THE CUSPIDALISATION OF SECTIONS OF ARITHMETIC FUNDAMENTAL GROUPS II

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Abstract. In this paper, which is a sequel to [Saïdi], we investigate the theory of cuspidalisation of sections of arithmetic fundamental groups of hyperbolic curves to cuspidally i-th and $2/p$-th step prosolvable arithmetic fundamental groups. As a consequence we exhibit two, necessary and sufficient, conditions for sections of arithmetic fundamental groups of hyperbolic curves over p-adic local fields to arise from rational points. We also exhibit a class of sections of arithmetic fundamental groups of p-adic curves which are orthogonal to Pic$^\wedge$, and which satisfy (unconditionally) one of the above conditions.

Contents

§**0. Introduction.** Let k be a characteristic zero field, X a proper, smooth, and geometrically connected hyperbolic (i.e., genus$(X) \geq 2$) algebraic curve over k. Let K_X be the function field of X, K_X^{sep} a separable closure of K_X, and $\overline{k} \subset K_X^{\text{sep}}$ the algebraic closure of k. Let $\pi_1(X)$ be the étale fundamental group of X which sits in the following exact sequence

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \xrightarrow{\text{pr}} G_k \to 1,$$

where $G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$, and $\pi_1(\overline{X})$ is the geometric étale fundamental group of X. Let $G_X \overset{\text{def}}{=} \text{Gal}(K_X^{\text{sep}}/K_X)$, and $\overline{G}_X \overset{\text{def}}{=} \text{Gal}(K_X^{\text{sep}}/K_X.\overline{k})$. Thus, we have exact sequences

$$1 \to \overline{G}_X \to G_X \to G_k \to 1,$$

and

$$1 \to \mathcal{I}_X \to G_X \to \pi_1(X) \to 1,$$

where $\mathcal{I}_X$ is the inertia subgroup. The theory of cuspidalisation of sections of arithmetic fundamental groups was initiated in [Saïdi], its ultimate aim is to reduce the Grothendieck anabelian section conjecture to its birational version. It can be formulated as follows (cf. loc. cit.).

The Cuspidalisation Problem for Sections of $\pi_1(X)$. Let $G_X \twoheadrightarrow H \twoheadrightarrow \pi_1(X)$ be a quotient of G_X. Given a section $s : G_k \to \pi_1(X)$ of the projection $\pi_1(X) \twoheadrightarrow G_k$, is it possible to **lift** s to a section $\tilde{s} : G_k \to H$ of the projection $H \twoheadrightarrow G_k$? i.e., is it possible to construct a section $\tilde{s}$ such that the following diagram is commutative

$$\begin{array}{ccc} G_k & \xrightarrow{\tilde{s}} & H \\ {\scriptstyle \text{id}}\downarrow & & \downarrow \\ G_k & \xrightarrow{s} & \pi_1(X) \end{array}$$

where the right vertical map is the projection $H \twoheadrightarrow \pi_1(X)$?

In [Saïdi] we investigated the cuspidalisation problem in the case $H \overset{\text{def}}{=} G_X^{(\text{c}-\text{ab})}$ is the maximal (geometrically) cuspidally abelian quotient of G_X. In this paper we generalise this theory to the (geometrically) *cuspidally i-th* as well as i/p-th; where p is a prime, *step prosolvable* quotient of G_X.

For $i \geq 0$, let $\mathcal{I}_X \twoheadrightarrow \mathcal{I}_{X,i}$ be the maximal i-th step prosolvable quotient of $\mathcal{I}_X$, and $G_X^{(i-\text{sol})} \overset{\text{def}}{=} G_X / \operatorname{Ker}(\mathcal{I}_X \twoheadrightarrow \mathcal{I}_{X,i})$ the maximal (geometrically) cuspidally i-th step prosolvable quotient of G_X ($G_X^{(\text{c}-\text{ab})} \overset{\text{def}}{=} G_X^{(1-\text{sol})}$). For $i \geq 1$, let $s_i : G_k \to G_X^{(i-\text{sol})}$ be a section of the projection $G_X^{(i-\text{sol})} \twoheadrightarrow G_k$. In §2 we investigate the problem of lifting s_i to a section $s_{i+1} : G_k \to G_X^{(i+1-\text{sol})}$ of the projection $G_X^{(i+1-\text{sol})} \twoheadrightarrow G_k$. We say that the field k satisfies the condition (**H**) if the following holds. The Galois cohomology groups $H^1(G_k, M)$ are *finite* for every finite G_k-module M. This condition is satisfied for instance if the Galois group G_k is (topologically) finitely generated (e.g. k is a p-adic local field). One of our main results in this paper is the following (cf. Theorem 2.3.8).

Theorem 1. *Assume $i \geq 1$, and k satisfies the condition* (**H**). *The section $s_i : G_k \to G_X^{(i-\text{sol})}$* **lifts** *to a section $s_{i+1} : G_k \to G_X^{(i+1-\text{sol})}$ of the projection $G_X^{(i+1-\text{sol})} \twoheadrightarrow G_k$* **if and only if** *for every $X' \to X$ a neighbourhood of the section s_i (i.e., corresponding to an open subgroup of $G_X^{(i-\text{sol})}$ containing $s_i(G_k)$) the class of $\operatorname{Pic}^1_{X'}$ in $H^1(G_k, \operatorname{Pic}^0_{X'})$ lies in the maximal divisible subgroup of $H^1(G_k, \operatorname{Pic}^0_{X'})$.*

Key to the proof of Theorem 1 is the description of the G_k-module structure, induced by s_i, of $\mathcal{I}_X[i+1] \overset{\text{def}}{=} \operatorname{Ker}(G_X^{(i+1-\text{sol})} \twoheadrightarrow G_X^{(i-\text{sol})})$ as the projective limit of the Tate modules of the jacobians of the neighbourhoods $\{X'\}$ as in the statement of Theorem 1 (cf. Proposition 1.1.5, and Lemma 2.3.2).

In §3 we investigate the following mod-p variant of Theorem 1, where p is a prime integer. Let $t \geq 0$, $i \geq 0$, and $\mathcal{I}_X \twoheadrightarrow \mathcal{I}_{X,i+1/p^t}$ the $i+1$-th quotient of the $\mathbb{Z}/p^t\mathbb{Z}$-derived series of $\mathcal{I}_X$ (cf. 1.2). Thus, $\mathcal{I}_{X,i+1/p^t}$ is i+1-step prosolvable with successive abelian quotients annihilated by p^t. Let $G_X^{(i+1/p^t-\text{sol})} \overset{\text{def}}{=} G_X / \operatorname{Ker}(\mathcal{I}_X \twoheadrightarrow \mathcal{I}_{X,i+1/p^t})$ be the maximal (geometrically) cuspidally $i+1/p^t$-th prosolvable quotient of G_X. Given a section $s : G_k \to \pi_1(X)$ of the projection $\pi_1(X) \twoheadrightarrow G_k$ we investigate the problem of lifting s to a section $s_{i+1} : G_k \to G_X^{(i+1/p^t-\mathrm{sol})}$ of the projection $G_X^{(i+1/p^t-\mathrm{sol})} \twoheadrightarrow G_k$ in the **case $\mathbf{i = 1}$** and **$\mathbf{t = 1}$**; the only case needed for applications in §4, and §5. For this purpose we introduce a certain quotient $G_X \twoheadrightarrow G_X^{(p,2)} \twoheadrightarrow G_X^{(2/p-\mathrm{sol})}$, and investigate the problem of lifting the section s to a section $\tilde{s} : G_k \to G_X^{(p,2)}$ of the projection $G_X^{(p,2)} \twoheadrightarrow G_k$ (this would give rise to a section of the projection $G_X^{(2/p-\mathrm{sol})} \twoheadrightarrow G_k$ which lifts s). The quotient $G_X^{(p,2)}$ sits in an exact sequence $1 \to \mathcal{I}_X[p,2] \to G_X^{(p,2)} \to G_X^{(1/p^2-\mathrm{sol})} \to 1$, where $\mathcal{I}_X[p,2]$ is abelian annihilated by p (cf. 3.3 for more details). In Theorem 3.4.11 we give necessary and sufficient conditions for the section s to lift to a section $\tilde{s} : G_k \to G_X^{(p,2)}$ (cf. loc. cit. for a more precise statement).

