On a concept of genericity for RLC networks

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Abstract

A recent definition of genericity for resistor-inductor-capacitor (RLC) networks is that the realisability set of the network has dimension one more than the number of elements in the network. We prove that such networks are minimal in the sense that it is not possible to realise a set of dimension n with fewer than $n-1$ elements. We provide an easily testable necessary and sufficient condition for genericity in terms of the derivative of the mapping from element values to impedance parameters, which is illustrated by several examples. We show that the number of resistors in a generic RLC network cannot exceed $k+1$ where k is the order of the impedance. With an example, we show that an impedance function of lower order than the number of reactive elements in the network need not imply that the network is non-generic. We prove that a network with a non-generic subnetwork is itself non-generic. Finally we show that any positive-real impedance can be realised by a generic network. In particular we show that sub-networks that are used in the important Bott-Duffin synthesis method are in fact generic.

Keywords: Network synthesis, Semi-algebraic set, Electrical networks, Minimality

1. Introduction

In recent years, there has been a resurgence of research activity in electric circuit theory, thanks in part to the invention of the inerter and the resulting analogy between electric circuits and passive mechanical networks [18]. The research has exposed several fundamental questions which have never been fully resolved. Of particular interest and importance is the concept of minimality. Notably, it is not known how to find an electric circuit to realise an arbitrary given impedance function minimally (i.e., using the least possible number of elements) [4, 9, 12]. Surprisingly, well-known networks which are apparently non-minimal in this sense, such as the Bott-Duffin realisation and its simplifications, have in fact recently been shown to be minimal for certain impedance functions [7, 8].

In this paper, we develop the notion of generic networks, as defined in [14]. The impedance of a given resistor-inductor-capacitor (RLC) network is the ratio of two polynomials

$$Z(s) = \frac{a_k s^k + a_{k-1}s^{k-1} + \cdots + a_0}{b_k s^k + b_{k-1}s^{k-1} + \cdots + b_0}, \tag{1}$$

where all coefficients are non-negative and at least one coefficient in the denominator is non-zero. Varying the element values (resistances, inductances and capacitances) over the real positive numbers generates a set of impedances characterised by the vector of coefficients $(a_0, a_1, \ldots, a_k, b_0, b_1, \ldots, b_k)$. This set can be viewed as a (real semi-algebraic) subset of $(2k+2)$-dimensional Euclidean space, which we call the realisability set of the network. In Lemma 2, we show that the dimension of the realisability set is no greater than one plus the number of elements in the network. A network is called *generic* if the dimension of the realisability set is exactly equal to one more than the number of elements in the network (Definition 1). It follows that generic networks are almost always minimal, in the sense that almost all of the impedances realized by a given generic network that contains n elements cannot also be realized by a network that contains strictly fewer than n elements.[1] In Theorem 1, we provide a necessary and sufficient condition for genericity in terms of the derivative of the mapping from element values to impedance parameters. As a corollary, we show that the number of resistors in a generic network is at most one more than the order of the impedance. The genericity concept is explored through several examples in Section V. Section VI then considers interconnection, and it is proved that a network with a non-generic subnetwork is itself non-generic (Theorem 2). Finally, we provide a proof that the Bott-Duffin networks are generic, and conclude that any positive-real impedance can be realised by a generic RLC network (Theorem 3).

We adopt the following notation throughout the paper:

$\mathbb{R}$	real numbers
$\mathbb{R}_{>0}$	positive real numbers
$\mathbb{R}_{\geq 0}$	non-negative real numbers
$\mathbb{R}^n$	(column) vectors of real numbers
$(x_1, \ldots, x_n)$	column vector

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[1]More precisely, the set of impedances in the realizability set that can be realized with strictly fewer elements is a subset of the realizability set whose codimension is at least one.

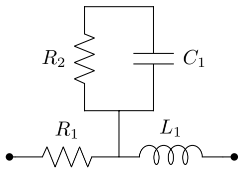

Figure 1: A network with non-coprime impedance polynomials.

2. Preliminaries

Consider an RLC two-terminal network $\mathcal{N}$ with $m \geq 1$ elements (resistors, capacitors or inductors) and corresponding parameters $\mathbf{E} = (E_1, \ldots, E_m)$. It follows from Kirchhoff's tree theorem [16, Section 7.2] that the driving-point impedance of $\mathcal{N}$ is a function of the element parameters of the form

$$Z(\mathbf{E}, s) = \frac{f(\mathbf{E}, s)}{g(\mathbf{E}, s)}, \tag{2}$$

with

$$\begin{aligned} f(\mathbf{E}, s) &= f_k(\mathbf{E})s^k + f_{k-1}(\mathbf{E})s^{k-1} + \cdots + f_0(\mathbf{E}), \\ g(\mathbf{E}, s) &= g_k(\mathbf{E})s^k + g_{k-1}(\mathbf{E})s^{k-1} + \cdots + g_0(\mathbf{E}), \end{aligned}$$

where $f_i(\mathbf{E}) = f_i(E_1, \ldots, E_m)$, $g_j(\mathbf{E}) = g_j(E_1, \ldots, E_m)$ for $0 \leq i, j \leq k$ are polynomials in $E_1, \ldots, E_m$ with non-negative integer coefficients, at least one $g_j(\mathbf{E})$ is not identically zero, and not both of $f_k(\mathbf{E})$ and $g_k(\mathbf{E})$ are identically zero. We refer to the integer k as the order of the impedance, which cannot exceed the number of reactive elements (capacitors and inductors) in the network. Following [16, Section 7.2], the polynomials $f(\mathbf{E}, s)$ and $g(\mathbf{E}, s)$ are obtained as follows. Let $\mathcal{N}'$ be obtained by connecting together the driving-point terminals in $\mathcal{N}$, and let $\tilde{f}(\mathbf{E}, s)$ (resp., $\tilde{g}(\mathbf{E}, s)$) be the Laurent polynomial given by the sum over all spanning trees in $\mathcal{N}'$ (resp., $\mathcal{N}$) of the product of the admittances of all elements in each spanning tree. Then $f(\mathbf{E}, s)$ (resp., $g(\mathbf{E}, s)$) is obtained by multiplying $\tilde{f}(\mathbf{E}, s)$ (resp., $\tilde{g}(\mathbf{E}, s)$) by the least common multiple of the denominators of the monomials in $\tilde{f}(\mathbf{E}, s)$ and $\tilde{g}(\mathbf{E}, s)$. Note that $f(\mathbf{E}, s)$ and $g(\mathbf{E}, s)$ are not guaranteed to be coprime. For example, the network in Fig. 1 yields polynomials

$$\begin{aligned} f(\mathbf{E}, s) &= R_2C_1L_1s^2 + (L_1 + R_1R_2C_1)s + R_1, \\ g(\mathbf{E}, s) &= R_2C_1s + 1, \end{aligned}$$

whereupon $g(\mathbf{E}, s)$ divides $f(\mathbf{E}, s)$.

Now, consider the candidate impedance function (1), where $a_i, b_j \in \mathbb{R}_{\geq 0}$ for $0 \leq i, j \leq k$. For the network $\mathcal{N}$ to realise the impedance function (1) it is necessary and sufficient that there exist $c \in \mathbb{R}_{>0}$ and $\mathbf{E}_0 \in \mathbb{R}^m_{>0}$ such that

$$a_i = cf_i(\mathbf{E}_0) \text{ and } b_i = cg_i(\mathbf{E}_0) \ \ (i = 0, \ldots, k). \tag{3}$$

We define the *realisability set* of $\mathcal{N}$ to be the set

$$\mathcal{S} = \{(a_0, \ldots, a_k, b_0, \ldots, b_k) \text{ such that (3) holds, } \mathbf{E}_0 \in \mathbb{R}^m_{>0} \text{ and } c \in \mathbb{R}_{>0}\big\}.$$

Let $\mathbf{x} = (x_1, \ldots, x_{m+1}) = (E_1, \ldots, E_m, c) \in \mathbb{R}^{m+1}_{>0}$ and define the function $\mathbf{h} : \mathbb{R}^{m+1}_{>0} \to \mathbb{R}^{2k+2}_{\geq 0}$ as follows:

$$\mathbf{h}(\mathbf{x}) = c\,(f_0, \ldots, f_k, g_0, \ldots, g_k)$$

Then $\mathcal{S}$ is the image of $\mathbb{R}^{m+1}_{>0}$ under $\mathbf{h}$.

The set $\mathcal{S}$ can also be seen to be the projection onto the first $2k + 2$ components of the real semi-algebraic set

$$\mathcal{S}_f = \{(a_0, \ldots, a_k, b_0, \ldots, b_k, \mathbf{E}_0, c) \text{ such that (3) holds, } \mathbf{E}_0 \in \mathbb{R}^m_{>0} \text{ and } c \in \mathbb{R}_{>0}\big\}$$

in $\mathbb{R}^{2k+m+3}_{\geq 0}$. Hence $\mathcal{S}$ is a real semi-algebraic set using the Tarski-Seidenberg theorem [2]. We use the notation $\pi_{\{r_1,\ldots,r_p\}}(\cdot)$ to denote the projection of a real semi-algebraic set onto the components with indices $r_1, \ldots, r_p$. Thus, $\mathcal{S} = \pi_{\{1,\ldots,2k+2\}}(\mathcal{S}_f)$.

