-TFM with the solution of the

116  $k_f - \varepsilon_f$  model, and is compared and contrasted against the newly proposed elliptic  
 117 relaxation model,  $\overline{v_f^2} - f$ . Moreover, results are also compared against the low Re  
 118 number model of Launder and Sharma [21] in order to facilitate a boundary layer  
 119 resolved comparison.

## 120 2. Numerical model

121 We begin at the RA-TFM of Fox [13], and as we are interested in extending the  
 122 fluid-phase turbulence modelling we present the fluid- and particle-phase governing  
 123 equations as well as the particle-phase fluctuation energy equations in Table 1. We  
 124 have neglected coupling through buoyancy due to the high density ratios simulated  
 125 in this work and therefore, we begin at the fluid-phase turbulence equations.

126 The turbulent kinetic energy transport equation for the fluid-phase takes the  
 127 form:

$$\begin{aligned} \frac{\partial(\alpha_f \rho_f k_f)}{\partial t} + \nabla \cdot (\alpha_f \rho_f k_f \mathbf{u}_f) = \nabla \cdot \left( \mu_t + \frac{\mu_{ft}}{\sigma_{fk}} \right) \nabla k_f + \alpha_f \rho_f \Pi_f - \alpha_f \rho_f \varepsilon_f \\ + 2\beta(k_{fp} - k_f). \end{aligned} \quad (1)$$

128 The turbulent kinetic energy dissipation transport equation reads:

$$\begin{aligned} \frac{\partial(\alpha_f \rho_f \varepsilon_f)}{\partial t} + \nabla \cdot (\alpha_f \rho_f \varepsilon_f \mathbf{u}_f) = \nabla \cdot \left( \mu_t + \frac{\mu_{ft}}{\sigma_{fk}} \right) \nabla \varepsilon_f + \frac{\varepsilon_f}{k_f} \left[ C_1 \alpha_f \rho_f \Pi_f - C_2 \alpha_f \rho_f \varepsilon_f \right] \\ + 2C_3 \beta (\varepsilon_{fp} - \varepsilon_f). \end{aligned} \quad (2)$$

129 The first term on the RHS is the fluid-phase turbulent kinetic energy/dissipation  
 130 flux. The second term,  $\Pi_f$  is the kinetic energy production due to mean shear with



131 the third term being the turbulent kinetic energy dissipation. The remaining term  
132 is the coupling terms due to velocity covariance (see Table 9) and is a measure of  
133 how correlated the two phases are. This provides the primary coupling mechanism  
134 in this work as the two phases are only coupled through drag.

135 These two equations make up the conventional  $k_f - \varepsilon_f$  turbulence model with  
136 model constants, largely taken from compressible turbulence modelling [43], found  
137 in Table 3. The complete set of equations that make up the RA-TFM are found in  
138 Table 1 and the equations associated with the low Re number model of Launder and  
139 Sharma [21] can be found in Table 2.

Table 1: RA-TFM governing equations and the particle fluctuation energy equations.

$$\frac{\partial(\alpha_p \rho_p)}{\partial t} + \nabla \cdot (\alpha_p \rho_p \mathbf{u}_p) = 0 \quad (3)$$

$$\frac{\partial(\alpha_f \rho_f)}{\partial t} + \nabla \cdot (\alpha_f \rho_f \mathbf{u}_f) = 0 \quad (4)$$

$$\begin{aligned} \frac{\partial(\alpha_p \rho_p \mathbf{u}_p)}{\partial t} + \nabla \cdot (\alpha_p \rho_p \mathbf{u}_p \mathbf{u}_p) = \nabla \cdot \left( 2(\mu_p + \mu_{pt}) \bar{\mathbf{S}}_p \right) + \beta \left[ (\mathbf{u}_f - \mathbf{u}_p) - \frac{\nu_{ft}}{\text{Sc}_{fs} \alpha_p \alpha_f} \nabla \alpha_p \right] \\ - \nabla p_p - \alpha_p \nabla p_f + \alpha_p \rho_p \mathbf{g} \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{\partial(\alpha_f \rho_f \mathbf{u}_f)}{\partial t} + \nabla \cdot (\alpha_f \rho_f \mathbf{u}_f \mathbf{u}_f) = \nabla \cdot \left( 2(\mu_f + \mu_{ft}) \bar{\mathbf{S}}_f \right) + \beta \left[ (\mathbf{u}_p - \mathbf{u}_f) + \frac{\nu_{ft}}{\text{Sc}_{fs} \alpha_p \alpha_f} \nabla \alpha_p \right] \\ - \alpha_f \nabla p_f + \alpha_f \rho_f \mathbf{g} \end{aligned} \quad (6)$$

$$\begin{aligned} \frac{\partial(\alpha_p \rho_p k_p)}{\partial t} + \nabla \cdot (\alpha_p \rho_p k_p \mathbf{u}_p) = \nabla \cdot \left( \mu_p + \frac{\mu_{pt}}{\sigma_{pk}} \right) \nabla k_p + \alpha_p \rho_p \Pi_p - \alpha_p \rho_p \varepsilon_p \\ + 2\beta(k_{fp} - k_p) \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{\partial(\alpha_p \rho_p \varepsilon_p)}{\partial t} + \nabla \cdot (\alpha_p \rho_p \varepsilon_p \mathbf{u}_p) = \nabla \cdot \left( \mu_p + \frac{\mu_{pt}}{\sigma_{p\varepsilon}} \right) \nabla \varepsilon_p + \frac{\varepsilon_p}{k_p} \left[ C_1 \alpha_p \rho_p \Pi_p - C_2 \alpha_p \rho_p \varepsilon_p \right] \\ + 2C_3 \beta (\varepsilon_{fp} - \varepsilon_p) \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{3}{2} \left[ \frac{\partial(\alpha_p \rho_p \Theta_p)}{\partial t} + \nabla \cdot (\alpha_p \rho_p \Theta_p \mathbf{u}_p) \right] = \nabla \cdot \left( \kappa_\Theta + \frac{3\mu_{pt}}{2Pr_{pt}} \right) \nabla \Theta_p + 2\mu_p \bar{\mathbf{S}}_p : \bar{\mathbf{S}}_p \\ - p_p \nabla \cdot \mathbf{u}_p + \alpha_p \rho_p \varepsilon_p - 3\beta \Theta_p - \gamma \end{aligned} \quad (9)$$

Table 2: Low Re number turbulence model of Launder and Sharma [21].

$$\frac{\partial(\alpha_f \rho_f k_f)}{\partial t} + \nabla \cdot (\alpha_f \rho_f k_f \mathbf{u}_f) = \nabla \cdot \left( \mu_f + \frac{\mu_{ft}}{\sigma_{fk}} \right) \nabla k_f + \alpha_f \rho_f \Pi_f - \alpha_f \rho_f \varepsilon_f + 2\beta(k_{fp} - k_f) \quad (10)$$

$$\frac{\partial(\alpha_f \rho_f \tilde{\varepsilon})}{\partial t} + \nabla \cdot (\alpha_f \rho_f \tilde{\varepsilon} \mathbf{u}_f) = \nabla \cdot \left( \mu_f + \frac{\mu_{f\epsilon}}{\sigma_{f\epsilon}} \right) \nabla \tilde{\varepsilon} + C_1 \alpha_f \rho_f \Pi_f \frac{\tilde{\varepsilon}_f}{k_f} - C_2 \alpha_f \rho_f f_2 \frac{\tilde{\varepsilon}_f^2}{k_f} + E + 2C_3 \beta (\varepsilon_{fp} - \tilde{\varepsilon}_f) \quad (11)$$

where

$$\mu_{ft} = c_\mu f_\mu \rho_f \frac{k_f^2}{\tilde{\varepsilon}}$$

$$\varepsilon_f = \tilde{\varepsilon} + D$$

$$f_\mu = \exp\left(\frac{-3.4}{(1 + R_T/50)^2}\right)$$

$$f_2 = 1 - 0.3 \exp(-R_T^2)$$

$$D = 2\mu_f (\nabla \sqrt{k_f})^2$$

$$E = 2\mu_f \mu_{ft} (\nabla^2 \mathbf{u}_f)^2$$

$$R_T = \frac{k_f^2}{\nu_f \tilde{\varepsilon}}$$

Table 3:  $k_f - \varepsilon_f$  model constants.