In §4 we assume k is a p-adic local field (finite extension of $\mathbb{Q}_p$). We observe in this case that if $\tilde{s} : G_k \to G_X^{(p,2)}$ is a section of the projection $G_X^{(p,2)} \twoheadrightarrow G_k$, and $s : G_k \to \pi_1(X)$ is the induced section of $\pi_1(X) \twoheadrightarrow G_k$, then s is *geometric* in the sense that it arises from a rational point $x \in X(k)$ (cf. Proposition 4.6). Further, we provide the following characterisation of sections $s : G_k \to \pi_1(X)$ which are geometric (cf. Theorem 4.5 where we prove a pro-Σ; $p \in \Sigma$ is a set of primes, variant of Theorem 2).

Theorem 2. *Assume k is a p-***adic local field**. *A section $s : G_k \to \pi_1(X)$ of the projection $\pi_1(X) \twoheadrightarrow G_k$ is* **geometric** *(cf. Definition 4.1) if and only if the following two conditions hold.*
(i) The section s has a **cycle class uniformly orthogonal to** Pic **mod-**p^2 *(cf. Definition 3.4.1).*
(ii) There **exists** *a section $s' : G_k \to G_X^{(1/p^2-\mathrm{sol})}$ of the projection $G_X^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ which* **lifts** *the section s (this holds if condition (i) is satisfied by Theorem 3.4.4) such that the following holds. For every $X' \to X$ a neighbourhood of the section s' (i.e., corresponding to an open subgroup of $G_X^{(1/p^2-\mathrm{sol})}$ containing $s'(G_k)$) the class of $\mathrm{Pic}^1_{X'}$ in $H^1(G_k, \mathrm{Pic}^0_{X'})$ is* **divisible by** *p.*

Condition (ii) in Theorem 2 is a necessary and sufficient condition for the section s' therein to lift to a section $\tilde{s} : G_k \to G_X^{(p,2)}$ (cf. Theorem 3.4.10).

As an application of Theorem 2 we prove the following p-adic absolute anabelian result (cf. Theorem 4.8).

Theorem 3. *Let p_X, p_Y be prime integers, and X (resp. Y) a proper, smooth, geometrically connected hyperbolic curve over a p_X-adic local field k_X (respectively, p_Y-adic local field k_Y). Let $p_X \in \Sigma_X$ (resp. $p_Y \in \Sigma_Y$) be a non-empty set of prime integers of cardinality ≥ 2, Π_X (resp. Π_Y) the geometrically pro-Σ_X (resp. pro-Σ_Y) arithmetic fundamental group of X (resp. Y), and $\varphi : \Pi_X \to \Pi_Y$ an isomorphism of profinite groups which fits in the following commutative diagram*

$$\begin{array}{ccc} G_X^{(p_X,2)} & \xrightarrow{\widetilde{\varphi}} & G_Y^{(p_Y,2)} \\ \downarrow & & \downarrow \\ \Pi_X & \xrightarrow{\varphi} & \Pi_Y \end{array}$$

where $\widetilde{\varphi}$ is an isomorphism of profinite groups. Here $G_X^{(p_X,2)}$ (resp. $G_Y^{(p_Y,2)}$) is the pro-Σ_X (resp. Σ_Y) version of the profinite group $G_X^{(p_X,2)}$ (resp. $G_Y^{(p_Y,2)}$) (cf. 3.3), *and the vertical maps are the natural projections. Then φ is geometric, i.e., arises from a uniquely determined isomorphism of schemes $X \xrightarrow{\sim} Y$.*

Finally, in §5 we investigate *local* sections of arithmetic fundamental groups of p-adic curves. These are sections which arise from sections of arithmetic fundamental groups of *formal fibres* (cf. Definition 5.2). A geometric section is necessarily local in this sense. Our main result is the following (cf. Theorem 5.3).

Theorem 4. *Assume k is a* **p-adic local field***, and $s : G_k \to \pi_1(X)$ is a* **local** *section of the projection $\pi_1(X) \twoheadrightarrow G_k$. Then s has* **a cycle class which is uniformly orthogonal to** $\mathrm{Pic}^{\wedge}$ *in the sense of [Saïdi], Definition 1.4.1(i).*

To the best of our knowledge, local sections of arithmetic fundamental groups of p-adic curves are the first non trivial (i.e., not known to be geometric a priori) examples of sections of arithmetic fundamental groups of p-adic curves which are orthogonal to $\mathrm{Pic}^{\wedge}$. In particular, local sections satisfy condition (i) in Theorem 2.

Notations. Throughout this paper $\mathfrak{Primes}$ denotes the set of all prime integers. For a profinite group H, we write H^{ab} for the maximal abelian quotient of H.

§1. Cuspidally i-th and i/p-th step prosolvable geometric fundamental groups. Let ℓ be an algebraically closed field of characteristic $l \geq 0$, X a proper smooth and connected hyperbolic curve (i.e., genus$(X) \geq 2$) over ℓ, and K_X its function field. Let η be a geometric point of X above its generic point; which determines a separable closure K_X^{sep} of K_X, and $\pi_1(X, \eta)$ the étale fundamental group of X with base point η.

Let $\emptyset \neq \Sigma \subseteq \mathfrak{Primes}$ be a set of prime integers. In case $\mathrm{char}(\ell) = l > 0$ we assume that $l \notin \Sigma$. Write $\Delta_X \overset{\mathrm{def}}{=} \pi_1(X, \eta)^{\Sigma}$ for the maximal pro-Σ quotient of $\pi_1(X, \eta)$. Let $\{x_s\}_{s=1}^n \subset X(\ell)$, $U \overset{\mathrm{def}}{=} X \setminus \{x_1, \cdots, x_n\}$ an open subscheme of X, $\Delta_U \overset{\mathrm{def}}{=} \pi_1(U, \eta)^{\Sigma}$ the maximal pro-Σ quotient of the étale fundamental group $\pi_1(U, \eta)$ of U with base point η, and $I_U \overset{\mathrm{def}}{=} \mathrm{Ker}(\Delta_U \twoheadrightarrow \Delta_X)$. We shall refer to I_U as the *cuspidal subgroup* of Δ_U with respect to the natural projection $\Delta_U \twoheadrightarrow \Delta_X$ (cf. [Mochizuki], Definition 1.5); it is the subgroup of Δ_U (normally) generated by the (pro-Σ) inertia subgroups at the points $\{x_i\}_{i=1}^n$. We have the following exact sequence

$$1 \to I_U \to \Delta_U \to \Delta_X \to 1.$$

1.1 The quotient $\Delta_{U,i}$. For a profinite group H, we denote by $\overline{[H, H]}$ the closed subgroup of H topologically generated by the commutator subgroup. Consider the derived series of I_U

$$..... \subseteq I_U(i+1) \subseteq I_U(i) \subseteq ...... \subseteq I_U(1) \subseteq I_U(0) = I_U, \tag{1.1}$$

where, for $i \geq 0$, $I_U(i+1) \overset{\mathrm{def}}{=} \overline{[I_U(i), I_U(i)]}$ is the $(i+1)$-th derived subgroup, which is a characteristic subgroup of I_U. Write

$$I_{U,i} \overset{\mathrm{def}}{=} I_U / I_U(i).$$

Thus, $I_{U,i}$ is the maximal i-th step prosolvable quotient of I_U, and $I_{U,1}$ is the maximal abelian quotient of I_U. There exists a natural exact sequence

$$1 \to I_U[i+1] \to I_{U,i+1} \to I_{U,i} \to 1 \tag{1.2}$$

where $I_U[i+1]$ is the subgroup $I_U(i)/I_U(i+1)$ of $I_{U,i+1}$ and $I_U[i+1]$ is abelian. Write

$$\Delta_{U,i} \overset{\text{def}}{=} \Delta_U/I_U(i).$$

We shall refer to $\Delta_{U,i}$ (resp. $\Delta_{U,1}$) as the maximal **cuspidally i-th step pro-solvable** (resp. maximal **cuspidally abelian**) quotient of Δ_U (with respect to the surjection $\Delta_U \twoheadrightarrow \Delta_X$). We have the following commutative diagram of exact sequences.

$$(1.3) \qquad \begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & I_U[i+1] & = & I_U[i+1] & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & I_{U,i+1} & \longrightarrow & \Delta_{U,i+1} & \longrightarrow & \Delta_X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\Delta_X}} & & \\
1 & \longrightarrow & I_{U,i} & \longrightarrow & \Delta_{U,i} & \longrightarrow & \Delta_X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

The profinite group $\Delta_{U,i}$; being a quotient of Δ_U, is topologically finitely generated (cf. [Grothendieck], Exposé X, Corollaire 3.10, recall $\mathrm{char}(\ell) \notin \Sigma$). Hence there exists a sequence of characteristic open subgroups

$$... \subseteq \Delta_{U,i}[j+1] \subseteq \Delta_{U,i}[j] \subseteq ... \subseteq \Delta_{U,i}[1] \overset{\text{def}}{=} \Delta_{U,i}$$

of $\Delta_{U,i}$ such that $\bigcap_{j\geq 1} \Delta_{U,i}[j] = \{1\}$. The open subgroup $\Delta_{U,i}[j] \subseteq \Delta_{U,i}$ corresponds to a finite (Galois) cover $X_{i,j}^U \to X$ between smooth connected and proper ℓ-curves which is étale above U. The geometric point η determines naturally a geometric point $\eta_{i,j}$ of $X_{i,j}^U$. Write $\Delta_{i,j}^U \overset{\text{def}}{=} \Delta_{X_{i,j}^U} \overset{\text{def}}{=} \pi_1(X_{i,j}^U, \eta_{i,j})^{\Sigma}$ for the maximal pro-Σ étale fundamental group of $X_{i,j}^U$ with base point $\eta_{i,j}$, and $(\Delta_{i,j}^U)^{\mathrm{ab}}$ for the maximal abelian quotient of $\Delta_{i,j}^U$. The following Proposition provides a description of the structure of the profinite group $I_U[i+1]$ (cf. sequence (1.2) and diagram (1.3)) in the case $i \geq 1$. A description of the structure of $I_U[1]$ is given in [Mochizuki] Proposition 1.14 (see also [Saïdi], 2.1).