3. A necessary and sufficient condition for genericity

The *dimension* $\dim(\mathcal{S})$ of a semi-algebraic set $\mathcal{S}$ is defined as the largest d such that there exists a one-to-one smooth map from the open cube $(-1, 1)^d \subset \mathbb{R}^d$ into $\mathcal{S}$ [1].

Lemma 1. *For a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ let $\pi = \pi_{\{r_1,\ldots,r_p\}}$ for some indices $r_1, \ldots, r_p$ with $p < n$, then $\dim(\pi(\mathcal{S})) \leq \dim(\mathcal{S})$ [1, Lemma 5.30].*

Lemma 2. *For an RLC two-terminal network $\mathcal{N}$ with $m \geq 1$ elements and realisability set $\mathcal{S}$ then $\dim(\mathcal{S}) \leq m + 1$.*

Proof. Given $E_{i,0} > 0$ for $1 \leq i \leq m$ and $c_0 > 0$ there exists $\epsilon > 0$ such that, for any given $(x_1, \ldots, x_{m+1}) \in (-1, 1)^{m+1}$, then $E_i = E_{i,0} + \epsilon x_i > 0$ and $c = c_0 + \epsilon x_{m+1} > 0$, whereupon $a_0, \ldots, a_k, b_0, \ldots, b_k$ are uniquely determined by the function $\mathbf{h}$. Since, in addition, $f_0, \ldots, f_k$ and $g_0, \ldots, g_k$ are all polynomials in $E_1, \ldots, E_m$, then it follows that there is a smooth one-to-one mapping from $(-1, 1)^{m+1}$ into some neighbourhood of any point in $\mathcal{S}_f$, which means that $\dim(\mathcal{S}_f) = m+1$. Note that this neighbourhood contains all points in $\mathcal{S}_f$ that are sufficiently close to the given point in the Euclidean metric. Such a neighbourhood in $\mathcal{S}_f$ is homeomorphic to the unit cube in $\mathbb{R}^{m+1}$, hence to the unit sphere in $\mathbb{R}^{m+1}$, hence not homeomorphic to a unit sphere in any other dimension [6, Theorem 2.26]. The result now follows from Lemma 1. □

We now state the definition of a generic network proposed in [14]. We note that [17] introduces a similar notion of a "non-redundant" system in the context of parameterized state space equations.

Definition 1. An RLC two-terminal network $\mathcal{N}$ containing m elements is *generic* if $\dim(\mathcal{S}) = m + 1$ where $\mathcal{S}$ is the realisability set of the network.

We introduce the matrix $D(\mathbf{E})$ defined by:

$$D(\mathbf{E}) = \begin{pmatrix} \frac{\partial f_0}{\partial E_1} & \cdots & \frac{\partial f_0}{\partial E_m} & f_0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial f_k}{\partial E_1} & \cdots & \frac{\partial f_k}{\partial E_m} & f_k \\ \frac{\partial g_0}{\partial E_1} & \cdots & \frac{\partial g_0}{\partial E_m} & g_0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial g_k}{\partial E_1} & \cdots & \frac{\partial g_k}{\partial E_m} & g_k \end{pmatrix} \tag{4}$$

and note that the derivative of $\mathbf{h}$ satisfies $D(\mathbf{E})\,\mathrm{diag}(c,\ldots,c,1) = \mathbf{h}'(\mathbf{x})$. The next theorem provides a easily testable method for determining the genericity of a given RLC network using the matrix $D(\mathbf{E})$. We note that [17, Proposition 1] provides a similar necessary and sufficient condition for a system to be "non-redundant".

Theorem 1. *Let $\mathcal{N}$ be an RLC two-terminal network comprising $m \geq 1$ elements with parameters $\mathbf{E} = (E_1,\ldots,E_m)$. Then $\mathcal{N}$ is generic if and only if there exists $\mathbf{E}_0 = (E_{1,0},\ldots,E_{m,0}) \in \mathbb{R}^m_{>0}$ such that $\mathrm{rank}(D(\mathbf{E}_0)) = m+1$.*

Proof. Assume there exists $\mathbf{E}_0 \in \mathbb{R}^m_{>0}$ such that $\mathrm{rank}(D(\mathbf{E}_0)) = m+1$ and note that $\mathrm{rank}(\mathbf{h}'(\mathbf{x}_0)) = m+1$ for $\mathbf{x}_0 = (\mathbf{E}_0, c)$ for any $c > 0$. Let A be a square submatrix of $\mathbf{h}'(\mathbf{x}_0)$ consisting of rows $l_1,\ldots,l_{m+1}$ for which $\det(A) \neq 0$. Let $\hat{\mathbf{h}}(\mathbf{x})$ be the restriction of $\mathbf{h}(\mathbf{x})$ to the components $l_1,\ldots,l_{m+1}$. Then, by the inverse function theorem [15, Theorem 9.24], $\hat{\mathbf{h}}(\mathbf{x})$ is a one-to-one mapping from a neighbourhood of $\mathbf{x}_0$ into $\mathbb{R}^{m+1}_{>0}$, which means that $\mathbf{h}(\mathbf{x})$ is a smooth one-to-one mapping from a neighbourhood of $\mathbf{x}_0$ into $\mathcal{S}$. Hence $\dim(\mathcal{S}) = m+1$ which means that $\mathcal{N}$ is generic.

Conversely, assume that $\dim(\mathcal{S}) = m+1$. Then there exists $\mathbf{x}_0 = (E_{1,0},\ldots,E_{m,0},c_0) \in \mathbb{R}^{m+1}_{>0}$ such that $\mathbf{h}(\mathbf{x})$ is a smooth one-to-one mapping from a neighbourhood of $\mathbf{x}_0$ into $\mathcal{S}$. Then there exists a differentiable inverse $\mathbf{w}(\mathbf{y})$ from a neighbourhood of $\mathbf{y}_0 = \mathbf{h}(\mathbf{x}_0)$ within $\mathcal{S}$ into a neighbourhood of $\mathbf{x}_0$. In particular $\mathbf{w}(\mathbf{h}(\mathbf{x})) = \mathbf{x}$ in a neighbourhood of $\mathbf{x}_0$. Using the chain rule [15, Theorem 9.15] $\mathbf{w}'(\mathbf{y}_0)\mathbf{h}'(\mathbf{x}_0) = I$, so $\mathrm{rank}(\mathbf{h}'(\mathbf{x}_0)) = m+1$. Writing $\mathbf{x}_0 = (\mathbf{E}_0, c)$ then $\mathrm{rank}(D(\mathbf{E}_0)) = m+1$, which completes the proof. □

In the next corollary, we provide an alternative necessary and sufficient condition for genericity in terms of the matrix

$$H(\mathbf{E},s) = \begin{pmatrix} \frac{\partial f}{\partial E_1} & \frac{\partial f}{\partial E_2} & \cdots & \frac{\partial f}{\partial E_m} & f \\ \frac{\partial g}{\partial E_1} & \frac{\partial g}{\partial E_2} & \cdots & \frac{\partial g}{\partial E_m} & g \end{pmatrix}. \tag{5}$$

Corollary 1. *If an RLC two-terminal network $\mathcal{N}$ contains elements with parameters $\mathbf{E} = (E_1,\ldots,E_m)$ and has impedance $f(\mathbf{E},s)/g(\mathbf{E},s)$, then $\mathcal{N}$ is generic if and only if there exists $\mathbf{E}_0 = (E_{1,0},\ldots,E_{m,0}) \in \mathbb{R}^m_{>0}$ such that*

$$\mathbf{x} \in \mathbb{R}^{m+1} \text{ and } H(\mathbf{E}_0,s)\mathbf{x} = 0 \quad \Rightarrow \quad \mathbf{x} = 0\,. \tag{6}$$

Proof. It can be easily verified that $H(\mathbf{E}_0,s)\mathbf{x}$ is a vector comprising two polynomials in s whose coefficients are given by the entries in the vector $D(\mathbf{E}_0)\mathbf{x}$, where $D(\mathbf{E})$ is defined in (4). In order for both polynomials in $H(\mathbf{E}_0,s)\mathbf{x}$ to be zero, each coefficient of each power of s has to be zero, from which we can conclude that $H(\mathbf{E}_0,s)\mathbf{x} = 0$ if and only if $D(\mathbf{E}_0)\mathbf{x} = 0$. By Theorem 1, the network $\mathcal{N}$ is generic if and only if the matrix $D(\mathbf{E})$ in (4) has full column rank for some $\mathbf{E}_0 = (E_{1,0},\ldots,E_{m,0}) \in \mathbb{R}^m_{>0}$. This is equivalent to

$$\mathbf{x} \in \mathbb{R}^{m+1} \text{ and } D(\mathbf{E}_0)\mathbf{x} = 0 \quad \Rightarrow \quad \mathbf{x} = 0\,.$$