$C_1$	$C_2$	$C_3$	$\beta_k$	$\beta_\varepsilon$	$C_{f\mu}$	$C_{p\mu}$	$\sigma_{f\epsilon}$	$\sigma_{fk}$
1.44	1.92	1	1	1	0.09	0.09	1.3	1

140 *2.1. Fluid-phase elliptic relaxation model*

141 We begin at the exact RA Reynolds stress transport equation for the fluid-phase  
 142 and for the sake of brevity the derivation is presented in the Appendix. The equation  
 143 then reads:

$$\begin{aligned}
 & \frac{\partial \langle \alpha_f \rangle \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f}{\partial t} + \nabla \cdot \langle \alpha_f \rangle \langle \mathbf{u}_f \rangle_f \otimes \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f = -\nabla \cdot \langle \alpha_f \rangle \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f \\
 & \quad - \underbrace{\langle \alpha_f \rangle \langle \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f \cdot \nabla \langle \mathbf{u}_f \rangle_f \rangle}_{\text{Production}} + \frac{1}{\rho_f} \nabla \cdot \langle \bar{\boldsymbol{\sigma}}_f \otimes \mathbf{u}_f''' \rangle - \frac{1}{\rho_f} \nabla \langle p_f \mathbf{u}_f''' \rangle \\
 & \quad + \underbrace{\frac{1}{\rho_f} \langle p_f \nabla \mathbf{u}_f''' \rangle}_{\text{pressure strain, } \phi_{yy}} - \underbrace{\frac{1}{\rho_f} \langle \bar{\boldsymbol{\sigma}}_f \cdot \nabla \mathbf{u}_f''' \rangle}_{\text{dissipation, } \varepsilon_{yy}} + \langle \alpha_f \rangle \beta \underbrace{(\langle \mathbf{u}_f''' \otimes \mathbf{u}_p'' \rangle_p - \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_p)}_{\text{velocity correlations}}.
 \end{aligned} \tag{12}$$

144 Firstly, as we intend to arrive at a transport equation for an ‘imaginary’ wall-normal  
 145 stress component a caveat is worth mentioning. If one considers the production  
 146 term in any classic eddy-viscosity model, the production term is proportional to the  
 147 mean flow gradient - importantly in the stream-wise direction. This means that  
 148 the turbulent kinetic energy is produced by the stream-wise mean flow gradients.  
 149 Consequently, in the wall-normal direction the production term vanishes.

150 The velocity correlations which arise due to phase coupling are modelled analo-  
 151 gously to those terms found in the  $k_f - \varepsilon_f$  transport equations. We set the covariance  
 152 of the fluctuations  $\langle \mathbf{u}_f''' \otimes \mathbf{u}_p'' \rangle_p = \overline{v_{fp}^2} = \beta_v \sqrt{v_p^2 v_f^2}$ , where  $\overline{v_p^2} = 2/3 k_p$  owing to its defi-  
 153 nition. The correlation factor,  $\beta_v = 1$  along with the correlation factors found in the  
 154 transport equations for turbulent kinetic energy and dissipation (see Table 9) are all  
 155 set to 1. This is a crude first approximation and the correlation factor should depend  
 156 on both mass loading and Stokes number. This is out of the scope of this work but  
 157 only a weak dependency through the relatively low mass loadings is expected.

158 Following the approach used in classic eddy-viscosity turbulence models, the di-  
 159 vergence terms appearing in the transport equation are closed by the eddy-viscosity  
 160 approximation [33]. This reads as:

$$\nabla \cdot \left[ \frac{\mu_{ft}}{\sigma_{fk}} \nabla \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f \right] \approx -\nabla \cdot \langle \alpha_f \rangle \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f + \frac{1}{\rho_f} \nabla \cdot \langle \bar{\boldsymbol{\sigma}}_f \otimes \mathbf{u}_f''' \rangle - \frac{1}{\rho_f} \nabla \langle p_f \mathbf{u}_f''' \rangle. \quad (13)$$

161 Finally, the terms left to close are the pressure strain and dissipation terms. These  
 162 terms are explicitly modelled in the  $\overline{v_f^2} - f$  model equations and are grouped into a  
 163 source term denoted  $k_f f$ ,

$$k_f f = \underbrace{\phi_{yy}}_{\text{pressure strain}} - \underbrace{\varepsilon_{yy}}_{\text{dissipation}} + \alpha_f \rho_f 6 \frac{\overline{v_f^2}}{k_f} \varepsilon_f. \quad (14)$$

164 The source term effectively redistributes turbulence energy from the stream-wise  
 165 Reynolds stress component to the wall-normal component. This is intuitive as pre-  
 166 viously discussed, when one considers a fully developed turbulent boundary layer  
 167 as the wall-normal Reynolds stress component's production is zero due to the mean  
 168 stream-wise flow gradient. This means that turbulence energy can only enter the  
 169 wall-normal component through redistribution. The original form of the source term  
 170 has been shown to overproduce in regions relatively far away from the wall and the  
 171 correction of Davidson et al. [7] is thus employed, this then reads

$$\overline{v_{f,source}^2} = \min \left\{ k_f f, -\frac{1}{T} \left[ (C_1 - 6) \overline{v_f^2} - \frac{2k_f}{3} (C_1 - 1) \right] + C_2 \Pi_f \right\}. \quad (15)$$

172 Finally, setting the wall-normal component of the fluid-phase Reynolds stress

173 tensor  $\langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f$  to  $\overline{v_f^2}$  a transport equation can be written as:

$$\begin{aligned} \frac{\partial(\alpha_f \rho_f \overline{v_f^2})}{\partial t} + \nabla \cdot (\alpha_f \rho_f \overline{v_f^2} \mathbf{u}_f) = \nabla \cdot \left( \mu_f + \frac{\mu_{ft}}{\sigma_{fk}} \right) \nabla \overline{v_f^2} + \overline{v_{f,source}^2} - \alpha_f \rho_f 6 \frac{\overline{v_f^2}}{k_f} \varepsilon_f \\ + 2\beta(\overline{v_{fp}^2} - \overline{v_f^2}). \end{aligned} \quad (16)$$

174 The reader should note that the third term on the RHS is a sink term that is  
 175 used to balance the source term  $k_f f$ . This is a modification proposed by Lien and  
 176 Kalitzin [26] and ensures that the source term  $k_f f \rightarrow 0$  as it approaches the wall.

177 Equation 16 contains no sensitivity to the wall, this is introduced through a  
 178 modified Helmholtz equation which forms an elliptic relaxation equation. The form  
 179 of this equation accounts for anisotropy close to walls and is also independent of  
 180 Reynolds number and  $y^+$  value which reads:

$$L^2 \frac{\partial^2 f}{\partial x^2} - f = \underbrace{\frac{C_1}{T} \left( \frac{\overline{v_f^2}}{k_f} - \frac{2}{3} \right)}_{\phi_{yy,S}} - \underbrace{C_2 \frac{\Pi_f}{k_f}}_{\phi_{yy,R}} - \frac{1}{T} \left( 6 \frac{\overline{v_f^2}}{k_f} - \frac{2}{3} \right). \quad (17)$$

181 The terms  $\phi_{yy,S}$  and  $\phi_{yy,R}$  are the so-called slow and rapid pressure-strain terms  
 182 [22, 33] with the final term being used to ensure far field behaviour i.e. that the  
 183 elliptic relaxation function diminishes away from walls.