Proposition 1.1.1. *Let $i \geq 1$. There exists a natural isomorphism*

$$I_U[i+1] \overset{\sim}{\to} \varprojlim_{j\geq 1} (\Delta_{i,j}^U)^{\mathrm{ab}}.$$

Proof of Proposition 1.1.1. Let G be a finite quotient of $\Delta_{U,i+1}$, which inserts in the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & I_{U,i+1} & \longrightarrow & \Delta_{U,i+1} & \longrightarrow & \Delta_X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & I & \longrightarrow & G & \longrightarrow & G^{\mathrm{et}} & \longrightarrow & 1
\end{array}$$

where the vertical maps are surjective. We assume, without loss of generality, that G is *not* a quotient of $\Delta_{U,i}$. The quotient G corresponds to a finite Galois cover $X_1 \to X$ with Galois group G, which factorizes as $X_1 \to X_1^{\text{et}} \to X$, where $X_1^{\text{et}} \to X$ is the maximal étale sub-cover with Galois group G^{et}, and $X_1 \to X_1^{\text{et}}$ is a (tamely) ramified Galois cover with group I. For $s \in \{1, \cdots, n\}$, let $I_{x_s} \subset G$ be an inertia subgroup associated to x_s. Thus, I_{x_s} is only defined up to conjugation, and I is an $(i+1)$-th step solvable group (normally) generated by the I_{x_s}'s. Moreover, I_{x_s} is cyclic of order $e_s \geq 1$ (coprime to $l = \text{char}(\ell)$) as the ramification is tame. The following claim follows immediately from the well-known structure of Δ_U (cf. [Grothendieck], Exposé X, Corollaire 3.10).

Claim 1.1.2. *There exists a finite quotient G' of $\Delta_{U,i}$, which inserts in the following commutative diagram (where the vertical maps are surjective)*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_{U,i} & \longrightarrow & \Delta_{U,i} & \longrightarrow & \Delta_X & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I' & \longrightarrow & G' & \longrightarrow & G'^{\text{et}} & \longrightarrow & 1 \end{array}$$

such that the following holds. The quotient $\Delta_{U,i} \twoheadrightarrow G'$ corresponds to a finite Galois cover $X_2 \to X$ with Galois group G', which factorizes as $X_2 \to X_2^{\text{et}} \to X$, where $X_2^{\text{et}} \to X$ is the maximal étale sub-cover with Galois group G'^{et}, and $X_2 \to X_2^{\text{et}}$ is a (tamely) ramified Galois cover with Galois group I' (an i-th step solvable group). Further, for $s \in \{1, \cdots, n\}$, $I'_{x_s} \subseteq I'$ an inertia subgroup associated to x_s, then I'_{x_s} is cyclic of order $f_s = e_s h_s$ a multiple of e_s.

Next, let $K_1 \overset{\text{def}}{=} K_{X_1}$ (resp. $K_2 \overset{\text{def}}{=} K_{X_2}$) be the function field of X_1 (resp. X_2). Let $L \overset{\text{def}}{=} K_1.K_2$ be the compositum of K_1 and K_2 (in K_X^{sep}), and $\widetilde{X}$ the normalisation of X in L. Thus, $\widetilde{X} \to X$ is a Galois cover with Galois group $H \subseteq G \times G'$ which is étale above U and factorizes as $\widetilde{X} \to \widetilde{X}^{\text{et}} \to X$, where $\widetilde{X}^{\text{et}} \to X$ is the maximal étale sub-cover with Galois group H^{et}, and $\widetilde{X} \to \widetilde{X}^{\text{et}}$ is a (tamely) ramified Galois cover with group I_H : the subgroup of H (normally) generated by the inertia subgroups at the points of $\widetilde{X}$ above the $\{x_s\}_{s=1}^n$. (Thus, we have an exact sequence $1 \to I_H \to H \to H^{\text{et}} \to 1$.)

Lemma 1.1.3. *The quotient $\Delta_U \twoheadrightarrow H$ factorizes as $\Delta_U \twoheadrightarrow \Delta_{U,i+1} \twoheadrightarrow H$.*

Proof of Lemma 1.1.3. Indeed, one verifies easily that I_H is a subgroup of $I \times I'$ and $I \times I'$ is $(i+1)$-th step solvable. $\square$

Next, let $\widetilde{I}$ be the maximal i-th step solvable quotient of I, which inserts in the exact sequence $1 \to I(i+1) \to I \to \widetilde{I} \to 1$, with $I(i+1)$ abelian (note that $I(i+1)$ is non trivial by our assumption that G is not a quotient of $\Delta_{U,i}$). Write $\widetilde{G} \overset{\text{def}}{=} G/I(i+1)$, which inserts in the exact sequence $1 \to \widetilde{I} \to \widetilde{G} \to G^{\text{et}} \to 1$. In particular, $\widetilde{G}$ is a quotient of $\Delta_{U,i}$. Let $\widetilde{H}$ be the image of H in $\widetilde{G} \times G'$. We have a commutative diagram of exact sequences where the vertical maps are natural inclusions.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_H(i+1) \overset{\text{def}}{=} H \cap (I(i+1) \times \{1\}) & \longrightarrow & H & \longrightarrow & \widetilde{H} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I(i+1) \times \{1\} & \longrightarrow & G \times G' & \longrightarrow & \widetilde{G} \times G' & \longrightarrow & 1 \end{array}$$

Lemma 1.1.4. *The group $\widetilde{H}$ is a quotient of $\Delta_{U,i}$. Moreover, the cover $\widetilde{X} \to X$ factorizes as $\widetilde{X} \to \widetilde{X}' \to X$, where $\widetilde{X}' \to X$ is Galois (étale above U) with Galois group $\widetilde{H}$, and $\widetilde{X} \to \widetilde{X}'$ is an* **abelian étale** *cover with Galois group $I_H(i+1)$.*

Proof of Lemma 1.1.4. The first assertion follows from the various definitions. Next, the Galois cover $X_1 \to X$ factorizes as $X_1 \to \widetilde{X}_1 \to X$ where $X_1 \to \widetilde{X}_1$ is Galois with Galois group $I(i+1)$, and $\widetilde{X}_1 \to X$ is Galois with group $\widetilde{G}$. Let $\widetilde{X}'$ be the normalisation of X in the compositum of the function fields of $\widetilde{X}_1$ and X_2. Thus, $\widetilde{X}' \to X$ is a Galois cover with Galois group $\widetilde{H}$, and we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ \widetilde{X}' & \longrightarrow & \widetilde{X}_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

of finite Galois covers. The ramification index in the Galois cover $X_2 \to X$ above a branched (closed) point $x_s \in X$ is divisible by the ramification index above x_s in the Galois cover $X_1 \to X$ (cf. the above condition that f_s is divisible by e_s). The fact that the morphism $\widetilde{X} \to X_2$, and a fortiori $\widetilde{X} \to \widetilde{X}'$; which is abelian with Galois group $I_H(i+1)$, is étale follows from Abhyankar's Lemma (cf. [Grothendieck], Exposé X, Lemma 3.6). $\square$