Hence $\mathcal{N}$ is generic if and only if (6) holds. □

Corollary 2. *Let $\mathcal{N}$ be a generic RLC network whose impedance takes the form of* (2)*. Then the number of resistors in $\mathcal{N}$ is less than or equal to $k+1$.*

Proof. Let n be the number of resistors in $\mathcal{N}$ and m be the total number of elements. Then in order that $\mathrm{rank}(D(\mathbf{E}_0)) = m+1$ it is necessary that $2k+2 \geq m+1$. Given that $k \leq m-n$ (the number of reactive elements in $\mathcal{N}$), the result follows. □

Note that if $\mathrm{rank}(D(\mathbf{E}_0)) = m+1$ for some $\mathbf{E}_0 \in \mathbb{R}^m_{>0}$, then $\mathrm{rank}(D(\mathbf{E}_0)) = m+1$ for almost all $\mathbf{E}_0 \in \mathbb{R}^m_{>0}$. More specifically, the set of all $\mathbf{E}_0 \in \mathbb{R}^m_{>0}$ such that $\mathrm{rank}(D(\mathbf{E}_0)) < m+1$ is a subset of $\mathbb{R}^m_{>0}$ of codimension greater than or equal to one. This follows since $\mathrm{rank}(D(\mathbf{E}_0)) < m+1$ if and only if $\det((D(\mathbf{E}_0))^T D(\mathbf{E}_0)) = 0$. But $\det((D(\mathbf{E}))^T D(\mathbf{E}))$ is a polynomial in $E_1,\ldots,E_m$, so it either vanishes identically or it vanishes on a subset of $\mathbb{R}^m_{>0}$ of codimension greater than or equal to one.

Similarly, it can be shown that if there exist $\mathbf{E}_0 \in \mathbb{R}^m_{>0}$ such that (6) holds, then (6) holds for almost all $\mathbf{E}_0 \in \mathbb{R}^m_{>0}$.

4. Examples

The necessary and sufficient condition in Theorem 1, together with the necessary condition in Corollary 2, provides an efficient way of verifying genericity of RLC networks which does not rely on obtaining the realisability conditions of the networks. Throughout this section we will say that $\mathrm{rank}(D) = p$, where the general expression for $D(\mathbf{E})$ is given in (4), if $p = \max_{\mathbf{E}_0 \in \mathbb{R}^m_{>0}}(\mathrm{rank}(D(\mathbf{E}_0)))$.

Example 1. The network in Fig. 2 is a first trivial example of a non-generic network. This can be verified through Corollary 2 or by considering that the network can be reduced to a network consisting of a single resistor, which defines a realisability set of dimension two.

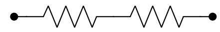

Figure 2: A simple non-generic network.

Example 2. The so-called "Ladenheim catalogue" is the set of all essentially distinct RLC networks with at most five elements of which at most two are reactive [13] [14]. To obtain the set, all basic graphs with at most five edges are listed and populated with the three types of components. A number of networks which contain a series or parallel connection of the same type of component are then trivially simplified: it can be shown in a similar way to Example 1 that such networks are all non-generic. This initial enumeration leads to 148 networks, out of which another 40 networks are eliminated which are capable of being "reduced" to simpler networks. An example of one of these 40 networks eliminated in the last step is shown in Fig. 3. The impedance of this network is a biquadratic, with

$$\begin{aligned} f_2 &= C_1C_2(R_1R_2 + R_1R_3 + R_2R_3)\,, & g_2 &= C_1C_2(R_2 + R_3)\,, \\ f_1 &= C_1(R_1 + R_3) + C_2(R_2 + R_3)\,, & g_1 &= C_1\,, \\ f_0 &= 1\,, & g_0 &= 0\,. \end{aligned}$$

Since $g_0 = 0$, it follows that $D(\mathbf{E})$ contains one row that is identically zero. Therefore $\mathrm{rank}(D) \leq 5$ and from Theorem 1 the network is non-generic. It can also be seen, through a Zobel

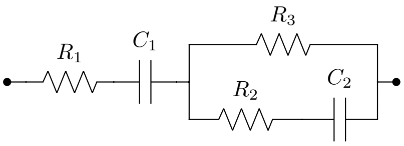

Figure 3: Non-generic network.

transformation (see [14]), that the network reduces to a generic four-element network. The remaining 108 networks in the catalogue have been shown to be generic in [14, Theorem 7.3] by directly determining the dimension of the realisability set of the networks. Using Theorem 1 it is possible to give an alternative proof of this result. For this it is sufficient to verify that the matrix $D(\mathbf{E})$ in (4) has full rank for one network in each of the 24 subfamilies in the Ladenheim catalogue. By way of illustration we prove here and in Example 3 the genericity of two networks from the catalogue. The impedance of the network in Fig. 4(a) (which has also been studied in [11]) is a biquadratic and it can be easily computed that the determinant of the 6×6 matrix $D(\mathbf{E})$ is

$$\begin{aligned} &- C_1L_1(C_1R_1R_2(R_1R_2 + R_2R_3) + L_1R_3(R_2 + R_3)) \\ &\times (C_1R_1R_2(R_2 + 2R_3)(R_1 + R_3) - L_1(R_2 + R_3)(2R_1 + R_3)), \end{aligned}$$

which is not identically zero, hence $\text{rank}(D) = 6$. Therefore, the network is generic by Theorem 1 and defines a realisability set $\mathcal{S}$ of dimension six.

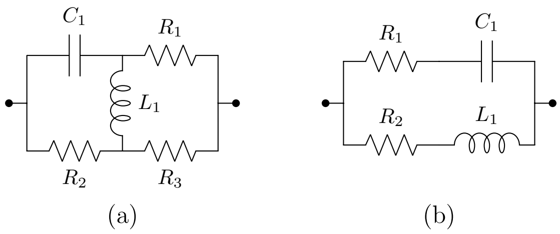

Figure 4: Two generic networks (networks #95 and #97, respectively) from the Ladenheim catalogue [14].

Example 3. The four-element network in Fig. 4(b) is another generic network from the Ladenheim catalogue which realises a biquadratic impedance. The determinant of the 5×5 submatrix obtained from $D(\mathbf{E})$ by removing the last row is equal to

$$R_2C_1(R_1R_2C_1 - L_1)\,,$$

which is not identically zero, hence $\text{rank}(D) = 5$. Therefore, the network is generic by Theorem 1 and defines a realisability set $\mathcal{S}$ of dimension five. Since all six impedance coefficients are non-zero, this means that they must be interdependent. We can in fact show (as also derived in [14]) that

$$(f_2g_0 + f_0g_2)(f_2g_0 + f_0g_2 - f_1g_1) + f_0f_2g_1^2 = 0\,.$$

Example 4. Fig. 5 can be obtained by adding a resistor to the generic network of Fig. 4(a). This network is no longer generic, by Corollary 2. In fact, it can be computed that the impedance is a biquadratic, so $D(\mathbf{E})$ has 6 rows, hence $\text{rank}(D) \leq 6$. This network has been considered in [10], [19].

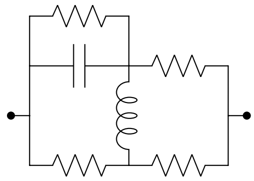

Figure 5: Non-generic network.

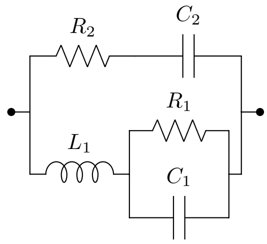

Figure 6: Three-reactive five-element generic network.

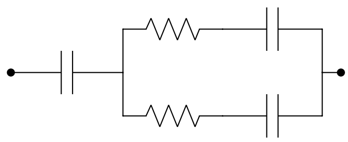

Figure 7: Three-reactive five-element non-generic network.

Example 5. The impedance of the three-reactive five-element network in Fig. 6 (which has been analysed in [11]) is a bicubic. The determinant of the 6×6 submatrix obtained by removing the last two rows of $D(\mathbf{E})$ is equal to

$$R_1^3L_1^2C_1^2C_2^3(R_1R_2C_1 - R_2^2C_2 - L_1)\,,$$

which is not identically zero, hence $\text{rank}(D) = 6$. Therefore, the network is generic by Theorem 1 and defines a realisability set of dimension six.

Example 6. The impedance of the three-reactive five-element network in Fig. 7 is a biquadratic, with $g_0 = 0$. This is an example where the order of the impedance $k = 2$ is strictly less than the number of reactive elements. In this case, $D(\mathbf{E})$ contains one row that is identically zero (as $g_0 = 0$), hence $\text{rank}(D) \leq 5$ necessarily, and the network is non-generic by Theorem 1.

Example 7. The seven-element network in Fig. 8 (see Fig. 3 in [7]) is another example where the order of the impedance ($k = 4$) is strictly less than the number of reactive elements in the network, as pointed out in [7]. This loss of order can be seen from Kirchhoff's tree theorem (see [16, Section 7.2]) since there can be no spanning tree of the network which contains all three capacitors. In this case it can be verified that $\text{rank}(D) = 8$. Hence the network is generic and defines a realisability set of dimension eight. Note that this means that a lower than expected order of the impedance need not imply that the network is non-generic.