184 In the original formulation of this equation as given by Durbin [8] the boundary  
 185 condition for  $f$  contains the wall distance to the fourth power in its denominator.  
 186 This lead to computational stiffness and numerical oscillations in the near-wall region.  
 187 This issue was resolved by Lien and Kalitzin [26] by introducing  $6 \frac{\overline{v_f^2}}{k_f}$  as a sink and  
 188 source in the  $k_f f$  source term, the  $\overline{v_f^2}$  transport equation and the elliptical relaxation  
 189 equation,  $f$ . This ensures that  $f$  tends to 0 at the wall enabling a Dirichlet boundary

190 condition to be prescribed. The turbulent viscosity is calculated from the solution  
 191 of the  $\overline{v_f^2} - f$  model and a correction is employed to ensure the correct velocity scale  
 192 is used as the wall is approached. The correction of Davidson et al. [7] is employed  
 193 and the definition of the turbulent viscosity now reads

$$\nu_{ft} = \min \left\{ C_{f\mu} k_f^2 / \varepsilon_f, C_\mu \overline{v_f^2} T \right\}, \quad (18)$$

194

195 where the turbulent time and length scales are defined as

$$T = \max \left( \frac{k_f}{\varepsilon_f}, 6 \sqrt{\frac{\nu_f}{\varepsilon_f}} \right), \quad (19)$$

$$L = \max \left( \frac{k_f^{3/2}}{\varepsilon_f}, C_\eta \frac{\nu_f^{3/4}}{\varepsilon_f^{1/4}} \right). \quad (20)$$

196

197 Both time and length scales are limited in regions close to the wall. This is achieved  
 198 by introducing a dependency on Kolmogorov scales which are only active in regions  
 199 very close to the wall i.e.  $y^+ < 5$ . This ensures that a singularity is not introduced  
 200 into the solution matrix and that the scales collapse at the wall. Another modification  
 201 close to the wall is to modify the “constant”  $C_{\varepsilon 1}$  by damping it in the near-wall region  
 202 by employing the following formulation

$$C_{\varepsilon 1} = 1.4 \left( 1 + 0.05 \sqrt{k_f / \overline{v_f^2}} \right). \quad (21)$$

203 To summarise, the  $\overline{v_f^2} - f$  model equations can be found in Table 4 with the

204 turbulence modelling constants, taken from the original model [8], found in Table 8.  
 205 The complete set of equations that make up the RA-TFM with the  $\overline{v_f^2} - f$  model is  
 206 then the equations found in Table 1 and the aforementioned equations.

Table 4:  $\overline{v_f^2} - f$  model equations.

$$\frac{\partial(\alpha_f \rho_f k_f)}{\partial t} + \nabla \cdot (\alpha_f \rho_f k_f \mathbf{u}_f) = \nabla \cdot \left( \mu_f + \frac{\mu_{ft}}{\sigma_{fk}} \right) \nabla k_f + \alpha_f \rho_f \Pi_f - \alpha_f \rho_f \varepsilon_f + 2\beta(k_{fp} - k_f) \quad (22)$$

$$\frac{\partial(\alpha_f \rho_f \varepsilon_f)}{\partial t} + \nabla \cdot (\alpha_f \rho_f \varepsilon_f \mathbf{u}_f) = \nabla \cdot \left( \mu_f + \frac{\mu_{ft}}{\sigma_{f\epsilon}} \right) \nabla \varepsilon_f + \frac{\varepsilon_f}{k_f} \left[ \frac{C_{\varepsilon 1} \alpha_f \rho_f \Pi_f - C_{\varepsilon 2} \alpha_f \rho_f \varepsilon_f}{T} \right] + 2C_3 \beta (\varepsilon_{fp} - \varepsilon_f) \quad (23)$$

$$\frac{\partial(\alpha_f \rho_f \overline{v_f^2})}{\partial t} + \nabla \cdot (\alpha_f \rho_f \overline{v_f^2} \mathbf{u}_f) = \nabla \cdot \left( \mu_f + \frac{\mu_{ft}}{\sigma_{fk}} \right) \nabla \overline{v_f^2} + \overline{v_{f,source}^2} - \alpha_f \rho_f 6 \frac{\overline{v_f^2}}{k_f} \varepsilon_f + 2\beta(\overline{v_{fp}^2} - \overline{v_f^2}) \quad (24)$$

$$L^2 \frac{\partial^2 f}{\partial x^2} - f = \frac{C_1}{T} \left( \frac{\overline{v_f^2}}{k_f} - \frac{2}{3} \right) - C_2 \frac{\Pi_f}{k_f} - \frac{1}{T} \left( 6 \frac{\overline{v_f^2}}{k_f} - \frac{2}{3} \right) \quad (25)$$

Table 5:  $\overline{v_f^2} - f$  model parameters.

$C_\mu$	$C_1$	$C_2$	$C_L$	$C_\eta$	$C_{\varepsilon 2}$	$C_3$	$\beta_k$	$\beta_\epsilon$	$\beta_v$	$C_{f\mu}$	$\sigma_k$	$\sigma_{f\epsilon}$
0.22	1.4	0.3	0.23	70	1.9	1	1	1	1	0.09	1	1.3

207 Wall boundary conditions for  $\varepsilon_f$  can be found by a Taylor expansion around the  
 208 no-slip condition at the wall [33] which reads as:

$$\varepsilon_f \rightarrow 2\nu_f \frac{k_f}{y^2} \quad (26)$$



209 For the remaining fluid-phase model variables the following boundary conditions  
210 at the wall are prescribed,  $\mathbf{u}_f = k_f = \overline{v_f^2} = f = 0$ . For the particulate phase  
211 a Neumann boundary condition is prescribed for the velocity and all turbulence  
212 statistics. For the simulations with the  $k_f - \varepsilon_f$  model the standard wall functions for  
213 both turbulence statistics are employed. At the inlet the velocity of both the fluid-  
214 and particle-phase are set at  $9.4\text{ms}^{-1}$ . A Neumann boundary condition is used for  $f$   
215 together with Dirichlet boundary conditions for all turbulent statistics. At the outlet  
216 the a Dirichlet boundary condition for pressure is set whilst a Neumann boundary  
217 condition is prescribed for all remaining variables. Both  $k_p$  and  $\varepsilon_p$  are initialised as  
218 1/3rd of their fluid counterpart with  $\Theta_p = 1.0 \times 10^{-8}\text{m}^2\text{s}^{-2}$ .

219 The RA-TFM and the recently derived  $\overline{v_f^2} - f$  turbulence model is implemented  
220 into the open-source toolbox OpenFOAM [53] and is denoted as `ratfmFoam` [36, 37]  
221 which is made available for public use. To handle the pressure-velocity coupling the  
222 Pressure Implicit with Splitting Operators (PISO) algorithm [10, 17] is used. The  
223 volume fraction is solved using Multi-dimensional Universal Limiter with Explicit  
224 Solution (MULES) [56] which is a flux-corrected transport algorithm which ensures  
225 robustness, stability and convergence. Time derivative terms are discretised using the  
226 first order accurate implicit Euler, gradients are discretised using the Gauss-Green  
227 scheme, convective terms are discretised using the first-order upwind scheme Finally,  
228 Laplacian schemes are discretised with the second order accurate central differencing  
229 scheme.

Table 6: Table of simulated cases

Case	Material	$d_p$ [ $\mu\text{m}$ ]	$\rho_p$ [ $\text{kg m}^{-3}$ ]	Mass loading, $\phi$	$St$
1	glass	50	2500	2%	0.57
2	copper	70	8800	10%	3

231 The cases used throughout are based on two experiments from Kulick et al. [20]  
 232 which include separately both glass and copper particles, the details of which can  
 233 be found in Table 6. For both cases the the channel half-width is  $H = 0.02\text{m}$  with  
 234 a corresponding length of  $5.2\text{m}$  and a wall friction velocity  $u_\tau = 0.49\text{ms}^{-1}$ . The  
 235 viscosity of gas is  $\nu_f = 15.11 \times 10^{-5}\text{m}^2\text{s}^{-1}$  with a density of  $\rho_f = 1.2\text{kg m}^{-3}$  The flow  
 236 is orientated vertically with a uniform body force of gravity acting in the direction  
 237 of the flow ( $g = 9.8\text{m s}^{-2}$ ), this configuration resulted in a centerline velocity of  
 238  $U_{cl} = 10.5\text{ms}^{-1}$ . The mass loading is defined as  $\phi = \frac{\alpha_p \rho_p}{\alpha_f \rho_f}$ , and assuming uniform  
 239 velocity at the inlet.