Going back to the proof of Proposition 1.1.1, the above discussion shows that the finite quotients $\Delta_{U,i+1} \twoheadrightarrow H$ as in Lemma 1.1.3 form a cofinal system of finite quotients of $\Delta_{U,i+1}$. Thus, $\Delta_{U,i+1} \xrightarrow{\sim} \varprojlim_{H} H$. Proposition 1.1.1 then follows from the facts that the various H above fit in an exact sequence $1 \to I_H(i+1) \to H \to \widetilde{H} \to 1$; $\Delta_{U,i} \xrightarrow{\sim} \varprojlim_{\widetilde{H}} \widetilde{H}$, and the above Galois covers $\widetilde{X} \to \widetilde{X}'$ with group $I_H(i+1)$ are étale abelian (cf. Lemma 1.1.4). This finishes the proof of Proposition 1.1.1. $\square$

Similarly, let $G_{K_X} \stackrel{\text{def}}{=} \text{Gal}(K_X^{\text{sep}}/K_X)$, and $G_X \stackrel{\text{def}}{=} G_{K_X}^{\Sigma}$ the maximal pro-Σ quotient of G_{K_X}. We have a natural exact sequence

$$1 \to \mathcal{I}_X \to G_X \to \Delta_X \to 1,$$

where $\mathcal{I}_X \stackrel{\text{def}}{=} \text{Ker}(G_X \twoheadrightarrow \Delta_X)$ is the *cuspidal subgroup* of G_X (with respect to the surjection $G_X \twoheadrightarrow \Delta_X$). Let $i \geq 0$ and write

$$\mathcal{I}_{X,i} \stackrel{\text{def}}{=} \mathcal{I}_X/\mathcal{I}_X(i).$$

Thus, $\mathcal{I}_{X,i}$ is the maximal i-th step prosolvable quotient of $\mathcal{I}_X$, and $\mathcal{I}_{X,1}$ is the maximal abelian quotient of $\mathcal{I}_X$. There exists a natural exact sequence

$$1 \to \mathcal{I}_X[i+1] \to \mathcal{I}_{X,i+1} \to \mathcal{I}_{X,i} \to 1, \tag{1.4}$$

where $\mathcal{I}_X[i+1]$ is the subgroup $\mathcal{I}_X(i)/\mathcal{I}_X(i+1)$ of $\mathcal{I}_{X,i+1}$, and $\mathcal{I}_X[i+1]$ is abelian. Write

$$G_{X,i} \stackrel{\text{def}}{=} G_X/\mathcal{I}_X(i).$$

We shall refer to $G_{X,i}$ (resp. $G_{X,1}$) as the maximal **cuspidally i-th step pro-solvable** (resp. maximal **cuspidally abelian**) quotient of G_X (with respect to the surjection $G_X \twoheadrightarrow \Delta_X$). We have the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{I}_X[i+1] & = & \mathcal{I}_X[i+1] & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathcal{I}_{X,i+1} & \longrightarrow & G_{X,i+1} & \longrightarrow & \Delta_X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{I}_{X,i} & \longrightarrow & G_{X,i} & \longrightarrow & \Delta_X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
\tag{1.5}
$$

of exact sequences.

Proposition 1.1.5. *There are natural isomorphisms* $G_{X,i} \overset{\sim}{\to} \varprojlim_U \Delta_{U,i}$, $\mathcal{I}_{X,i} \overset{\sim}{\to} \varprojlim_U I_{U,i}$, *and* $\mathcal{I}_X[i+1] \overset{\sim}{\to} \varprojlim_U I_U[i+1]$, *where the projective limit is taken over all non-empty open subschemes* $U \subseteq X$. *Moreover, for* $i \geq 1$, *we have a natural isomorphism*

$$\mathcal{I}_X[i+1] \overset{\sim}{\to} \varprojlim_U (\varprojlim_{j\geq 1} (\Delta_{i,j}^U)^{\text{ab}}),$$

where $(\Delta_{i,j}^U)^{\text{ab}}$ *is as in the discussion preceding Proposition 1.1.1.*

Proof. Follows from the various definitions and Proposition 1.1.1. $\square$

1.2. The quotient $\Delta_U \twoheadrightarrow \Delta_U^{p,i+1}$. In this subsection we discuss a certain variant of the theory in 1.1, we use the same notations as in loc. cit.. For a profinite group H, a prime integer p, and an integer $t \geq 1$, write

$$..... \subseteq H(i+1/p^t) \subseteq H(i/p^t) \subseteq ...... \subseteq H(1/p^t) \subseteq H(0/p^t) = H$$

for the $\mathbb{Z}/p^t\mathbb{Z}$-derived series of H, where $H(i+1/p^t) \overset{\text{def}}{=} \overline{< [H(i/p^t), H(i/p^t)], H(i/p^t)^{p^t} >}$ is the $i+1/p^t$-th derived subgroup, which is a characteristic subgroup of H. Write

$$H_{i/p^t} \overset{\text{def}}{=} H/H(i/p^t).$$

We will refer to H_{i/p^t} as the maximal i/p^t-th step *prosolvable* quotient of H, and H_{1/p^t} as the maximal *abelian* annihilated by p^t quotient of H. There exists a natural exact sequence

$$1 \to H[i+1/p^t] \to H_{i+1/p^t} \to H_{i/p^t} \to 1$$

where $H[i+1/p^t]$ is the subgroup $H(i/p^t)/H(i+1/p^t)$ of H_{i+1/p^t}, and $H[i+1/p^t]$ is abelian annihilated by p^t.

Next, let $p \in \Sigma$, and consider the $\mathbb{Z}/p\mathbb{Z}$-derived series of I_U

$$(1.6) \qquad ..... \subseteq I_U(i+1/p) \subseteq I_U(i/p) \subseteq ...... \subseteq I_U(1/p) \subseteq I_U(0/p) = I_U$$

(cf. the above discussion in the case $t = 1$). Then $I_{U,i/p} \overset{\text{def}}{=} I_U/I_U(i/p)$ is the maximal i/p-th step prosolvable quotient of I_U, and $I_{U,1/p}$ is the maximal abelian annihilated by p quotient of I_U. Write $\Delta_{U,i/p} \overset{\text{def}}{=} \Delta_U/I_U(i/p)$, which inserts in the exact sequence

$$1 \to I_{U,i/p} \to \Delta_{U,i/p} \to \Delta_X \to 1.$$

We shall refer to $\Delta_{U,i/p}$ (resp. $\Delta_{U,1/p}$) as the maximal **cuspidally i/p-th step prosolvable** (resp. maximal **cuspidally abelian annihilated by** p) quotient of Δ_U (with respect to the surjection $\Delta_U \twoheadrightarrow \Delta_X$).

Next, we define a certain quotient $\Delta_U \twoheadrightarrow \Delta_U^{p,i+1}$ of Δ_U, which dominates $\Delta_{U,i+1/p}$. Let $i \geq 0$, and G a finite quotient of $\Delta_{U,i+1/p}$ which inserts in the following commutative diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & I_{U,i+1/p} & \longrightarrow & \Delta_{U,i+1/p} & \longrightarrow & \Delta_X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & I & \longrightarrow & G & \longrightarrow & G^{\text{et}} & \longrightarrow & 1
\end{array}$$

Thus, the quotient G corresponds to a finite Galois cover $X_1' \to X$ with Galois group G, which factorizes as $X_1' \to {X_1'}^{\text{et}} \to X$, where ${X_1'}^{\text{et}} \to X$ is the maximal étale sub-cover with Galois group G^{et}, and $X_1' \to {X_1'}^{\text{et}}$ is a tamely ramified Galois cover with Galois group I. Moreover, I is an $(i+1)$-th step solvable group whose successive abelian quotients are annihilated by p. We will assume for the remaining discussion, without loss of generality, that G is *not* a quotient of $\Delta_{U,i/p}$.

Let $s \in \{1, \cdots, n\}$, and $I_{x_s} \subset G$ an inertia subgroup associated to x_s. Then I_{x_s} is cyclic of order p^t, with $t \leq i+1$, as follows from the structure of I. Write $\Delta_{U,1/p^{i+1}} \overset{\text{def}}{=} \Delta_U/I_U(1/p^{i+1})$ for the maximal cuspidally abelian annihilated by p^{i+1} quotient of Δ_U (with respect to the surjection $\Delta_U \twoheadrightarrow \Delta_X$). The following claim follows immediately from the well-known structure of Δ_U (cf. [Grothendieck], Exposé X, Corollaire 3.10).