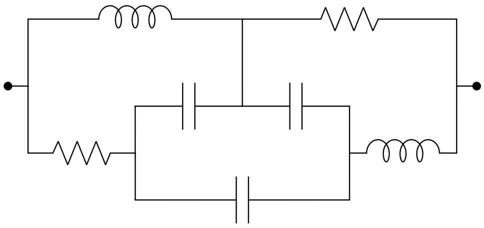

Figure 8: Five-reactive element generic network from [7] of fourth order.

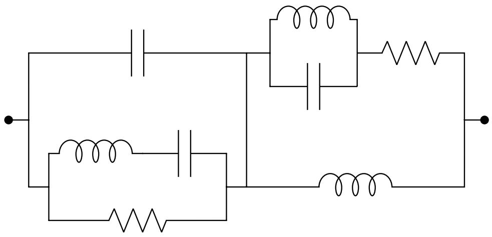

Figure 9: Bott-Duffin network for the realisation of a biquadratic.

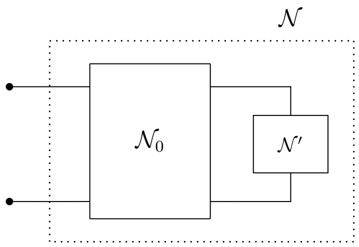

Figure 10: Two-terminal network $\mathcal{N}$ with a two-terminal subnetwork $\mathcal{N}'$.

Example 8. The network in Fig. 9 has the same structure as the Bott-Duffin construction for the biquadratic minimum function $Z(s)$ with $Z(j\omega_1) = j\omega_1 X_1$, where $\omega_1 > 0$ and $X_1 > 0$ [3]. Assuming that all network elements can vary freely, it is interesting to see whether the network is generic. The network has eight elements and its impedance is of sixth order. It can be computed that $\mathrm{rank}(D) = 9$, hence the network is generic and defines a realisability set of dimension nine. It can also be verified that by adding a resistor in series or in parallel to the network in Fig. 9 the resulting network is still generic (with a realisability set of dimension ten).

5. Interconnection of generic networks

In this section we look at the genericity of interconnections of networks, and prove: (i) the result that having a non-generic subnetwork embedded within a network leads to non-genericity of the overall network; and (ii) that any positive-real impedance can be realised by a generic network.

Lemma 3. *Consider an RLC two-terminal network $\mathcal{N}$ with the structure shown in Fig. 10, in which the network $\mathcal{N}_0$ comprises $m \geq 1$ elements with parameters $\hat{\mathbf{E}} = (\hat{E}_1, \ldots, \hat{E}_m)$ and the network $\mathcal{N}'$ comprises $n \geq 1$ elements with parameters $\bar{\mathbf{E}} = (\bar{E}_1, \ldots, \bar{E}_n)$, whereupon $\mathcal{N}$ comprises $n + m$ elements with parameters $\mathbf{E} = (\hat{\mathbf{E}}, \bar{\mathbf{E}})$. If the driving-point impedance of $\mathcal{N}'$ is $f(\bar{\mathbf{E}}, s)/g(\bar{\mathbf{E}}, s)$, then the impedance of $\mathcal{N}$ takes the form*

$$Z(\mathbf{E}, s) = \frac{u(\hat{\mathbf{E}}, s) f(\bar{\mathbf{E}}, s) + v(\hat{\mathbf{E}}, s) g(\bar{\mathbf{E}}, s)}{w(\hat{\mathbf{E}}, s) f(\bar{\mathbf{E}}, s) + z(\hat{\mathbf{E}}, s) g(\bar{\mathbf{E}}, s)}, \tag{7}$$

where $u(\hat{\mathbf{E}}, s)$, $v(\hat{\mathbf{E}}, s)$, $w(\hat{\mathbf{E}}, s)$ and $z(\hat{\mathbf{E}}, s)$ are polynomials in s whose coefficients are polynomials in $\hat{E}_1, \ldots, \hat{E}_m$, while $f(\bar{\mathbf{E}}, s)$ and $g(\bar{\mathbf{E}}, s)$ are polynomials in s whose coefficients are polynomials in $\bar{E}_1, \ldots, \bar{E}_n$.

Proof. Let G be the undirected graph with edges corresponding to the network elements $\hat{E}_1, \ldots, \hat{E}_m$ of $\mathcal{N}$ and one edge corresponding to network $\mathcal{N}'$. Let $\widetilde{G}$ be the graph obtained by connecting together the vertices corresponding to the driving-point terminals in G. Denote by $f_G(\mathbf{E}, s)$ the Laurent polynomial given by the sum over all spanning trees in G of the product of the admittances of all edges in each spanning tree, and similarly for $f_{\widetilde{G}}(\mathbf{E}, s)$. Then, by Kirchhoff's matrix tree theorem, the impedance of $\mathcal{N}$ is equal to $f_{\widetilde{G}}(\mathbf{E}, s)/f_G(\mathbf{E}, s)$. Given that the admittance of one of the edges in G and $\widetilde{G}$ is $g(\bar{\mathbf{E}}, s)/f(\bar{\mathbf{E}}, s)$, it follows that the impedance of $\mathcal{N}$ takes the form (7). □

Theorem 2. *Let $\mathcal{N}$ and $\mathcal{N}'$ be as in Lemma 3. If the subnetwork $\mathcal{N}'$ is non-generic then $\mathcal{N}$ is non-generic.*

Proof. Let $f(\bar{\mathbf{E}}, s)$, $g(\bar{\mathbf{E}}, s)$, $u(\hat{\mathbf{E}}, s)$, $v(\hat{\mathbf{E}}, s)$, $w(\hat{\mathbf{E}}, s)$ and $z(\hat{\mathbf{E}}, s)$ be as in Lemma 3. Then the impedance $Z(\mathbf{E}, s) = a(\mathbf{E}, s)/b(\mathbf{E}, s)$ of $\mathcal{N}$ takes the form (7), and we can write

$$\begin{pmatrix} a(\mathbf{E}, s) \\ b(\mathbf{E}, s) \end{pmatrix} = M(\hat{\mathbf{E}}, s) \begin{pmatrix} f(\bar{\mathbf{E}}, s) \\ g(\bar{\mathbf{E}}, s) \end{pmatrix}, \tag{8}$$

where

$$M(\hat{\mathbf{E}}, s) = \begin{pmatrix} u(\hat{\mathbf{E}}, s) & v(\hat{\mathbf{E}}, s) \\ w(\hat{\mathbf{E}}, s) & z(\hat{\mathbf{E}}, s) \end{pmatrix}$$

is a matrix of polynomials in s whose coefficients are polynomials in $\hat{\mathbf{E}} = (\hat{E}_1, \ldots, \hat{E}_m)$, while $f(\bar{\mathbf{E}}, s)$ and $g(\bar{\mathbf{E}}, s)$ are polynomials in s whose coefficients are polynomials in $\bar{\mathbf{E}} = (\bar{E}_1, \ldots, \bar{E}_n)$. By Corollary 1, the network $\mathcal{N}$ is generic if and only if there exists $\mathbf{E}_0 = (\hat{\mathbf{E}}_0, \bar{\mathbf{E}}_0) \in \mathbb{R}^{m+n}_{>0}$ such that

$$\mathbf{x} \in \mathbb{R}^{m+n+1} \text{ and } H(\mathbf{E}_0, s)\mathbf{x} = 0 \quad \Rightarrow \quad \mathbf{x} = 0, \tag{9}$$

where

$$H(\mathbf{E}, s) = \begin{pmatrix} \frac{\partial a}{\partial \hat{E}_1} & \cdots & \frac{\partial a}{\partial \hat{E}_m} & \frac{\partial a}{\partial \bar{E}_1} & \cdots & \frac{\partial a}{\partial \bar{E}_n} & a \\ \frac{\partial b}{\partial \hat{E}_1} & \cdots & \frac{\partial b}{\partial \hat{E}_m} & \frac{\partial b}{\partial \bar{E}_1} & \cdots & \frac{\partial b}{\partial \bar{E}_n} & b \end{pmatrix}.$$

Since $M(\hat{\mathbf{E}}, s)$ is independent of $\bar{\mathbf{E}}$, it follows from (8) that

$$H(\mathbf{E}, s) = \left(* \;\middle|\; M(\hat{\mathbf{E}}, s) \bar{H}(\bar{\mathbf{E}}, s) \right),$$

where the first block of the matrix corresponds to the partial derivatives of $a(\mathbf{E}, s)$ and $b(\mathbf{E}, s)$ with respect to $\hat{E}_1, \ldots, \hat{E}_m$, and

$$\bar{H}(\bar{\mathbf{E}}, s) = \begin{pmatrix} \frac{\partial f}{\partial \bar{E}_1} & \cdots & \frac{\partial f}{\partial \bar{E}_n} & f \\ \frac{\partial g}{\partial \bar{E}_1} & \cdots & \frac{\partial g}{\partial \bar{E}_n} & g \end{pmatrix}.$$

Since $\mathcal{N}'$ is non-generic, given any $\bar{\mathbf{E}}_0 \in \mathbb{R}^n_{>0}$ there exists $0 \neq \mathbf{y} \in \mathbb{R}^{n+1}$ such that $H(\bar{\mathbf{E}}_0, s)\mathbf{y} = 0$. It follows that, for any given $\mathbf{E}_0 \in \mathbb{R}^{m+n}_{>0}$, there exists $0 \neq \mathbf{y} \in \mathbb{R}^{n+1}$ such that

$$H(\mathbf{E}_0, s)\begin{pmatrix}0\\ \mathbf{y}\end{pmatrix} = \left(\, * \;\middle|\; M(\hat{\mathbf{E}}_0, s)\bar{H}(\bar{\mathbf{E}}_0, s)\right)\begin{pmatrix}0\\ \mathbf{y}\end{pmatrix} = 0\,,$$

which contradicts (9). Therefore $\mathcal{N}$ is non-generic. □

Corollary 3. *A necessary condition for the series or parallel connection of two networks $\mathcal{N}_1$ and $\mathcal{N}_2$ to be generic is that $\mathcal{N}_1$ and $\mathcal{N}_2$ are generic.*

Proof. This follows from Theorem 2. □

It is worth noting that the necessary condition in Corollary 3 is not sufficient for a series connection of two networks to be generic. The networks in Figs. 2 and 3 are simple examples of non-generic networks consisting of a series connection of two generic networks.