Table 7: Properties of each mesh,  $f_x, f_y$  refer to mesh stretching with Mesh 1 [ $f_x = 1.1, f_y = 1.1$ ] and Mesh 2 [ $f_x = 1.2, f_y = 1.2$ ].

Mesh	$\Delta x_{min}, \Delta x_{max}$ [m]	$\Delta y_{min}, \Delta y_{max}$ [m]	Mesh size	Comp time
1	$1.2 \times 10^{-3}, 0.02$	$1.2 \times 10^{-5}, 1.2 \times 10^{-3}$	202,761	32 hrs
"	"	"	"	20 hrs
2	$7 \times 10^{-4}, 9 \times 10^{-4}$	$7 \times 10^{-4}, 9 \times 10^{-4}$	66,481	4 hrs

240 Owing to the different modelling approaches used throughout two different meshes  
 241 are employed and are detailed in Table 7. Mesh 1 is associated with the  $\overline{v_f^2} - f$  model  
 242 and the low Re Number model of Launder and Sharma [21] and is resolved to  $y^+ < 1$

243 ensuring that the resolution of the boundary layer is captured. Mesh 2 is associated  
244 with the  $k_f - \varepsilon_f$  model and is resolved up to  $y^+ > 30$  ensuring that the wall functions  
245 can be applied across the correct section of the boundary layer (i.e. log-layer). The  
246 final column refers to the computational time spent for a typical run consisting of 30  
247 seconds of real flow time. For ease of reference the  $\overline{v_f^2} - f$  formulation will hereafter be  
248 referred to as V2F, the low Re number formulation as LE and the  $k_f - \varepsilon_f$  formulation  
249 as KE.

250 **3. Results and Discussion**

251 *3.1. Mean fluid stream-wise velocity profiles*

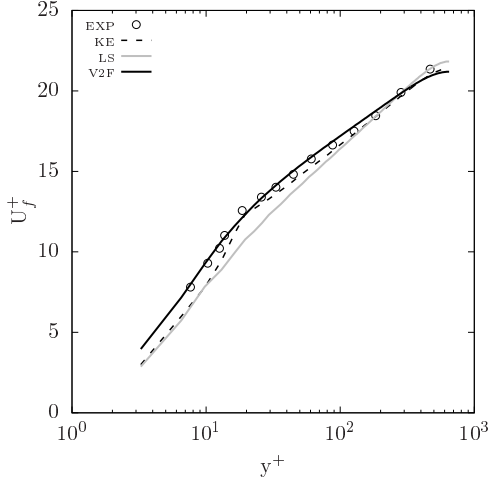


Figure 1: C1 - Mean fluid velocity profile.

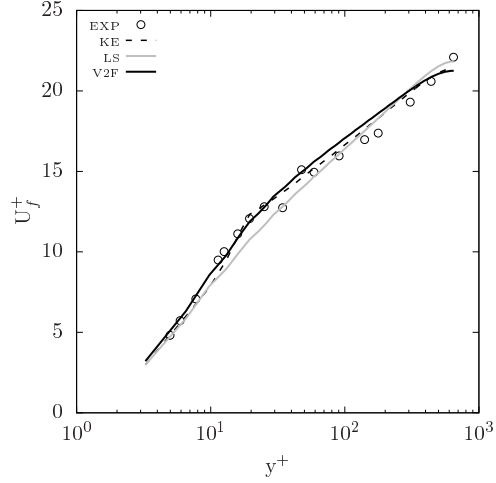


Figure 2: C2 - Mean fluid velocity profile.

252 Figures 1 & 2 show the mean fluid velocity profiles for each case. It is evident  
 253 from both plots that the prediction of both V2F & KE models are in good agreement  
 254 with the experimental data of Kulick et al. [20]. For both C1 & C2 the mean fluid ve-  
 255 locity profile remains unchanged, behaviour that is consistent with the experimental  
 256 observations. Moreover, the experimental uncertainty was reported by the authors  
 257 to be  $\approx 2\%$  and it can be seen that across both profiles the numerical prediction lies  
 258 well within this range.

259 This behaviour is not apparent in the predictions from the LE model as there is  
 260 an underestimation of the fluid velocity. It is interesting to observe that the LE and  
 261 KE predictions are similar outside the range of  $10 < y^+ < 100$  across both plots.  
 262 Over the transition region i.e. buffer layer to log-layer, the damping function tends

263 to over-predict the turbulence viscosity. The ‘kink’ is not reproduced leading to a  
 264 flattening of the velocity profile but despite this the overall prediction is satisfactory.

265 When comparing both the V2F & KE model predictions there is only a small dis-  
 266 crepancy between each result. This disparity is at its most obvious across the viscous  
 267 and buffer layer i.e.  $y^+ < 20$  in Fig. 1. Owing to the wall function the turbulence  
 268 statistics are integrated to the wall, with a presumed log-layer relationship, from the  
 269 first computational cell at  $y^+ \approx 30$ . This resulted in an over-prediction of turbulence  
 270 viscosity which is felt as an under-prediction in the mean velocity profile. This trend  
 271 is seen across the profile for both plots as the KE consistently under-predicts the  
 272 mean velocity profile in comparison with V2F although this difference is small.

273 *3.2. Mean particle stream-wise velocity profiles*

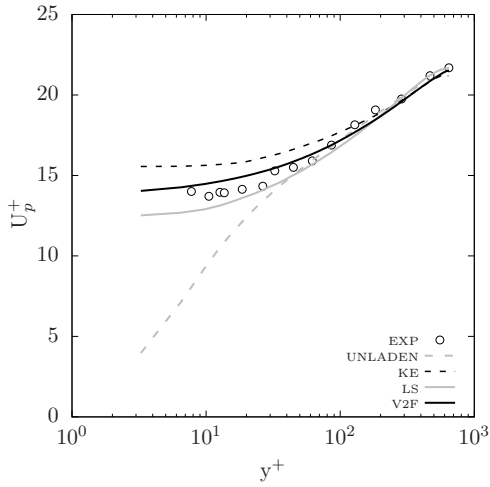


Figure 3: C1 - Mean particle velocity profile.

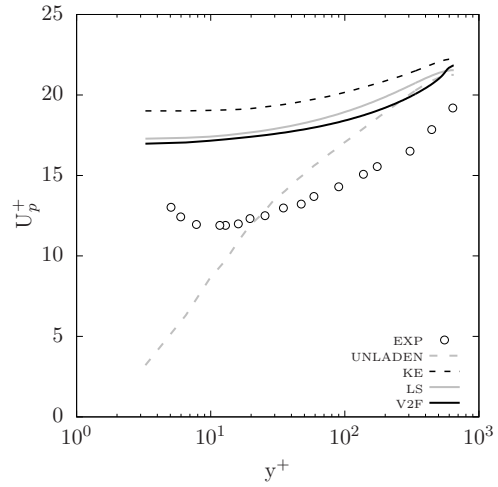


Figure 4: C2 - Mean particle velocity profile.

274 The mean particle velocity profiles are shown in Figs. 3 & 4. Focusing on the  
 275 former it can be seen that V2F, LE & KE models accurately predict the trend seen

276 in the experimental observations. The trend is characterised by a flatter profile as  
277 the particles approach the wall. As the particle velocities need not be zero at the  
278 wall unlike in the fluid-phase, a large slip value exists. The particles deviate from the  
279 fluid-phase velocities at around  $y^+ < 100$  and maintain their momentum, leading to  
280 a flattening of the profile as the wall is approached.