Claim 1.2.1. *There exists a finite quotient G' of $\Delta_{U,1/p^{i+1}}$ which inserts in the following commutative diagram (where the vertical maps are surjective)*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & I_{U,1/p^{i+1}} & \longrightarrow & \Delta_{U,1/p^{i+1}} & \longrightarrow & \Delta_X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & I' & \longrightarrow & G' & \longrightarrow & {G'}^{\text{et}} & \longrightarrow & 1
\end{array}$$

such that the following holds. The quotient $\Delta_{U,1/p^{i+1}} \twoheadrightarrow G'$ corresponds to a Galois cover $X_2' \to X$ with Galois group G' which factorizes as $X_2' \to {X_2'}^{\text{et}} \to X$, where ${X_2'}^{\text{et}} \to X$ is the maximal étale sub-cover with Galois group ${G'}^{\text{et}}$, and $X_2' \to {X_2'}^{\text{et}}$ is a tamely ramified cover with Galois group I' (an abelian group annihilated by

p^{i+1}). *Further, for* $s \in \{1, \cdots, n\}$, *and* $I'_{x_s} \subseteq I'$ *an inertia subgroup associated to* x_s, *then* I'_{x_s} *is cyclic of order* p^{i+1}.

Let $K'_1 \overset{\text{def}}{=} K_{X'_1}$ (resp. $K'_2 \overset{\text{def}}{=} K_{X'_2}$) be the function field of X'_1 (resp. X'_2), $L' \overset{\text{def}}{=} K'_1.K'_2$ the compositum of K'_1 and K'_2, and Y the normalisation of X in L'. Thus, $Y \to X$ is a Galois cover with Galois group $H \subseteq G \times G'$ which is étale above U. Note that H maps onto G, G', and the quotient $\Delta_U \twoheadrightarrow H$ doesn't factorize through $\Delta_U \twoheadrightarrow \Delta_{U,i+1/p}$ if $i \geq 1$.

Let $I'' \overset{\text{def}}{=} \operatorname{Ker}(I \twoheadrightarrow I^{\text{ab}})$. Thus, I'' is an i-th step solvable group whose successive quotients are annihilated by p, and is a characteristic subgroup of I. Write $\widetilde{G} \overset{\text{def}}{=} G/I''$, and let $\widetilde{H}$ be the image of H in the quotient $\widetilde{G} \times G'$ of $G \times G'$. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_H \overset{\text{def}}{=} H \cap (I'' \times \{1\}) & \longrightarrow & H & \longrightarrow & \widetilde{H} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I'' \times \{1\} & \longrightarrow & G \times G' & \longrightarrow & \widetilde{G} \times G' & \longrightarrow & 1 \end{array}$$

where the vertical maps are natural inclusions.

Lemma 1.2.2. *The group* $\widetilde{H}$ *is a quotient of* $\Delta_{U,1/p^{i+1}}$. *Moreover, the Galois cover* $Y \to X$ *factorizes as* $Y \to Y' \to X$, *where* $Y' \to X$ *is a tamely ramified Galois cover with Galois group* $\widetilde{H}$, *and* $Y \to Y'$ *is an* **étale** *Galois cover with Galois group* $I_H \subseteq I''$: *an* i**-th step solvable** *group whose successive abelian quotients are* **annihilated by** p.

Proof. The first assertion follows from the fact that the inertia subgroup of $\widetilde{H}$ is a subgroup of $I^{\text{ab}} \times I'$. The proof of the second assertion is similar to the proof of Lemma 1.1.4 using Abhyankar's Lemma (cf. loc. cit.). □

The profinite group $\Delta_{U,1/p^{i+1}}$; being a quotient of Δ_U, is topologically finitely generated. Hence there exists a sequence of characteristic open subgroups

$$... \subseteq \Delta_{U,1/p^{i+1}}[j+1] \subseteq \Delta_{U,1/p^{i+1}}[j] \subseteq ... \subseteq \Delta_{U,1/p^{i+1}}[1] \overset{\text{def}}{=} \Delta_{U,1/p^{i+1}}$$

of $\Delta_{U,1/p^{i+1}}$ such that $\bigcap_{j\geq 1} \Delta_{U,1/p^{i+1}}[j] = \{1\}$. The open subgroup $\Delta_{U,1/p^{i+1}}[j] \subseteq \Delta_{U,1/p^{i+1}}$ corresponds to a finite Galois cover $(X'_{i+1,j})^U \to X$ between smooth connected and proper ℓ-curves, with Galois group $G^U_{i+1,j} \overset{\text{def}}{=} \Delta_{U,1/p^{i+1}}/\Delta_{U,1/p^{i+1}}[j]$, and which restricts to an étale cover $(V_{i+1,j})^U \to U$. The geometric point η determines a geometric point $\eta'_{i+1,j}$ of $(X'_{i+1,j})^U$ and $(V_{i+1,j})^U$. Write $(\Delta'_{i+1,j})^U = \Delta_{(X'_{i+1,j})^U} \overset{\text{def}}{=} \pi_1((X'_{i+1,j})^U, \eta'_{i+1,j})^\Sigma$ for the maximal pro-Σ étale fundamental group of $(X'_{i+1,j})^U$ with base point $\eta'_{i+1,j}$, and $((\Delta'_{i+1,j})^U)_{i/p}$ for the maximal i/p-th step prosolvable quotient of $(\Delta'_{i+1,j})^U$. Consider the following push-out diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1((V_{i+1,j})^U, \eta'_{i+1,j})^\Sigma & \longrightarrow & \Delta_U & \longrightarrow & G^U_{i+1,j} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & ((\Delta'_{i+1,j})^U)_{i/p} & \longrightarrow & \widetilde{G}^U_{i+1,j} & \longrightarrow & G^U_{i+1,j} & \longrightarrow & 1 \end{array}$$

($\ker(\pi_1((V'_{i+1,j})^U, \eta'_{i+1,j})^\Sigma \twoheadrightarrow ((\Delta'_{i+1,j})^U)_{i/p})$ is a normal subgroup of Δ_U).

Lemma 1.2.3. *With the above notations, G is a quotient of $\widetilde{G}^U_{i+1,j}$ for some $j \geq 1$.*

Proof. Follows from Lemma 1.2.2 and the various definitions. $\square$

Let

$$\Delta_U^{p,i+1} \stackrel{\text{def}}{=} \varprojlim_{j\geq 1} \widetilde{G}^U_{i+1,j},$$

where $\widetilde{G}^U_{i+1,j}$ is as in Lemma 1.2.3 (the $\{\widetilde{G}^U_{i+1,j}\}_{j\geq 1}$ form a projective system). Thus, it follows from the various definitions that we have a natural exact sequence

$$1 \to \varprojlim_{j\geq 1}((\Delta'_{i+1,j})^U)_{i/p} \to \Delta_U^{p,i+1} \to \Delta_{U,1/p^{i+1}} \to 1, \tag{1.7}$$

where the $\{((\Delta'_{i+1,j})^U)_{i/p}\}_{j\geq 1}$ are defined as above.

Proposition 1.2.4. *The profinite group $\Delta_{U,i+1/p}$ is a* **quotient** *of $\Delta_U^{p,i+1}$.*

Proof. Follows from the above discussion (cf. Lemma 1.2.3). $\square$

Similarly, write $\mathcal{I}_{X,i+1/p} \stackrel{\text{def}}{=} \mathcal{I}_X/\mathcal{I}_X(i+1/p)$. Thus, $\mathcal{I}_{X,i+1/p}$ is the maximal $i+1/p$-th step prosolvable quotient of $\mathcal{I}_X$, and $\mathcal{I}_{X,1/p}$ is the maximal abelian annihilated by p quotient of $\mathcal{I}_X$. Write

$$G_{X,i+1/p} \stackrel{\text{def}}{=} G_X/\mathcal{I}_X(i+1/p).$$

We shall refer to $G_{X,i+1/p}$ (resp. $G_{X,1/p}$) as the maximal **cuspidally $i+1/p$-th step prosolvable** (resp. maximal **cuspidally abelian annihilated by p**) quotient of G_X (with respect to the surjection $G_X \twoheadrightarrow \Delta_X$). Also, write $G_{X,1/p^{i+1}} \stackrel{\text{def}}{=} G_X/\mathcal{I}_X(1/p^{i+1})$, and $G_X^{p,i+1} \stackrel{\text{def}}{=} \varprojlim_U(\Delta_U^{p,i+1})$ where the limit is taken over all open subschemes $U \subseteq X$. We have the following exact sequence

$$1 \to \varprojlim_U(\varprojlim_{j\geq 1}((\Delta'_{i+1,j})^U)_{i/p}) \to G_X^{p,i+1} \to G_{X,1/p^{i+1}} \to 1. \tag{1.8}$$

Lemma 1.2.5. *The profinite group $G_{X,i+1/p}$ is a* **quotient** *of $G_X^{p,i+1}$.*

Proof. Follows from the various definitions, and Proposition 1.2.4. $\square$

§2. Cuspidalisation of sections of cuspidally i-th step prosolvable arithmetic fundamental groups. In this section k is a field with $\text{char}(k) = l \geq 0$, X is a proper smooth and geometrically connected hyperbolic (i.e., genus$(X) \geq 2$) curve over k, and K_X its function field. Let η be a geometric point of X above its generic point; it determines an algebraic closure $\overline{k}$ of k, and a geometric point $\overline{\eta}$ of $\overline{X} \stackrel{\text{def}}{=} X \times_k \overline{k}$.