We now prove three lemmas that are needed in the proof of Theorem 3.

Lemma 4. *Consider an RLC two-terminal network $\mathcal{N}$ with the structure shown in Fig. 11, where the subnetwork $\mathcal{N}_1$ is generic and does not have an impedance zero at the origin. Then $\mathcal{N}$ is generic.*

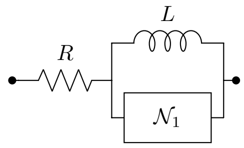

Figure 11: Two-terminal network with a generic subnetwork $\mathcal{N}_1$.

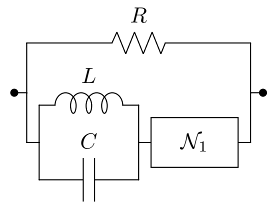

Figure 12: Two-terminal network with a generic subnetwork $\mathcal{N}_1$.

Proof. Denote the element parameters in $\mathcal{N}_1$ by $\hat{\mathbf{E}} = (\hat{E}_1, \ldots, \hat{E}_m)$, and let the impedance of $\mathcal{N}_1$ be $f(\hat{\mathbf{E}}, s)/g(\hat{\mathbf{E}}, s)$. Then $\mathcal{N}$ comprises elements with parameters $\mathbf{E} = (R, L, \hat{\mathbf{E}})$, and the impedance of $\mathcal{N}$ takes the form $Z(\mathbf{E}, s) = a(\mathbf{E}, s)/b(\mathbf{E}, s)$ where

$$\begin{aligned} a(\mathbf{E}, s) &= R(f(\hat{\mathbf{E}}, s) + sLg(\hat{\mathbf{E}}, s)) + sLf(\hat{\mathbf{E}}, s), \text{ and} \\ b(\mathbf{E}, s) &= f(\hat{\mathbf{E}}, s) + sLg(\hat{\mathbf{E}}, s)\,. \end{aligned}$$

Since $\mathcal{N}_1$ is generic, it follows from Corollary 1 that there exists $\hat{\mathbf{E}}_0 = (\hat{E}_{1,0}, \ldots, \hat{E}_{m,0}) \in \mathbb{R}^m_{>0}$ such that

$$\mathbf{y} \in \mathbb{R}^{m+1} \text{ and } \hat{H}(\hat{\mathbf{E}}_0, s)\mathbf{y} = 0 \quad \Rightarrow \quad \mathbf{y} = 0\,, \tag{10}$$

where

$$\hat{H}(\hat{\mathbf{E}}, s) = \begin{pmatrix} \frac{\partial f}{\partial \hat{E}_1} & \cdots & \frac{\partial f}{\partial \hat{E}_m} & f \\ \frac{\partial g}{\partial \hat{E}_1} & \cdots & \frac{\partial g}{\partial \hat{E}_m} & g \end{pmatrix}. \tag{11}$$

To prove that $\mathcal{N}$ is generic we need to show that there exists $\mathbf{E}_0 = (R_0, L_0, \hat{\mathbf{E}}_0) \in \mathbb{R}^{m+2}_{>0}$ such that

$$\mathbf{x} \in \mathbb{R}^{m+3} \text{ and } H(\mathbf{E}_0, s)\mathbf{x} = 0 \;\Rightarrow\; \mathbf{x} = 0\,, \tag{12}$$

where

$$H(\mathbf{E}, s) = \begin{pmatrix} \frac{\partial a}{\partial R} & \frac{\partial a}{\partial L} & \frac{\partial a}{\partial \hat{E}_1} & \cdots & \frac{\partial a}{\partial \hat{E}_m} & a \\ \frac{\partial b}{\partial R} & \frac{\partial b}{\partial L} & \frac{\partial b}{\partial \hat{E}_1} & \cdots & \frac{\partial b}{\partial \hat{E}_m} & b \end{pmatrix}.$$

To show this, we note that since $a(\mathbf{E}, s)$ and $b(\mathbf{E}, s)$ depend on $\hat{\mathbf{E}}$ through $f(\hat{\mathbf{E}}, s)$ and $g(\hat{\mathbf{E}}, s)$, by the chain rule (12) is equivalent to

$$\mathbf{x} \in \mathbb{R}^{m+3} \text{ and } M(\mathbf{E}_0, s)\left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & \hat{H}(\hat{\mathbf{E}}_0, s) \end{array}\right)\mathbf{x} = 0 \;\Rightarrow\; \mathbf{x} = 0 \tag{13}$$

where I_2 is the two-by-two identity matrix, and $M(\mathbf{E}, s)$ takes the form

$$\begin{pmatrix} sLg(\hat{\mathbf{E}}, s) + f(\hat{\mathbf{E}}, s) & (Rg(\hat{\mathbf{E}}, s) + f(\hat{\mathbf{E}}, s))s & sL + R & sRL \\ 0 & sg(\hat{\mathbf{E}}, s) & 1 & sL \end{pmatrix}.$$

Since (10) holds, it then suffices to show that, for any given real scalars u, v and polynomials $w(s)$, $z(s)$ of degree less than or equal to n, then

$$\begin{aligned} &u(sLg(\hat{\mathbf{E}}_0, s) + f(\hat{\mathbf{E}}_0, s)) + v(Rg(\hat{\mathbf{E}}_0, s) + f(\hat{\mathbf{E}}_0, s))s \\ &\quad + w(s)(sL + R) + sRLz(s) = 0 \text{ and} \end{aligned} \tag{14}$$

$$svg(\hat{\mathbf{E}}_0, s) + w(s) + sLz(s) = 0 \tag{15}$$

imply that $u = v = w(s) = z(s) = 0$. Subtracting (15) multiplied by R from (14) we obtain

$$u(sLg(\hat{\mathbf{E}}_0, s) + f(\hat{\mathbf{E}}_0, s)) + vsf(\hat{\mathbf{E}}_0, s) + sLw(s) = 0\,. \tag{16}$$

We let $s = 0$ in (15) and (16) to conclude that $w(0)$=0 and u=0 (since $f(\hat{\mathbf{E}}_0, 0) \neq 0$). Equation (16) now reduces to $vf(\hat{\mathbf{E}}_0, s) + Lw(s) = 0$, and again by setting $s = 0$ we can conclude that $v = 0$. Finally, $w(s) = z(s) = 0$ easily follows from (15) and (16). We have thus shown that (13) holds, and it follows that $\mathcal{N}$ is generic. □

Lemma 5. *Consider an RLC two-terminal network $\mathcal{N}$ with the structure shown in Fig. 12, where the subnetwork $\mathcal{N}_1$ is generic. Then $\mathcal{N}$ is generic.*

Proof. Denote the element parameters in $\mathcal{N}_1$ by $\hat{\mathbf{E}} = (\hat{E}_1, \ldots, \hat{E}_m)$, and let the impedance of $\mathcal{N}_1$ be $f(\hat{\mathbf{E}}, s)/g(\hat{\mathbf{E}}, s)$. Then $\mathcal{N}$ comprises elements with parameters $\mathbf{E} = (R, L, C, \hat{\mathbf{E}})$, and the impedance of $\mathcal{N}$ takes the form $Z(\mathbf{E}, s) = a(\mathbf{E}, s)/b(\mathbf{E}, s)$ where

$$\begin{aligned} a(\mathbf{E}, s) &= R(Lsg(\hat{\mathbf{E}}, s) + (1 + \alpha s^2)f(\hat{\mathbf{E}}, s)), \text{ and} \\ b(\mathbf{E}, s) &= Lsg(\hat{\mathbf{E}}, s) + (1 + \alpha s^2)(f(\hat{\mathbf{E}}, s) + Rg(\hat{\mathbf{E}}, s)), \end{aligned}$$

where $\alpha = LC$. By Corollary 1, $\mathcal{N}$ is generic if and only if there exists $\mathbf{E}_0 = (R_0, L_0, C_0, \hat{\mathbf{E}}_0)$ such that

$$\mathbf{x} \in \mathbb{R}^{m+4} \text{ and } H(\mathbf{E}_0, s)\mathbf{x} = 0 \quad \Rightarrow \quad \mathbf{x} = 0\,, \tag{17}$$

where

$$H(\mathbf{E}, s) = \begin{pmatrix} \frac{\partial a}{\partial R} & \frac{\partial a}{\partial L} & \frac{\partial a}{\partial C} & \frac{\partial a}{\partial \hat{E}_1} & \cdots & \frac{\partial a}{\partial \hat{E}_m} & a \\ \frac{\partial b}{\partial R} & \frac{\partial b}{\partial L} & \frac{\partial b}{\partial C} & \frac{\partial b}{\partial \hat{E}_1} & \cdots & \frac{\partial b}{\partial \hat{E}_m} & b \end{pmatrix}.$$