281 In Fig. 3 the profile predicted by the V2F model is in good agreement with  
282 the experimental data. This is also true for the KE and LE models up until the  
283 near-wall region is approached. Over the range  $y^+ < 100$  the KE prediction deviates  
284 from the experimental results as the momentum is over-predicted. The contrary  
285 is true for the LE model in which the particles remain correlated with the carrier  
286 flow up until  $y^+ \approx 50$  and then begin to deviate resulting in a under-estimation  
287 of the particle velocities. The cause of this behaviour is attributed to the particle  
288 fluctuation energy calculation. For the KE model this results is an underestimation  
289 of the energy exchange and for the LE model an overestimation, this behaviour will  
290 be discussed further in Sec. 3.5.

291 Looking at Fig. 4 it can be seen that there is an over-estimation in the mean  
292 particle velocities across all three models. This discrepancy was also predicted in the  
293 E-L results of Yamamoto et al. [54] and Wang and Squires [52]. A recent study by  
294 Vreman [51] suggests that this global reduction in the particle velocities is due to  
295 the so-called “non-uniform feedback force” which is exacerbated by wall roughness.  
296 This results in an additional drag force exerted on the particles leading to increased  
297 turbulence attenuation.

298 This additional force would result in a much flatter profile as shown in Vreman

299 [51] and lead to results that closely align with the experimental data in Fig. 4. As  
 300 wall roughness has not been modelled in this study, and similar results have been  
 301 reported by other researchers using higher resolution methods i.e. E-L [54, 52], it is  
 302 plausible to conclude that this is the source of the overestimation. It is instructive  
 303 to note that despite this, the qualitative behaviour of the profile is captured by the  
 304 numerical models resulting in a comparable trend across the profile.

305 *3.3. Fluid stream-wise turbulence intensity*

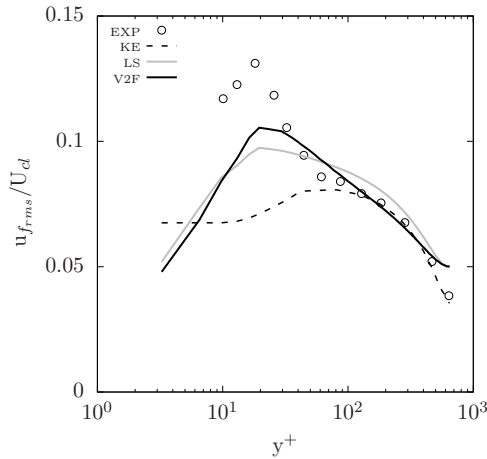


Figure 5: C1 - Fluid stream-wise turbulence intensity profile.

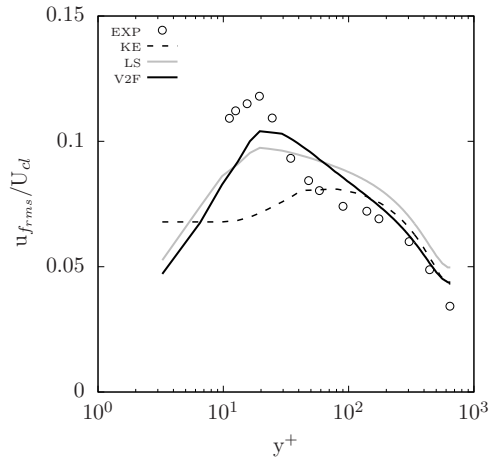


Figure 6: C2 - Fluid stream-wise turbulence intensity profile.

306 Figures 5 & 6 show the fluid-phase turbulence intensity for each case. When  
 307 comparing the V2F & KE model across both cases it is apparent that there is a clear  
 308 difference between the two. The V2F model is capable of predicting a strong peak  
 309 at  $y^+ \approx 20$  and then dissipating off into the core of the channel. This is, of course,  
 310 not seen in the KE model result as the first computational cell is placed at  $y^+ > 30$ .

311 This then omits the presence of the peak and results in a near constant value of  
312  $u_{f_{rms}}$  as the wall is approached. In the core of the flow, over the region ( $y^+ > 70$ ),  
313 there is better agreement with the experimental data as the transport terms begin  
314 to dominate.

315 The LE model performs well in comparison with the experimental data. Resolving  
316 the near wall region enables the peak to be predicted although it is not as pronounced  
317 or concentrated as the peak predicted by the V2F model. There is a “spreading” of  
318 the turbulent kinetic energy across the range,  $10 < y^+ < 100$ , smearing the transition  
319 region. This is suspected to be the cause of the degradation of the mean velocity  
320 profile in Figs. 1 & 2. This could be improved through tweaking of the damping  
321 function but as highlighted in the introduction this is an inherent shortcoming of the  
322 approach and there are several different ad-hoc solutions to dampening the viscosity  
323 but the underlying issues remain.

324 The KE models dependency on the wall function results in a deterioration of the  
325 turbulence intensity prediction. This will be shown to have important consequences  
326 when predicting the particle fluctuation energy behaviour. The V2F model shows  
327 excellent agreement across both plots with the under prediction being confined to the  
328 turbulence peak and dissipation towards the wall. It has been suggested [9, 8] that  
329 the  $\overline{v_f^2} - f$  model performs best at high  $Re$  number. In this work a relatively low  $Re$   
330 number of 14,000 is simulated which could be the cause of the under-prediction. This  
331 could be improved with a manipulation of the turbulence constant i.e.  $C_2$  although  
332 this remains out of the scope of this study.

333 Kulick et al. [20] reports turbulence attenuation in C2. As discussed in Sec. 3.2



334 this is due to the lack of wall roughness modelled in this work. Across both Figs.  
 335 the behaviour is similar with the velocity covariance terms contributing little to the  
 336 prediction. This finding is also consistent with those of Yamamoto et al. [54], Wang  
 337 and Squires [52] in which negligible attenuation was reported.

338 *3.4. Fluid wall-normal turbulence intensity*

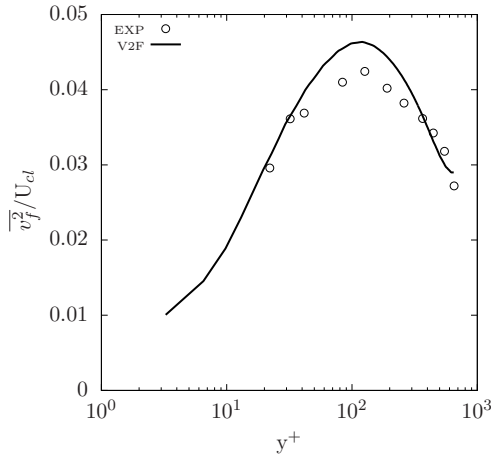


Figure 7: C1 - Fluid wall-normal turbulence intensity profile.

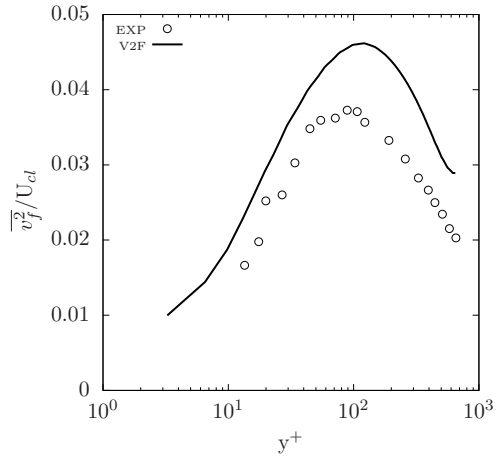


Figure 8: C2 - Fluid wall-normal turbulence intensity profile.

339 Figures 7 & 8 show the fluctuating wall-normal component. This component is  
 340 explicitly modelled in the V2F model as  $\overline{v_f^2}$  and is crucial in enabling the resolution  
 341 of the boundary layer. As it can be seen from Fig. 7 the distribution is in good  
 342 agreement with the experimental predictions. The V2F model shows the correct  
 343 dampening of the wall-normal component through the elliptic relaxation equation  
 344 and enables a strong turbulence production peak as seen in Sec. 3.3. For C2 the  
 345 wall-normal intensity was also attenuated in the same way the stream-wise intensity

346 was. As previously discussed no attenuation was reported in these results.