2.1. Let $\Sigma \subseteq \mathfrak{Primes}$ be a non-empty set of prime integers. In case $\text{char}(k) = l > 0$ we assume that $l \notin \Sigma$. Write $\Delta_X \stackrel{\text{def}}{=} \pi_1(\overline{X},\overline{\eta})^{\Sigma}$ for the maximal pro-Σ quotient of $\pi_1(\overline{X},\overline{\eta})$, and $\Pi_X \stackrel{\text{def}}{=} \pi_1(X,\eta)/\operatorname{Ker}(\pi_1(\overline{X},\overline{\eta}) \twoheadrightarrow \pi_1(\overline{X},\overline{\eta})^{\Sigma})$. Thus, we have an exact sequence

$$1 \to \Delta_X \to \Pi_X \xrightarrow{\text{pr}_{X,\Sigma}} G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k) \to 1. \tag{2.1}$$

We shall refer to $\pi_1(X,\eta)^{(\Sigma)} \stackrel{\text{def}}{=} \Pi_X$ as the **geometrically pro-Σ arithmetic fundamental group** of X.

2.1.1. Let $U \subseteq X$ be a nonempty open subscheme. Write $\Delta_U \overset{\text{def}}{=} \pi_1(\overline{U}, \overline{\eta})^{\Sigma}$ for the maximal pro-Σ quotient of the fundamental group $\pi_1(\overline{U}, \overline{\eta})$ of $\overline{U}$ with base point $\overline{\eta}$, and $\Pi_U \overset{\text{def}}{=} \pi_1(U, \eta)/\operatorname{Ker}(\pi_1(\overline{U}, \overline{\eta}) \twoheadrightarrow \pi_1(\overline{U}, \overline{\eta})^{\Sigma})$. Thus, we have an exact sequence $1 \to \Delta_U \to \Pi_U \to G_k \to 1$. Let $I_U \overset{\text{def}}{=} \operatorname{Ker}(\Pi_U \twoheadrightarrow \Pi_X) = \operatorname{Ker}(\Delta_U \twoheadrightarrow \Delta_X)$ be the cuspidal subgroup of Π_U (with respect to the surjection $\Pi_U \twoheadrightarrow \Pi_X$). We have the following exact sequence

$$1 \to I_U \to \Pi_U \to \Pi_X \to 1. \tag{2.2}$$

Let $i \geq 0$ be an integer, $I_{U,i}$ the maximal i-th step prosolvable quotient of I_U (cf. 1.1), and $\Pi_U^{(i-\text{sol})} \overset{\text{def}}{=} \Pi_U/\operatorname{Ker}(I_U \twoheadrightarrow I_{U,i})$. We shall refer to $\Pi_U^{(i-\text{sol})}$ as the maximal (geometrically) **cuspidally i-th step prosolvable quotient** of Π_U (with respect to the surjection $\Pi_U \twoheadrightarrow \Pi_X$).

2.1.2. Similarly, we have an exact sequence of absolute Galois groups $1 \to G_{\overline{k}.K_X} \to G_{K_X} \to G_k \to 1$, where $G_{\overline{k}.K_X} \overset{\text{def}}{=} \pi_1(\operatorname{Spec}(\overline{k}.K_X), \eta)$, and $G_{K_X} \overset{\text{def}}{=} \pi_1(\operatorname{Spec}(K_X), \eta)$. Let $G_{\overline{X}} \overset{\text{def}}{=} G^{\Sigma}_{\overline{k}.K_X}$ be the maximal pro-Σ quotient of $G_{\overline{k}.K_X}$, $G_X \overset{\text{def}}{=} G_{K_X}/\operatorname{Ker}(G_{\overline{k}.K_X} \twoheadrightarrow G^{\Sigma}_{\overline{k}.K_X})$, and $\mathcal{I}_X \overset{\text{def}}{=} \operatorname{Ker}(G_X \twoheadrightarrow \Pi_X) = \operatorname{Ker}(G_{\overline{X}} \twoheadrightarrow \Delta_X)$ the cuspidal subgroup of G_X (with respect to the surjection $G_X \twoheadrightarrow \Pi_X$). Let $\mathcal{I}_{X,i}$ be the maximal i-th step prosolvable quotient of $\mathcal{I}_X$. By pushing the exact sequence $1 \to \mathcal{I}_X \to G_X \to \Pi_X \to 1$ by the characteristic quotient $\mathcal{I}_X \twoheadrightarrow \mathcal{I}_{X,i}$ we obtain an exact sequence

$$1 \to \mathcal{I}_{X,i} \to G_X^{(i-\text{sol})} \to \Pi_X \to 1. \tag{2.3}$$

We will refer to the quotient $G_X^{(i-\text{sol})}$ as the maximal (geometrically) **cuspidally i-th step prosolvable quotient** of G_X (with respect to the surjective homomorphism $G_X \twoheadrightarrow \Pi_X$). There exist natural isomorphisms

$$G_X^{(i-\text{sol})} \overset{\sim}{\to} \varprojlim_U \Pi_U^{(i-\text{sol})}, \qquad \mathcal{I}_{X,i} \overset{\sim}{\to} \varprojlim_U I_{U,i}\ ,$$

where the limit is taken over all open subschemes $U \subseteq X$.

2.2. Let $i \geq 0$. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{I}_X[i+1] & = & \mathcal{I}_X[i+1] & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & G_{\overline{X},i+1} & \longrightarrow & G_X^{(i+1-\text{sol})} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G_{\overline{X},i} & \longrightarrow & G_X^{(i-\text{sol})} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{2.4}$$

and similarly, for $U \subseteq X$ nonempty open, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & I_U[i+1] & = & I_U[i+1] & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Delta_{U,i+1} & \longrightarrow & \Pi_U^{(i+1-\mathrm{sol})} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_{U,i} & \longrightarrow & \Pi_U^{(i-\mathrm{sol})} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
\tag{2.5}
$$

(recall the definition of $\mathcal{I}_X[i+1]$ and $I_U[i+1]$ from §1).

Assume that the lower horizontal sequence in diagram (2.4) splits. Let $s : G_k \to G_X^{(i-\mathrm{sol})}$ be a section of the projection $G_X^{(i-\mathrm{sol})} \twoheadrightarrow G_k$, which induces a section $s_U : G_k \to \Pi_U^{(i-\mathrm{sol})}$ of the projection $\Pi_U^{(i-\mathrm{sol})} \twoheadrightarrow G_k$, $\forall U \subseteq X$ open.

The cuspidalisation problem for sections of cuspidally i-th step prosolvable arithmetic fundamental groups. *Let $i \geq 0$. Given a section $s_U : G_k \to \Pi_U^{(i-\mathrm{sol})}$ as above, is it possible to construct a section $\tilde{s}_U : G_k \to \Pi_U^{(i+1-\mathrm{sol})}$ of the projection $\Pi_U^{(i+1-\mathrm{sol})} \twoheadrightarrow G_k$ which* **lifts** *the section s_U, i.e., which fits in a commutative diagram*

$$
\begin{array}{ccc}
G_k & \xrightarrow{\tilde{s}_U} & \Pi_U^{(i+1-\mathrm{sol})} \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\
G_k & \xrightarrow{s_U} & \Pi_U^{(i-\mathrm{sol})}
\end{array}
$$

where the right vertical map is the natural surjection?

Similarly, is it possible to construct a section $\tilde{s} : G_k \to G_X^{(i+1-\mathrm{sol})}$ of the projection $G_X^{(i+1-\mathrm{sol})} \twoheadrightarrow G_k$ which **lifts** *the section s, i.e., which fits in a commutative diagram*

$$
\begin{array}{ccc}
G_k & \xrightarrow{\tilde{s}} & G_X^{(i+1-\mathrm{sol})} \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\
G_k & \xrightarrow{s} & G_X^{(i-\mathrm{sol})}
\end{array}
$$

where the right vertical map is the natural surjection?

2.3. The above cuspidalisation problem in the case $i = 0$ has been investigated in [Saïdi]. Next, we will investigate this problem in the case $i \geq 1$.