We let

$$U = \begin{pmatrix} 1 & 0 \\ -1 & R \end{pmatrix},\ V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & C & L \end{pmatrix},\ \text{and } W = \begin{pmatrix} V^{-1} & 0 \\ 0 & I_{m+1} \end{pmatrix},$$

where I_{m+1} is the $m+1$ by $m+1$ identity matrix, and we note that $F(\mathbf{E}, s) = UH(\mathbf{E}, s)W$ where

$$F(\mathbf{E}, s) = \begin{pmatrix} \frac{\partial a}{\partial R} & \frac{\partial a}{\partial L} & \frac{\partial a}{\partial \alpha} & \frac{\partial a}{\partial \hat{E}_1} & \cdots & \frac{\partial a}{\partial \hat{E}_m} & a \\ \frac{\partial c}{\partial R} - b & \frac{\partial c}{\partial L} & \frac{\partial c}{\partial \alpha} & \frac{\partial c}{\partial \hat{E}_1} & \cdots & \frac{\partial c}{\partial \hat{E}_m} & c \end{pmatrix},$$

and where

$$c(\mathbf{E}, s) = Rb(\mathbf{E}, s) - a(\mathbf{E}, s) = R^2(1 + \alpha s^2)g(\hat{\mathbf{E}}, s).$$

It follows that $\mathcal{N}$ is generic if and only if there exists $\mathbf{E}_0 = (R_0, L_0, C_0, \hat{\mathbf{E}}_0)$ such that

$$\mathbf{x} \in \mathbb{R}^{m+4} \text{ and } F(\mathbf{E}_0, s)\mathbf{x} = 0 \quad \Rightarrow \quad \mathbf{x} = 0\,. \tag{18}$$

Since $\mathcal{N}_1$ is generic, then there exists $\hat{\mathbf{E}}_0 = (\hat{E}_{1,0}, \ldots, \hat{E}_{m,0}) \in \mathbb{R}^m_{>0}$ such that (10) holds, where $\hat{H}(\hat{\mathbf{E}}, s)$ is as in (11). Similar to Lemma 4, by applying the chain rule we find that

$$F(\mathbf{E}, s) = M(\mathbf{E}, s)\left(\begin{array}{c|c} I_3 & 0 \\ \hline 0 & \hat{H}(\hat{\mathbf{E}}, s) \end{array}\right),$$

where $M(\mathbf{E}, s)$ takes the form

$$\begin{pmatrix} a(\mathbf{E}, s)/R & Rsg(\hat{\mathbf{E}}, s) & Rs^2 f(\hat{\mathbf{E}}, s) & R(1+\alpha s^2) & RLs \\ -b(\mathbf{E}, s) & 0 & R^2 s^2 g(\hat{\mathbf{E}}, s) & 0 & R^2(1+\alpha s^2) \end{pmatrix}.$$

It therefore suffices to show that, for any given real scalars u, v, w and polynomials $y(s)$, $z(s)$ of degree less than or equal to n, then there exists $R_0, L_0, \alpha_0 \in \mathbb{R}$ such that

$$\begin{aligned}&(L_0 sg(\hat{\mathbf{E}}_0, s) + (1 + \alpha_0 s^2)f(\hat{\mathbf{E}}_0, s))u + R_0 sg(\hat{\mathbf{E}}_0, s)v \\ &\quad + R_0 s^2 f(\hat{\mathbf{E}}_0, s)w + R_0(1 + \alpha_0 s^2)y(s) + R_0 L_0 sz(s) = 0, \text{ and}\end{aligned} \tag{19}$$

$$\begin{aligned}&-(L_0 sg(\hat{\mathbf{E}}_0, s) + (1 + \alpha_0 s^2)(f(\hat{\mathbf{E}}_0, s) + R_0 g(\hat{\mathbf{E}}_0, s)))u \\ &\quad + R_0^2 s^2 g(\hat{\mathbf{E}}_0, s)w + R_0^2(1 + \alpha_0 s^2)z(s) = 0,\end{aligned} \tag{20}$$

imply that $u = v = w = y(s) = z(s) = 0$. Since $g(\hat{\mathbf{E}}_0, s)$ cannot vanish identically on the imaginary axis, then we can pick $\alpha_0 > 0$ such that $g(\hat{\mathbf{E}}_0, j/\sqrt{\alpha_0}) \neq 0$. Substituting $s = j/\sqrt{\alpha_0}$ in (20) we obtain $-g(\hat{\mathbf{E}}_0, j/\sqrt{\alpha_0})(L_0 ju + R_0^2 w/\sqrt{\alpha_0}) = 0$, the only real solution of which is $u = w = 0$. From (20) it now follows that $z(s) = 0$. Equation (19) now reduces to

$$R_0 sg(\hat{\mathbf{E}}_0, s)v + R_0(1 + \alpha_0 s^2)y(s) = 0$$

from which we conclude, by substituting $s = j/\sqrt{\alpha_0}$, that $v = 0$. From the same equation we then conclude that $y(s) = 0$. We have therefore shown that (18) holds, hence $\mathcal{N}$ is generic. □

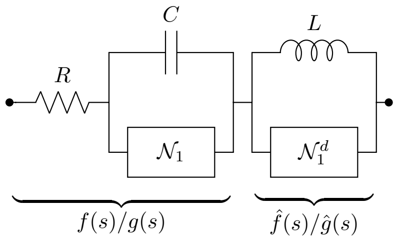

Figure 13: Two-terminal network with generic subnetworks $\mathcal{N}_1$ and $\mathcal{N}_1^d$.

Our final lemma show that the network of Fig. 13 is generic if $\mathcal{N}_1$ is generic and has no impedance pole at the origin.

Lemma 6. *Let $\mathcal{N}$ be an RLC two-terminal network with the structure shown in Fig. 13, where $\mathcal{N}_1^d$ denotes the network dual of $\mathcal{N}_1$. Let $\mathcal{N}_1$ comprise elements with parameters $\tilde{\mathbf{E}} = (\tilde{E}_1, \ldots \tilde{E}_m)$, and let the impedance of $\mathcal{N}_1$ take the form $q(\tilde{\mathbf{E}}, s)/d(\tilde{\mathbf{E}}, s)$. If there exists $\tilde{\mathbf{E}}_0 \in \mathbb{R}^m_{>0}$ such that (i) $\deg(q(\tilde{\mathbf{E}}_0, s)) \geq \deg(d(\tilde{\mathbf{E}}_0, s))$; (ii) $d(\tilde{\mathbf{E}}_0, 0) \neq 0$; (iii) $d(\tilde{\mathbf{E}}_0, s)$ and $q(\tilde{\mathbf{E}}_0, s)$ are coprime polynomials; and (iv) $\mathbf{x} \in \mathbb{R}^{m+1}$ and $\tilde{H}(\tilde{\mathbf{E}}_0, s)\mathbf{x} = 0 \Rightarrow \mathbf{x} = 0$, where*

$$\tilde{H}(\tilde{\mathbf{E}}, s) = \begin{pmatrix} \frac{\partial q}{\partial \tilde{E}_1} & \cdots & \frac{\partial q}{\partial \tilde{E}_m} & q \\ \frac{\partial d}{\partial \tilde{E}_1} & \cdots & \frac{\partial d}{\partial \tilde{E}_m} & d \end{pmatrix},$$

then $\mathcal{N}$ is generic.