347 *3.5. Particle fluctuation energy*

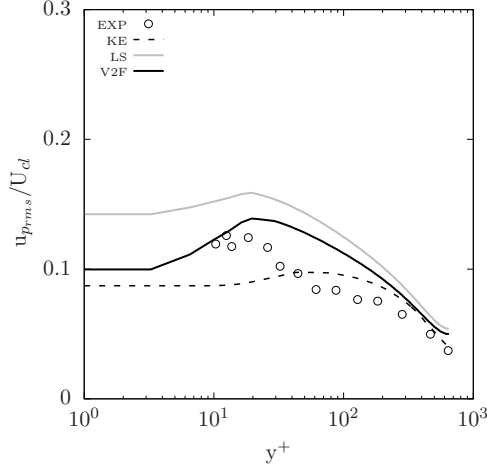


Figure 9: C1 - Particle fluctuation energy profile.

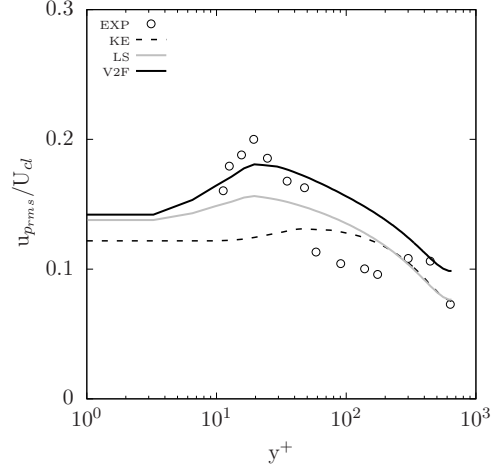


Figure 10: C2 - Particle fluctuation energy profile.

348 In the RA-TFM we explicitly account for two contributions to the particle fluctu-  
 349 ation energy [12],  $\kappa_p = 3\Theta_p + k_p$  where  $\Theta_p$  represents the small-scale kinetic collisional  
 350 energy i.e. uncorrelated energy and  $k_p$  represents the large-scale turbulent kinetic  
 351 energy i.e. correlated energy. Broadly speaking  $\Theta_p$  is relevant at high  $St$  number  
 352 and high mass loading, and  $k_p$  is relevant at low  $St$  number and low mass loading.  
 353 This distinction has already proven crucial in the literature [12, 18, 38, 39, 50].

354 Figures 9 & 10 show the particle fluctuation energy for each case. As is evident  
 355 from both plots the V2F model outperforms the KE model. This is a direct con-  
 356 sequence of the poor prediction in the fluid turbulent kinetic energy. Owing to the  
 357 relatively low  $St$  number in the core of the flow the particles are tightly correlated

358 therefore they are governed by the velocity covariance term which arises due to cou-  
359 pling through drag. The fluctuation energy distribution is dominated by  $k_p$  up until  
360 the near-wall region is approached - this is confirmed by comparing the distribution  
361 with that of Fig. 5. For C2 this is not strictly true as the  $St$  number is larger in  
362 the core of the flow resulting in a contribution acting across the half-width of the  
363 channel, this can be seen by comparing the two figures.

364 In the near-wall region the  $St$  number increases dramatically. This ensures that  
365 the particles become uncorrelated with the main carrier flow and  $\Theta_p$  is produced  
366 in the region  $y^+ < 10$ . Additionally, an energy cascade exists in which the large-  
367 scale particle turbulent kinetic energy dissipation,  $\varepsilon_p$  appears in the  $\Theta_p$  transport  
368 equation through a source term. The particle turbulence kinetic energy dissipation  
369 is then highest in the near-wall region thus contributing to the loss of correlation  
370 with the carrier flow.

371 The LE model overestimates the fluctuation energy in C1 and the profile begins  
372 to flatten out as the wall is approach. As shown in §3.2, the particle velocity profile  
373 was under predicted which is in line with the behaviour of the particle fluctuation  
374 energy. This was caused by an overproduction of  $\Theta_p$  as the energy transfer was  
375 overproduced, this resulted in an excessively large value of  $\Theta_p$  in the near-wall region.  
376 It is not obvious why this occurred as the velocity profile predicted for C2 is in good  
377 agreement with the experimental data. It can be speculated that the source of the  
378 error is the velocity covariance term as we set the correlation factor to 1 or the  
379 kinetic theory constitutive equations as we have employed standard expressions from  
380 the literature, although this is far from conclusive.

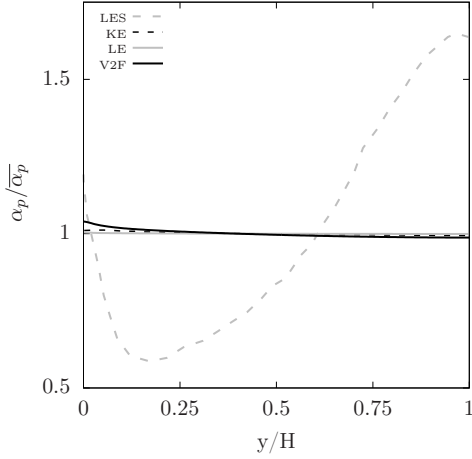


Figure 11: C1 - Volume fraction distribution normalised by mean values. The E-L results of Yamamoto et al. [54] displaying the normalised particle number density function.

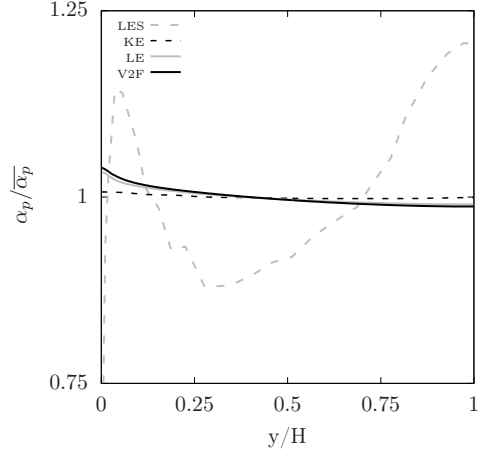


Figure 12: C2 - Volume fraction distribution normalised by mean values. The E-L results of Yamamoto et al. [54] displaying the normalised particle number density function.

382 Figures 11 & 12 show the volume fraction distribution for both cases. Addi-  
 383 tionally, the E-L results of Yamamoto et al. [54] have been displayed for qualitative  
 384 understanding. Across both plots the predictions found herein are at odds with the  
 385 E-L results. Some similarities can be drawn e.g. an accumulation in the near-wall  
 386 region in C2 but in general it is difficult to draw any conclusions from the data. The  
 387 volume fraction is a difficult statistic to predict in E-L and E-E simulations so this  
 388 result is not unexpected.

389 It is clear from both plots that the V2F model predicts an accumulation of parti-  
 390 cles in the near-wall. The particles tend to drift across the channel width and reside  
 391 in the near-wall region - characteristic behaviour of turbophoresis. As the force is  
 392 determined by the fluctuating wall normal component, of which is explicitly modelled

393 in the V2F model and coupled to the particle-phase correlated energy, the particles  
394 are able to drift down the gradients of turbulent kinetic energy.

395 The KE model predictions reveal a slightly different picture. In C1 an accumula-  
396 tion of particles in the near-wall region is seen but the sharp peak is not replicated,  
397 instead these particles are found in the main core of the flow. The prediction for C2  
398 reveals a breakdown in the volume fraction distribution in comparison to the V2F  
399 model. The particles are nearly uniformly distributed with a higher concentration  
400 in the main core of the flow. Due to the higher  $St$  number in C2 the particles are  
401 less correlated with the carrier flow, therefore in order to migrate towards the wall a  
402 larger dispersion is required. As the wall-normal component has not been explicitly  
403 modelled the particles can not overcome the turbulent kinetic energy gradient and  
404 remain in the main core of flow. Moreover, this can be a symptom of the mesh  
405 resolution as the wall function constraint ensures the near-wall region can not be  
406 resolved. The LE model results show no accumulation for C1 but do so for C2. The  
407 second result corroborates the findings from the V2F model in that the resolution of  
408 the boundary layer can lead to an accumulation of particles in the boundary layer.