We use the notations in 2.2. Recall the definition of the characteristic open subgroups $\{\Delta_{U,i}[j]\}_{j\geq 1}$ such that $\bigcap_{j\geq 1} \Delta_{U,i}[j] = \{1\}$ (cf. discussion before Proposition 1.1.1). Write $\widehat{\Pi}_U[i,j] \overset{\mathrm{def}}{=} \widehat{\Pi}_U[i,j][s_U] \overset{\mathrm{def}}{=} \Delta_{U,i}[j].s_U(G_k)$. Thus, $\widehat{\Pi}_U[i,j] \subseteq \Pi_U^{(i-\mathrm{sol})}$

is an open subgroup which contains the image $s_U(G_k)$ of the section s_U. Write $\Pi_U[i,j] \overset{\text{def}}{=} \Pi_U[i,j][s_U]$ for the inverse image of $\widehat{\Pi}_U[i,j]$ in Π_U. Thus, $\Pi_U[i,j] \subseteq \Pi_U$ is an open subgroup which corresponds to an étale cover $V_{i,j} \to U$, where $V_{i,j}$ is a geometrically irreducible k-curve (since $\Pi_U[i,j]$ maps onto G_k via the natural projection $\Pi_U \twoheadrightarrow G_k$ by the very definition of $\Pi_U[i,j]$).

Write $X^U_{i,j}$ (resp. $\overline{X}^U_{i,j}$) for the smooth compactification of $V_{i,j}$ (resp. $\overline{V}_{i,j} \overset{\text{def}}{=} V_{i,j} \times_k \overline{k}$). We have an exact sequence $1 \to \pi_1(\overline{X}^U_{i,j}, \overline{\eta}_{i,j}) \to \pi_1(X^U_{i,j}, \eta_{i,j}) \to G_k \to 1$, where $\eta_{i,j}$ (resp. $\overline{\eta}_{i,j}$) is a geometric point naturally induced by η. Write $\pi_1(\overline{X}^U_{i,j}, \overline{\eta}_{i,j})^{\Sigma,\text{ab}}$ for the maximal pro-Σ abelian quotient of $\pi_1(\overline{X}^U_{i,j}, \overline{\eta}_{i,j})$, and $\pi_1(X^U_{i,j}, \eta_{i,j})^{(\Sigma,\text{ab})} \overset{\text{def}}{=} \pi_1(X^U_{i,j}, \eta_{i,j})/\operatorname{Ker}(\pi_1(\overline{X}^U_{i,j}, \overline{\eta}_{i,j}) \twoheadrightarrow \pi_1(\overline{X}^U_{i,j}, \overline{\eta}_{i,j})^{\Sigma,\text{ab}})$ for the geometrically pro-Σ abelian fundamental group of $X^U_{i,j}$. Consider the following pull-back diagram.

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & I_U[i+1] & \longrightarrow & \mathcal{H}_{U,i} \overset{\text{def}}{=} \mathcal{H}_{U,i}[s_U] & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle s_U} & & \\
1 & \longrightarrow & I_U[i+1] & \longrightarrow & \Pi_U^{(i+1-\text{sol})} & \longrightarrow & \Pi_U^{(i-\text{sol})} & \longrightarrow & 1
\end{array}
\tag{2.6}
$$

Lemma 2.3.1. *The section* $s_U : G_k \to \Pi_U^{(i-\text{sol})}$ **lifts** *to a section* $\tilde{s}_U : G_k \to \Pi_U^{(i+1-\text{sol})}$ *of the projection* $\Pi_U^{(i+1-\text{sol})} \twoheadrightarrow G_k$ **if and only if** *the group extension* $1 \to I_U[i+1] \to \mathcal{H}_{U,i} \to G_k \to 1$ **splits**.

Proof. Follows immediately from the diagram (2.6). □

Lemma 2.3.2. *Assume $i \geq 1$. Then we have natural identifications* $I_U[i+1] \overset{\sim}{\to} \varprojlim_{j\geq 1} \pi_1(\overline{X}^U_{i,j}, \overline{\eta}_{i,j})^{\Sigma,\text{ab}}$, *and* $\mathcal{H}_{U,i} \overset{\sim}{\to} \varprojlim_{j\geq 1} \pi_1(X^U_{i,j}, \eta_{i,j})^{(\Sigma,\text{ab})}$.

Proof. Follows from the various definitions and Proposition 1.1.1. □

Definition 2.3.3. We say that the field k satisfies the condition $(\mathbf{H}_\Sigma)$ if the following holds. The Galois cohomology groups $H^1(G_k, M)$ are *finite* for every finite G_k-module M whose cardinality is divisible only by primes in Σ.

Lemma 2.3.4. *Assume that $i \geq 1$, and k satisfies the condition $(\mathbf{H}_\Sigma)$. Then the group extension* $1 \to I_U[i+1] \to \mathcal{H}_{U,i} \to G_k \to 1$ **splits if and only if** *the group extensions* $1 \to \pi_1(\overline{X}^U_{i,j}, \overline{\eta}_{i,j})^{\Sigma,\text{ab}} \to \pi_1(X^U_{i,j}, \eta_{i,j})^{(\Sigma,\text{ab})} \to G_k \to 1$ **split**, $\forall j \geq 1$.

Proof. The only if part follows immediately from Lemma 2.3.2. Conversely, assume that the group extension $\pi_1(X^U_{i,j}, \eta_{i,j})^{(\Sigma,\text{ab})}$ splits, $\forall j \geq 1$. Write $\mathcal{H}_{U,i} = \varprojlim_{G} G$ as the projective limit of finite quotients G which insert into a commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & I_U[i+1] & \longrightarrow & \mathcal{H}_{U,i} \overset{\text{def}}{=} \mathcal{H}_{U,i}[s_U] & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \overline{G} & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 1
\end{array}
$$

where the vertical maps are surjective. Let $\widetilde{G}$ be the pull-back of the group extension G by the surjective homomorphism $G_k \twoheadrightarrow H$. Thus, we have an exact sequence

$1 \to \overline{G} \to \widetilde{G} \to G_k \to 1$, and $\mathcal{H}_{U,i} = \varprojlim_{\widetilde{G}} \widetilde{G}$. Given a (geometrically finite) quotient $\mathcal{H}_{U,i} \twoheadrightarrow \widetilde{G}$ as above, it factorizes as $\mathcal{H}_{U,i} \twoheadrightarrow \pi_1(X_{i,j}^U, \eta_{i,j})^{(\Sigma,\mathrm{ab})} \twoheadrightarrow \widetilde{G}$ for some $j \geq 1$ (cf. Lemma 2.3.2). In particular, the group extension $\widetilde{G}$ splits by our assumption that the group extensions $\pi_1(X_{i,j}^U, \eta_{i,j})^{(\Sigma,\mathrm{ab})}$ split, $\forall j \geq 1$. The set $\mathrm{Sect}(G_k, \mathcal{H}_{U,i})$ of continuous splittings of the group extension $\mathcal{H}_{U,i}$ is naturally identified with the inverse limit $\varprojlim_{\widetilde{G}} \mathrm{Sect}(G_k, \widetilde{G})$ of the sets of continuous splittings of the group extensions $\widetilde{G}$ as above. For a (geometrically finite) quotient $\widetilde{G}$ of $\mathcal{H}_{U,i}$ as above the set $\mathrm{Sect}(G_k, \widetilde{G})$ is non-empty (cf. above discussion) and is, up to conjugation by elements of $\overline{G}$, a torsor under the Galois cohomology group $H^1(G_k, \overline{G})$, which is finite by our assumption that k satisfies the condition $(\mathbf{H}_\Sigma)$. Thus, the set $\mathrm{Sect}(G_k, \widetilde{G})$ is a non-empty finite set. Hence the set $\mathrm{Sect}(G_k, \mathcal{H}_{U,i})$ is non-empty being the projective limit of non-empty finite sets. $\square$

For $j \geq 1$, let $J_{i,j}^U \overset{\mathrm{def}}{=} \mathrm{Pic}_k^0(X_{i,j}^U)$ be the jacobian of $X_{i,j}^U$, and $(J_{i,j}^1)^U \overset{\mathrm{def}}{=} \mathrm{Pic}_k^1(X_{i,j}^U)$. Thus, $(J_{i,j}^1)^U$ is a torsor under $J_{i,j}^U$.

Lemma 2.3.5. *The group extension* $1 \to \pi_1(\overline{X}_{i,j}^U, \overline{\eta}_{i,j})^{\Sigma,\mathrm{ab}} \to \pi_1(X_{i,j}^U, \eta_{i,j})^{(\Sigma,\mathrm{ab})} \to G_k \to 1$ **splits if and only if** *the class of* $(J_{i,j}^1)^U$ *in* $H^1(G_k, J_{i,j}^U)$ *lies in the maximal* Σ*-divisible subgroup of* $H^1(G_k, J_{i,j}^U)$*, i.e., the maximal subgroup of* $H^1(G_k, J_{i,j}^U)$ *which is divisible by integers whose prime factors are in* Σ.