Proof. Let $\bar{\mathbf{E}} = (\bar{E}_1, \ldots, \bar{E}_m)$ denote the parameters of the elements in $\mathcal{N}_1^d$. Since the network dual of $\mathcal{N}_1^d$ is $\mathcal{N}_1$, then there exists $\bar{\mathbf{E}}_0 \in \mathbb{R}^m_{>0}$ such that the impedance of $\mathcal{N}_1^d$ takes the form $\bar{q}(\bar{\mathbf{E}}, s)/\bar{d}(\bar{\mathbf{E}}, s)$ where $\bar{q}(\bar{\mathbf{E}}_0, s)$ and $\bar{d}(\bar{\mathbf{E}}_0, s)$ are coprime and $\bar{q}(\bar{\mathbf{E}}_0, s)/\bar{d}(\bar{\mathbf{E}}_0, s) = d(\tilde{\mathbf{E}}_0, s)/q(\tilde{\mathbf{E}}_0, s)$. It is then easily shown that

$$\mathbf{y} \in \mathbb{R}^{m+1} \text{ and } \bar{H}(\bar{\mathbf{E}}_0, s)\mathbf{y} = 0 \quad \Rightarrow \quad \mathbf{y} = 0,$$

where

$$\bar{H}(\bar{\mathbf{E}}, s) = \begin{pmatrix} \frac{\partial \bar{q}}{\partial \bar{E}_1} & \cdots & \frac{\partial \bar{q}}{\partial \bar{E}_m} & \bar{q} \\ \frac{\partial \bar{d}}{\partial \bar{E}_1} & \cdots & \frac{\partial \bar{d}}{\partial \bar{E}_m} & \bar{d} \end{pmatrix}.$$

Next, let $\check{\mathbf{E}} = (R, C, \tilde{\mathbf{E}})$, $\hat{\mathbf{E}} = (L, \bar{\mathbf{E}})$ and $\mathbf{E} = (R, C, L, \tilde{\mathbf{E}}, \bar{\mathbf{E}})$, and note that the impedance of $\mathcal{N}$ takes the form $a(\mathbf{E}, s)/b(\mathbf{E}, s)$ where

$$\begin{aligned} a(\mathbf{E}, s) &= f(\check{\mathbf{E}}, s)\hat{g}(\hat{\mathbf{E}}, s) + \hat{f}(\hat{\mathbf{E}}, s)g(\check{\mathbf{E}}, s) \\ b(\mathbf{E}, s) &= g(\check{\mathbf{E}}, s)\hat{g}(\hat{\mathbf{E}}, s), \end{aligned}$$

where $f(\check{\mathbf{E}}, s)/g(\check{\mathbf{E}}, s)$ and $\hat{f}(\hat{\mathbf{E}}, s)/\hat{g}(\hat{\mathbf{E}}, s)$ are the impedances of the two subnetworks indicated in Fig. 13. In particular,

$$\begin{aligned} f(\check{\mathbf{E}}, s) &= d(\tilde{\mathbf{E}}, s)R + q(\tilde{\mathbf{E}}, s)(1 + sRC), \\ g(\check{\mathbf{E}}, s) &= d(\tilde{\mathbf{E}}, s) + sCq(\tilde{\mathbf{E}}, s), \\ \hat{f}(\hat{\mathbf{E}}, s) &= sL\bar{q}(\bar{\mathbf{E}}, s), \qquad \text{and} \\ \hat{g}(\hat{\mathbf{E}}, s) &= \bar{q}(\bar{\mathbf{E}}, s) + sL\bar{d}(\bar{\mathbf{E}}, s). \end{aligned}$$

Now, let $\check{\mathbf{E}}_0 = (R_0, C_0, \tilde{\mathbf{E}}_0)$ and $\hat{\mathbf{E}}_0 = (L_0, \bar{\mathbf{E}}_0)$ where $R_0, L_0, C_0 \in \mathbb{R}$ and $L_0 \neq C_0$. Since $d(\tilde{\mathbf{E}}_0, 0) \neq 0$; $q(\tilde{\mathbf{E}}_0, s), d(\tilde{\mathbf{E}}_0, s)$ are coprime polynomials; and $q(\tilde{\mathbf{E}}_0, s)/d(\tilde{\mathbf{E}}_0, s) = \bar{d}(\bar{\mathbf{E}}_0, s)/\bar{q}(\bar{\mathbf{E}}_0, s)$, then $g(\check{\mathbf{E}}_0, s)$ and $\hat{g}(\hat{\mathbf{E}}_0, s)$ are also coprime polynomials. It is also easily verified that $\hat{f}(\hat{\mathbf{E}}_0, 0) = 0$, $\hat{g}(\hat{\mathbf{E}}_0, 0) \neq 0$, $f(\check{\mathbf{E}}_0, 0) \neq 0$, and $g(\check{\mathbf{E}}_0, 0) \neq 0$. Furthermore, denoting $\deg(g(\check{\mathbf{E}}_0, s))$ by n, then $\deg(\hat{g}(\hat{\mathbf{E}}_0, s)) = n$, $\deg(f(\check{\mathbf{E}}_0, s)) = n$, and $\deg(\hat{f}(\hat{\mathbf{E}}_0, s)) \leq n$.

We will now show that,

$$\mathbf{x} \in \mathbb{R}^{2m+4} \text{ and } H(\mathbf{E}_0, s)\mathbf{x} = 0 \quad \Rightarrow \quad \mathbf{x} = 0\,, \tag{21}$$

where $\mathbf{E}_0 = (R_0, C_0, L_0, \tilde{\mathbf{E}}_0, \bar{\mathbf{E}}_0)$ and

$$H(\mathbf{E}, s) = \begin{pmatrix} \frac{\partial a}{\partial R} & \frac{\partial a}{\partial C} & \frac{\partial a}{\partial \tilde{E}_1} & \cdots & \frac{\partial a}{\partial \tilde{E}_m} & \frac{\partial a}{\partial L} & \frac{\partial a}{\partial \bar{E}_1} & \cdots & \frac{\partial a}{\partial \bar{E}_m} & a \\ \frac{\partial b}{\partial R} & \frac{\partial b}{\partial C} & \frac{\partial b}{\partial \tilde{E}_1} & \cdots & \frac{\partial b}{\partial \tilde{E}_m} & \frac{\partial b}{\partial L} & \frac{\partial b}{\partial \bar{E}_1} & \cdots & \frac{\partial b}{\partial \bar{E}_m} & b \end{pmatrix}. \tag{22}$$

By the chain rule, $H(\mathbf{E}_0, s)$ may be expressed as

$$H(\mathbf{E}_0, s) = M(\mathbf{E}_0, s)\underbrace{\left(\begin{array}{c|c} \check{H}(\check{\mathbf{E}}_0, s) & 0 \\ \hline 0 & \hat{H}(\hat{\mathbf{E}}_0, s) \end{array}\right)}_{N}, \tag{23}$$

where

$$\check{H}(\check{\mathbf{E}}, s) = \begin{pmatrix} \frac{\partial f}{\partial R} & \frac{\partial f}{\partial C} & \frac{\partial f}{\partial \tilde{E}_1} & \cdots & \frac{\partial f}{\partial \tilde{E}_m} \\ \frac{\partial g}{\partial R} & \frac{\partial g}{\partial C} & \frac{\partial g}{\partial \tilde{E}_1} & \cdots & \frac{\partial g}{\partial \tilde{E}_m} \end{pmatrix},$$

$$\hat{H}(\hat{\mathbf{E}}, s) = \begin{pmatrix} \frac{\partial \hat{f}}{\partial L} & \frac{\partial \hat{f}}{\partial \bar{E}_1} & \cdots & \frac{\partial \hat{f}}{\partial \bar{E}_m} & \hat{f} \\ \frac{\partial \hat{g}}{\partial L} & \frac{\partial \hat{g}}{\partial \bar{E}_1} & \cdots & \frac{\partial \hat{g}}{\partial \bar{E}_m} & \hat{g} \end{pmatrix}, \quad \text{and}$$

$$M(\mathbf{E}_0, s) = \begin{pmatrix} \hat{g}(\hat{\mathbf{E}}_0, s) & \hat{f}(\hat{\mathbf{E}}_0, s) & g(\check{\mathbf{E}}_0, s) & f(\check{\mathbf{E}}_0, s) \\ 0 & \hat{g}(\hat{\mathbf{E}}_0, s) & 0 & g(\check{\mathbf{E}}_0, s) \end{pmatrix}. \tag{24}$$

We therefore need to show that

$$\mathbf{x} \in \mathbb{R}^{2m+4} \text{ and } M(\mathbf{E}_0, s)N\mathbf{x} = 0 \quad \Rightarrow \quad \mathbf{x} = 0\,. \tag{25}$$

Consider a fixed but arbitrary $\mathbf{x} \in \mathbb{R}^{2m+4}$, let $\mathbf{y} = N\mathbf{x}$, and note that $\mathbf{y}$ takes the form $(u(s), v(s), w(s), z(s))$, where $u(s)$, $v(s)$, $w(s)$ and $z(s)$ are polynomials of degree less than or equal to n and $w(0) = 0$ (since $\hat{f}(\hat{\mathbf{E}}, 0) = 0$). We will first show that if $M(\mathbf{E}_0, s)\mathbf{y} = 0$ then $\mathbf{y} = \alpha(f(\check{\mathbf{E}}_0, s), g(\check{\mathbf{E}}_0, s), -\hat{f}(\hat{\mathbf{E}}_0, s), -\hat{g}(\hat{\mathbf{E}}_0, s))$ for some real constant α. The matrix equation $M(\mathbf{E}_0, s)\mathbf{y} = 0$ yields the following two polynomial equations:

$$\hat{g}(\hat{\mathbf{E}}_0, s)u(s) + \hat{f}(\hat{\mathbf{E}}_0, s)v(s) + g(\check{\mathbf{E}}_0, s)w(s) + f(\check{\mathbf{E}}_0, s)z(s) = 0\,, \tag{26}$$

$$\hat{g}(\hat{\mathbf{E}}_0, s)v(s) + g(\check{\mathbf{E}}_0, s)z(s) = 0\,. \tag{27}$$