#### 409 4. Conclusions

410 This work has proposed a generic approach for accounting for near-wall induced  
411 non-homogeneity in Eulerian-Eulerian simulations. An E-E elliptic relaxation model,  
412 namely the  $\overline{v_f^2} - f$  model, has been derived within a Reynolds-Averaged Two-Fluid  
413 model framework and applied to a downward-facing vertical channel. Predictions  
414 are validated against the benchmark experimental data of Kulick et al. [20] and  
415 compared against the conventional  $k_f - \varepsilon_f$  turbulence model. From this work the  
416 following conclusions can be drawn:

- 417 1. The E-E elliptic relaxation model shows improved prediction of fluid- and  
418 particle-phase turbulence statistics when compared with the conventional  $k_f -$   
419  $\varepsilon_f$  formulation;
- 420 2. The new modelling has been validated against benchmark experimental data  
421 with differing mass loading and Stokes number as well as being corroborated  
422 with Euler-Lagrange results;
- 423 3. The elliptic relaxation model has shown a high level of validation, in line with  
424 those from Euler-Lagrange, offering a viable way of achieving accurate results  
425 at a lower computational cost;
- 426 4. The use of single-phase wall functions in E-E simulations can result in an  
427 under-prediction of the velocity covariance coupling term which impedes on  
428 the particle fluctuation energy prediction. This is expected to be exacerbated  
429 with increasing mass loading;
- 430 5. The elliptic relaxation model enabled the migration of particles towards the

431 near-wall region, a result that was not replicated using the conventional tur-  
432 bulence model;

433 6. The approach presented herein offers a novel way of accounting for the near-wall  
434 region in E-E simulations.

## 435 **5. Code repository**

436 The source code of the `ratfmFoam` solver and the supplementary data used in this  
437 work can be downloaded from [\[36\]](#).

## 438 **6. Acknowledgements**

439 This work has benefited from a PhD Scholarship from the College of Engineering,  
440 Mathematics and Physical Sciences at the University of Exeter. The authors would  
441 also like to acknowledge the time used on ISCA HPC and Advanced Computing  
442 Facility at the University of Exeter.

443 **7. Appendix**

444 We begin at the fluid-phase momentum equation derived from a collisional Boltz-  
 445 mann equation like the one presented in Fox [13]. Here we couple the phases through  
 446 drag and include a body force due to gravity. This results in the equation,

$$\frac{\partial(\alpha_f \mathbf{u}_f)}{\partial t} + \nabla \cdot (\alpha_f \mathbf{u}_f \otimes \mathbf{u}_f + \alpha_f \bar{\mathbf{P}}_f) = \frac{\alpha_p \rho_p}{\rho_f} \mathcal{A} + \alpha_f \mathbf{g} \quad (27)$$

447 Taking the Reynolds-Average (RA) of Eq. 27 gives:

$$\begin{aligned} \frac{\partial \langle \alpha_f \rangle \langle \mathbf{u}_f \rangle_f}{\partial t} + \nabla \cdot \left( \langle \alpha_f \rangle \langle \mathbf{u}_f \rangle_f \otimes \langle \mathbf{u}_f \rangle_f + \langle \alpha_f \rangle \langle \mathbf{u}_f'' \otimes \mathbf{u}_f'' \rangle_f + \langle \alpha_f \rangle \langle \bar{\mathbf{P}}_f \rangle \right) \\ = \frac{\langle \alpha_p \rangle \rho_p}{\rho_f} \langle \mathcal{A} \rangle_f + \langle \alpha_f \rangle \mathbf{g} \end{aligned} \quad (28)$$

448 where  $\langle \alpha_f \rangle$  represents the RA fluid-phase volume fraction and  $\langle \mathbf{u}_f \rangle_f = \langle \alpha_f \mathbf{u}_f \rangle / \langle \alpha_f \rangle$   
 449 is the Phase-Averaged (PA) fluid-phase velocity. Now grouping the stress terms as,  
 450  $\langle \bar{\mathcal{P}}_f \rangle_f = \langle \bar{\mathbf{P}}_f \rangle_f + \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f$  and multiplying through by the PA fluid-phase velocity  
 451 one arrives at:

$$\begin{aligned} \frac{\partial \langle \alpha_f \rangle \langle \mathbf{u}_f \rangle_f \otimes \langle \mathbf{u}_f \rangle_f}{\partial t} + \nabla \cdot \left( \langle \alpha_f \rangle \langle \mathbf{u}_f \rangle_f \otimes \langle \mathbf{u}_f \rangle_f \otimes \langle \mathbf{u}_f \rangle_f + \langle \alpha_f \rangle \langle \mathbf{u}_f \rangle_f \otimes \langle \bar{\mathcal{P}}_f \rangle_f \right) \\ = \frac{\langle \alpha_p \rangle \rho_p}{\rho_f} \langle \mathbf{u}_f \rangle_f \otimes \langle \mathcal{A} \rangle_f + \langle \alpha_f \rangle \langle \mathbf{u}_f \rangle_f \otimes \mathbf{g} \end{aligned} \quad (29)$$

452 As we want to derive an equation for the Reynolds stress tensor we now find the  
 453 transport equation for the fluid-phase velocity tensor product. Note this is prior to  
 454 Reynolds-Averaging. Beginning at Eq. 27 we can multiply through by the fluid-phase  
 455 velocity, which reads as:

$$\frac{\partial(\alpha_f \mathbf{u}_f \otimes \mathbf{u}_f)}{\partial t} + \nabla \cdot (\alpha_f \mathbf{u}_f \otimes \mathbf{u}_f \otimes \mathbf{u}_f + \alpha_f \mathbf{u}_f \otimes \bar{\mathbf{P}}_f) = \frac{\alpha_p \rho_p}{\rho_f} \mathbf{u}_f \otimes \mathcal{A} + \alpha_f \mathbf{u}_f \otimes \mathbf{g} \quad (30)$$



456 Now invoking the relation for the fluid-phase pressure-stress tensor,

$$\bar{\mathbf{P}}_f = \frac{1}{\rho_f \alpha_f} (p_f \mathbf{I} - \bar{\boldsymbol{\sigma}}_f), \quad \langle \bar{\mathbf{P}}_f \rangle = \frac{1}{\rho_f \alpha_f} (\langle p_f \rangle \mathbf{I} - \langle \bar{\boldsymbol{\sigma}}_f \rangle) \quad (31)$$

457 and the momentum coupling,

$$\mathcal{A} = \frac{1}{\tau_p} (\mathbf{u}_p - \mathbf{u}_f), \quad \langle \mathcal{A} \rangle_f = \frac{1}{\tau_p} (\langle \mathbf{u}_p \rangle_f - \langle \mathbf{u}_f \rangle_f) \quad (32)$$

458 and then subtracting Eq. 29 from the RA of Eq. 30, the transport equation for  
459 the fluid-phase Reynolds-Stress can be written as

$$\begin{aligned} \frac{\partial \langle \alpha_f \rangle \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f}{\partial t} + \nabla \cdot \langle \alpha_f \rangle \langle \mathbf{u}_f \rangle_f \otimes \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f &= -\nabla \cdot \langle \alpha_f \rangle \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f \\ &\quad - \underbrace{\langle \alpha_f \rangle (\langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_f \cdot \nabla \langle \mathbf{u}_f \rangle_f)}_{\text{Production}} + \frac{1}{\rho_f} \nabla \cdot \langle \bar{\boldsymbol{\sigma}}_f \otimes \mathbf{u}_f''' \rangle - \frac{1}{\rho_f} \nabla \langle p_f \mathbf{u}_f''' \rangle \\ &\quad + \underbrace{\frac{1}{\rho_f} \langle p_f \nabla \mathbf{u}_f''' \rangle}_{\text{pressure strain, } \phi_{yy}} - \underbrace{\frac{1}{\rho_f} \langle \bar{\boldsymbol{\sigma}}_f \cdot \nabla \mathbf{u}_f''' \rangle}_{\text{dissipation, } \varepsilon_{yy}} + \underbrace{\langle \alpha_f \rangle \beta (\langle \mathbf{u}_f''' \otimes \mathbf{u}_p''' \rangle_p - \langle \mathbf{u}_f''' \otimes \mathbf{u}_f''' \rangle_p)}_{\text{velocity correlations}} \end{aligned} \quad (33)$$

460 where the fluid-phase velocity fluctuations are defined as  $\mathbf{u}_f''' = \mathbf{u}_f - \langle \mathbf{u}_f \rangle_f$ .

461 **Nomenclature**

$U_{cl}$	centreline velocity, [ $\text{ms}^{-1}$ ]
$C_D$	drag coefficient, [-]
$\mathbf{g}$	gravity, [ $\text{ms}^{-2}$ ]
$\mathbf{n}$	unit vector normal to the wall, [-]
$\text{Re}_p$	particle Reynolds number, [-]
$d_p$	particle diameter, [m]
$\mathbf{u}_i$	velocity, [ $\text{ms}^{-1}$ ]
$\mathbf{u}_p''$	particle velocity fluctuation w.r.t PA velocity, [ $\text{ms}^{-1}$ ]
$\mathbf{u}_f'''$	fluid velocity fluctuation w.r.t PA velocity, [ $\text{ms}^{-1}$ ]
$p_i$	pressure, [Pa]
$g_0$	radial distribution coefficient, [-]
$t$	time, [s]
$k_i$	turbulent kinetic energy, [ $\text{m}^2\text{s}^{-2}$ ]

462 *Greek letters*

$\alpha_i$	volume fraction, [-]
$\alpha_{p,max}$	maximum particle volume fraction, [-]
$\beta$	momentum exchange coefficient, [ $\text{kgm}^{-3}\text{s}^{-1}$ ]
$\Delta x$	length of the cell in the x direction, [m]
$\Delta y$	length of the cell in the y direction, [m]
$\varepsilon_i$	turbulent kinetic energy dissipation, [ $\text{m}^2\text{s}^{-3}$ ]
$\Theta_p$	granular temperature, [ $\text{m}^2\text{s}^{-2}$ ]
$\kappa_p$	particle fluctuation energy, [ $\text{m}^2\text{s}^{-2}$ ]
$\kappa_{\Theta s}$	diffusion coefficient for granular energy, [ $\text{kgm}^{-1}\text{s}^{-1}$ ]
$\mu_i$	shear viscosity, [ $\text{kgm}^{-1}\text{s}^{-1}$ ]
$\mu_{i,t}$	turbulent shear viscosity, [ $\text{kgm}^{-1}\text{s}^{-1}$ ]
$\nu_i$	kinematic viscosity, [ $\text{m}^2\text{s}^{-1}$ ]
$\nu_{i,t}$	turbulent kinematic viscosity, [ $\text{m}^2\text{s}^{-1}$ ]
$\rho_i$	density, [ $\text{kgm}^{-3}$ ]
$\bar{\sigma}_f$	fluid phase stress tensor, [ $\text{kgm}^{-1}\text{s}^{-2}$ ]
$\bar{\sigma}_p$	particle phase stress tensor, [ $\text{kgm}^{-1}\text{s}^{-2}$ ]
$\tau_p$	particle relaxation time, [s]
$\tau_f$	characteristic flow time, [s]

463 *Subscripts*

f	fluid
i	general index
p	particle
x	x direction
y	y direction
yy	wall normal component
z	z direction

464 *Superscripts*

"	PA particle velocity fluctuation
'''	PA fluid velocity fluctuation

465 *Special notation*

$\langle \cdot \rangle$	Reynolds averaging operator
$\langle \cdot \rangle_i$	phase averaging operator associated with phase i

Table 8: Model characteristics & turbulence variables.

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$$\beta = \frac{\rho_p \alpha_p}{\tau_d} = \frac{3}{4} \frac{\alpha_p \alpha_f \rho_f \mathbf{u}_r}{d_p} C_d$$

$$C_d = \begin{cases} \frac{24}{Re_p} \left[ 1 + 0.15 Re_p^{0.287} \right] & \text{if } Re_p < 1000 \\ 0.44 & \text{if } Re_p \geq 1000 \end{cases}$$

$$\kappa_p = k_p + 3/2 \Theta_p$$

$$u_{p_{rms}} = \sqrt{(2/3) \kappa_p}$$

$$u_{f_{rms}} = \sqrt{(2/3) k_f}$$

$$\tau_p = \frac{\rho_p d_p^2}{18 \rho_f \nu_f}$$

$$\tau_f = \frac{k_f}{\varepsilon_f}$$

$$St = \tau_d / \tau_f$$

$$e = 0.9$$

$$\Pi_p = 2\nu_{pt} \bar{\mathbf{S}}_p : \bar{\mathbf{S}}_p + \frac{2}{3} k_p \nabla \cdot \mathbf{u}_p$$

$$\Pi_f = 2\nu_{ft} \bar{\mathbf{S}}_f : \bar{\mathbf{S}}_f + \frac{2}{3} k_f \nabla \cdot \mathbf{u}_f$$

Table 9: Definition of variables.

$$\kappa_p = k_p + 1.5\Theta_p$$

$$\mu_f = \rho_f \nu_f$$

$$\mu_{ft} = \alpha_f \rho_f \nu_{ft} = \alpha_f \rho_f C_{f\mu} \frac{k_f^2}{\varepsilon_f}$$

$$\mu_p = \alpha_p \rho_p \nu_p = \frac{2\mu_{pdil}}{(1+e)g_0} \left[ 1 + \frac{4}{5}(1+e)g_0\alpha_p \right]^2 + \frac{4}{5}\alpha_p^2 \rho_p d_p g_0 (1+e) \left( \frac{\Theta_p}{\pi} \right)^{1/2}$$

$$\mu_{pdil} = \frac{5\sqrt{\pi}}{96} \rho_p d_p \Theta_p^{1/2}$$

$$\mu_{pt} = \alpha_p \rho_p \nu_{pt} = \alpha_p \rho_p C_{p\mu} \frac{k_p^2}{\varepsilon_p}$$

$$p_p = \rho_p \alpha_p \Theta_p + 2(1+e)\rho_p \alpha_p^2 g_0 \Theta_p$$

$$\gamma = \frac{12(1-e^2)g_0}{\sqrt{\pi}d_p} \alpha_p^2 \rho_p \Theta_p^{3/2}$$

$$\kappa_\Theta = \frac{2}{(1+e)g_0} \left[ 1 + \frac{6}{5}(1+e)g_0\alpha_p \right]^2 \kappa_{\Theta,dil} + 2\alpha_p^2 \rho_p d_p g_0 (1+e) \left( \frac{\Theta_p}{\pi} \right)^{1/2}$$

$$\kappa_{\Theta,dil} = \frac{75}{384} \sqrt{\pi} \rho_p d_p \Theta_p^{1/2}$$

$$g_0 = \left[ 1 - \left( \frac{\alpha_p}{\alpha_{p,max}} \right)^{\frac{1}{3}} \right]^{-1}$$

$$\bar{\mathbf{S}}_p = \frac{1}{2} [\nabla \mathbf{u}_p + (\nabla \mathbf{u}_p)^T] - \frac{1}{3} \nabla \cdot \mathbf{u}_p \mathbf{I}$$

$$\bar{\mathbf{S}}_f = \frac{1}{2} [\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^T] - \frac{1}{3} \nabla \cdot \mathbf{u}_f \mathbf{I}$$

$$k_{fp} = \beta_k \sqrt{k_f k_p}$$

$$\varepsilon_{fp} = \beta_\varepsilon \sqrt{\varepsilon_f \varepsilon_p}$$

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