Equation (27) can be written as $z(s)/v(s) = -\hat{g}(\hat{\mathbf{E}}_0, s)/g(\check{\mathbf{E}}_0, s)$, from which we conclude that $v(s) = \alpha g(\check{\mathbf{E}}_0, s)$ for some real constant α, since $g(\check{\mathbf{E}}_0, s)$ and $\hat{g}(\hat{\mathbf{E}}_0, s)$ are coprime polynomials with $\deg(g(\check{\mathbf{E}}_0, s)) = \deg(\hat{g}(\hat{\mathbf{E}}_0, s)) = n$, while $\deg(v(s))$, $\deg(z(s)) \leq n$. From (27) it then follows that $z(s) = -\alpha\hat{g}(\hat{\mathbf{E}}_0, s)$. Equation (26) now reduces to

$$\hat{g}(\hat{\mathbf{E}}_0, s)(u(s) - \alpha f(\check{\mathbf{E}}_0, s)) + g(\check{\mathbf{E}}_0, s)(w(s) + \alpha\hat{f}(\hat{\mathbf{E}}_0, s)) = 0\,. \tag{28}$$

We recall that $w(0) = \hat{f}(\hat{\mathbf{E}}_0, 0) = 0$ and $\hat{g}(\hat{\mathbf{E}}_0, 0) \neq 0$. Therefore, for $s = 0$, (28) yields $\hat{g}(\hat{\mathbf{E}}_0, 0)(u(0) - \alpha f(\check{\mathbf{E}}_0, 0)) = 0$, from which we conclude that $u(s) - \alpha f(\check{\mathbf{E}}_0, s)$ is divisible by s. But by writing (28) as

$$\frac{w(s) + \alpha\hat{f}(\hat{\mathbf{E}}_0, s)}{u(s) - \alpha f(\check{\mathbf{E}}_0, s)} = -\frac{\hat{g}(\hat{\mathbf{E}}_0, s)}{g(\check{\mathbf{E}}_0, s)}$$

we can conclude that $u(s) - \alpha f(\check{\mathbf{E}}_0, s)$ is also divisible by $g(\check{\mathbf{E}}_0, s)$, since $g(\check{\mathbf{E}}_0, s)$ and $\hat{g}(\hat{\mathbf{E}}_0, s)$ are coprime polynomials and $\deg(u(s) - \alpha f(\check{\mathbf{E}}_0, s)) \leq n$. Therefore $u(s) - \alpha f(\check{\mathbf{E}}_0, s)$ is divisible by $sg(\check{\mathbf{E}}_0, s)$ (which has degree $n + 1$), from which it follows that $u(s) = \alpha f(\check{\mathbf{E}}_0, s)$ necessarily. Equation (26) finally gives $w(s) = -\alpha\hat{f}(\hat{\mathbf{E}}_0, s)$.

At this point we have shown that

$$\mathbf{x} \in \mathbb{R}^{2m+4} \text{ and } M(\mathbf{E}_0, s)N\mathbf{x} = 0 \Rightarrow N\mathbf{x} = \alpha\begin{pmatrix} f(\check{\mathbf{E}}_0, s) \\ g(\check{\mathbf{E}}_0, s) \\ -\hat{f}(\hat{\mathbf{E}}_0, s) \\ -\hat{g}(\hat{\mathbf{E}}_0, s) \end{pmatrix}. \tag{29}$$

If we partition $\mathbf{x}$ into two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ each of dimension $m + 2$, the right-hand side of (29) may be written as

$$\left(\begin{array}{c|c} \check{H}(\check{\mathbf{E}}_0, s) & 0 \\ \hline 0 & \hat{H}(\hat{\mathbf{E}}_0, s) \end{array}\right)\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} - \alpha\begin{pmatrix} f(\check{\mathbf{E}}_0, s) \\ g(\check{\mathbf{E}}_0, s) \\ -\hat{f}(\hat{\mathbf{E}}_0, s) \\ -\hat{g}(\hat{\mathbf{E}}_0, s) \end{pmatrix} = 0\,,$$

which is equivalent to

$$\left(\begin{array}{c|c} \check{H}(\check{\mathbf{E}}_0, s) & \begin{matrix} f(\check{\mathbf{E}}_0, s) \\ g(\check{\mathbf{E}}_0, s) \end{matrix} \end{array}\right)\begin{pmatrix} \mathbf{x}_1 \\ -\alpha \end{pmatrix} = 0\,, \tag{30}$$

$$\left(\begin{array}{c|c} \hat{H}(\hat{\mathbf{E}}_0, s) & \begin{matrix} -\hat{f}(\hat{\mathbf{E}}_0, s) \\ -\hat{g}(\hat{\mathbf{E}}_0, s) \end{matrix} \end{array}\right)\begin{pmatrix} \mathbf{x}_2 \\ -\alpha \end{pmatrix} = 0\,. \tag{31}$$

Since (i) $\mathbf{x} \in \mathbb{R}^{m+1}$ and $\tilde{H}(\tilde{\mathbf{E}}_0, s)\mathbf{x} = 0 \Rightarrow \mathbf{x} = 0$, and (ii) $\mathbf{y} \in \mathbb{R}^{m+1}$ and $\bar{H}(\bar{\mathbf{E}}_0, s)\mathbf{y} = 0 \Rightarrow \mathbf{y} = 0$, then it follows from the proof of Lemma 4 that

$$\mathbf{t}_1 \in \mathbb{R}^{m+3} \text{ and } \left(\begin{array}{c|c} \check{H}(\check{\mathbf{E}}_0, s) & \begin{matrix} f(\check{\mathbf{E}}_0, s) \\ g(\check{\mathbf{E}}_0, s) \end{matrix} \end{array}\right)\mathbf{t}_1 = 0 \quad \Rightarrow \quad \mathbf{t}_1 = 0\,,$$

$$\text{and} \qquad \mathbf{t}_2 \in \mathbb{R}^{m+2} \text{ and } \hat{H}(\hat{\mathbf{E}}_0, s)\mathbf{t}_2 = 0 \quad \Rightarrow \quad \mathbf{t}_2 = 0\,.$$

Therefore we can conclude from (30) that $\mathbf{x}_1 = 0$ and $\alpha = 0$, and from (31) that $\mathbf{x}_2 = 0$. Therefore (25) holds and, by Corollary 1, the network $\mathcal{N}$ is generic. □

Lemma 6 may be generalised to the series connection of two RLC two-terminal networks $\mathcal{N}_1$ and $\mathcal{N}_2$. Namely, under the following assumptions we may conclude that the series connection of $\mathcal{N}_1$ and $\mathcal{N}_2$ is generic:

- The two networks are generic;
- One of the two networks has an impedance zero at the origin, and the other does not;
- The two networks do not have any coincident impedance poles for almost all element values.

Finally in this paper, we provide a proof of the genericity of the Bott-Duffin networks. We note that, if the impedance function is a biquadratic, then the Bott-Duffin method leads to a generic network with the structure of Fig. 9, as already discussed in Example 8. However, it remains to consider the cases for which the impedance is not biquadratic.

Theorem 3. *Any positive-real impedance can be realised by a generic RLC network.*

Proof. The Bott-Duffin theorem states that any positive-real impedance function can be realised by an RLC network [3]. It therefore suffices to show that each of the steps involved in the construction of such a network $\mathcal{N}$ preserves genericity (see [5] for a textbook explanation of the Bott-Duffin procedure).

To obtain a network $\mathcal{N}$ to realise an arbitrary given positive-real function $Z(s)$, the steps in the Bott-Duffin procedure (coupled with the so-called *Foster preamble*) are as follows:

1. Subtract any imaginary axis impedance poles (resulting in an impedance of lower order).
2. Subtract a constant equal to the smallest value of $\mathrm{Re}(Z(j\omega))$ for $\omega \in \mathbb{R} \cup \infty$, resulting in a network whose impedance $\hat{Z}(s)$ has no imaginary axis impedance poles and satisfies one of the following properties:
   (a) $\hat{Z}(s)$ has an admittance pole at the origin or at infinity;
   (b) $\hat{Z}(s)$ has an admittance pole elsewhere on the imaginary axis;
   (c) $\hat{Z}(s)$ is a minimum function.

   In each case, the impedance can then be reduced to one of lower order.

The network realisations corresponding to cases 1, 2a, 2b and 2c each take the form of one of the networks described in Lemmas 4–6, or can be obtained from such networks through a combination of frequency inversion and duality transformations (in certain cases it is necessary to replace the resistor by a short or open circuit). That genericity is preserved in each case can be shown using Lemmas 4–6 and minor modifications thereof. The Bott-Duffin procedure continues inductively until the resulting impedance has order zero. This final impedance can be realised by a single resistor, which itself is generic. This establishes the genericity of all of the other networks in the inductive procedure, whereupon we conclude that $\mathcal{N}$ is generic. $\square$

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