Proof. This is a well known fact (cf. [Harari-Szamuely], Theorem 1.2, and Proposition 2.1). $\square$

Proposition 2.3.6. *We use the same notations as above. Assume that* $i \geq 1$*, and* k *satisfies the condition* $(\mathbf{H}_\Sigma)$*. Then the section* $s_U : G_k \to \Pi_U^{(i-\mathrm{sol})}$ **lifts** *to a section* $\tilde{s}_U : G_k \to \Pi_U^{(i+1-\mathrm{sol})}$ *of the projection* $\Pi_U^{(i+1-\mathrm{sol})} \twoheadrightarrow G_k$ **if and only if** *the class of* $(J_{i,j}^1)^U$ *in* $H^1(G_k, J_{i,j}^U)$ *lies in the maximal* Σ*-divisible subgroup of* $H^1(G_k, J_{i,j}^U)$*,* $\forall j \geq 1$.

Proof. Follows from Lemmas 2.3.4 and 2.3.5. $\square$

Recall the section $s : G_k \to G_X^{(i-\mathrm{sol})}$ of the projection $G_X^{(i-\mathrm{sol})} \twoheadrightarrow G_k$ which induces the section $s_U : G_k \to \Pi_U^{(i-\mathrm{sol})}$ of $\Pi_U^{(i-\mathrm{sol})} \twoheadrightarrow G_k$, $\forall U \subseteq X$ open (cf. 2.2).

Lemma 2.3.7. *Assume that* k *satisfies the condition* $(\mathbf{H}_\Sigma)$*. Then the section* $s : G_k \to G_X^{(i-\mathrm{sol})}$ **lifts** *to a section* $\tilde{s} : G_k \to G_X^{(i+1-\mathrm{sol})}$ *of the projection* $G_X^{(i+1-\mathrm{sol})} \twoheadrightarrow G_k$ **if and only if** *for each nonempty open subscheme* $U \subseteq X$ *the section* $s_U : G_k \to \Pi_U^{(i-\mathrm{sol})}$ *lifts to a section* $\tilde{s}_U : G_k \to \Pi_U^{(i+1-\mathrm{sol})}$ *of the projection* $\Pi_U^{(i+1-\mathrm{sol})} \twoheadrightarrow G_k$.

Proof. Similar to the proof of Lemma 2.3.4, using the facts that k satisfies the condition $(\mathbf{H}_\Sigma)$, and $G_X^{(i-\mathrm{sol})} \overset{\sim}{\to} \varprojlim_U \Pi_U^{(i-\mathrm{sol})}$. $\square$

The following is our main result in this section.

Theorem 2.3.8. *We use the same notations as above. Assume that* $i \geq 1$*, and* k *satisfies the condition* $(\mathbf{H}_\Sigma)$*. Then the section* $s : G_k \to G_X^{(i-\mathrm{sol})}$ **lifts** *to a section* $\tilde{s} : G_k \to G_X^{(i+1-\mathrm{sol})}$ *of the projection* $G_X^{(i+1-\mathrm{sol})} \twoheadrightarrow G_k$ **if and only if** $\forall U \subseteq X$

nonempty open subscheme the class of $(J^1_{i,j})^U$ *in* $H^1(G_k, J^U_{i,j})$ *lies in the maximal* Σ*-divisible subgroup of* $H^1(G_k, J^U_{i,j})$, $\forall j \geq 1$.

Proof. Follows from Lemmas 2.3.4, 2.3.5, and 2.3.7. $\square$

§3. Lifting of sections to cuspidally $2/p$-th step prosolvable arithmetic fundamental groups. In this section we investigate a certain mod-p variant of the cuspidalisation problem investigated in §2 (as well as in [Saïdi]). Throughout §3 we use the same notations as in §2. Let $p \in \Sigma$ be a prime integer.

3.1. Let $U \subseteq X$ be a nonempty open subscheme. Let $i \geq 0$, $t \geq 1$, be integers, and $I_{U,i/p^t}$ the maximal i/p^t-th step prosolvable quotient of I_U (cf. 1.2). By pushing the exact sequence (2.2) by the surjective homomorphism $I_U \twoheadrightarrow I_{U,i/p^t}$ we obtain an exact sequence $1 \to I_{U,i/p^t} \to \Pi_U^{(i/p^t\text{-sol})} \to \Pi_X \to 1$. We shall refer to $\Pi_U^{(i/p^t\text{-sol})}$ as the maximal (geometrically) **cuspidally i/p^t-th step prosolvable quotient** of Π_U (with respect to the surjection $\Pi_U \twoheadrightarrow \Pi_X$). We have a commutative diagram of exact sequence.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & I_{U,i/p^t} & = & I_{U,i/p^t} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Delta_{U,i/p^t} & \longrightarrow & \Pi_U^{(i/p^t\text{-sol})} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{3.1}$$

Similarly, by pushing the exact sequence $1 \to \mathcal{I}_X \to G_X \to \Pi_X \to 1$ by the surjective homomorphism $\mathcal{I}_X \twoheadrightarrow \mathcal{I}_{X,i/p^t}$ we obtain an exact sequence $1 \to \mathcal{I}_{X,i/p^t} \to G_X^{(i/p^t\text{-sol})} \to \Pi_X \to 1$. We will refer to $G_X^{(i/p^t\text{-sol})}$ as the maximal (geometrically) **cuspidally i/p^t-th step prosolvable quotient** of G_X (with respect to the surjective homomorphism $G_X \twoheadrightarrow \Pi_X$). There exist natural isomorphisms

$$G_X^{(i/p^t\text{-sol})} \xrightarrow{\sim} \varprojlim_U \Pi_U^{(i/p^t\text{-sol})}, \qquad \mathcal{I}_{X,i/p^t} \xrightarrow{\sim} \varprojlim_U I_{U,i/p^t},$$

where the limits are over all open subschemes $U \subseteq X$, and a commutative diagram.

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{I}_{X,i/p^t} & = & \mathcal{I}_{X,i/p^t} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & G_{\overline{X},i/p^t} & \longrightarrow & G_X^{(i/p^t\text{-sol})} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
\tag{3.2}
$$

3.2. Assume that the lower horizontal exact sequence in diagram (3.2) splits. Let $s : G_k \to \Pi_X$ be a section of the projection $\Pi_X \twoheadrightarrow G_k$.

The lifting problem to sections of cuspidally $i + 1/p^t$-th step prosolvable arithmetic fundamental groups. *Let $i \geq 0$, $t \geq 1$, be integers. Given a section $s : G_k \to \Pi_X$ as above is it possible to construct a section $s_{U,i+1} : G_k \to \Pi_U^{(i+1/p^t\text{-sol})}$ of the projection $\Pi_U^{(i+1/p^t\text{-sol})} \twoheadrightarrow G_k$ which* **lifts** *the section s, i.e., which fits in a commutative diagram*

$$
\begin{array}{ccc}
G_k & \xrightarrow{s_{U,i+1}} & \Pi_U^{(i+1/p^t\text{-sol})} \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\
G_k & \xrightarrow{s} & \Pi_X
\end{array}
$$

where the right vertical map is the natural surjection?

Similarly, is it possible to construct a section $s_{i+1} : G_k \to G_X^{(i+1/p^t\text{-sol})}$ of the projection $G_X^{(i+1/p^t\text{-sol})} \twoheadrightarrow G_k$ which **lifts** *the section s, i.e., which fits in a commutative diagram*

$$
\begin{array}{ccc}
G_k & \xrightarrow{s_{i+1}} & G_X^{(i+1/p^t\text{-sol})} \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\
G_k & \xrightarrow{s} & \Pi_X
\end{array}
$$

where the right vertical map is the natural surjection?

3.3. The quotients $G_X \twoheadrightarrow G_X^{(p,i+1)}$, $\Pi_U \twoheadrightarrow \Pi_U^{(p,i+1)}$, and lifting of sections. Next, recall the notations in 1.2 and the discussion therein, especially the definition of the quotient $\Delta_U \twoheadrightarrow \Delta_U^{p,i+1}$ (cf. the discussion after Lemma 1.2.3). The kernel of the surjective homomorphism $\Delta_U \twoheadrightarrow \Delta_U^{p,i+1}$ is a normal subgroup of Π_U (as one easily verifies). Write $\Pi_U^{(p,i+1)} \overset{\text{def}}{=} \Pi_U / \operatorname{Ker}(\Delta_U \twoheadrightarrow \Delta_U^{p,i+1})$. Thus, we have an exact sequence

$$1 \to \Delta_U^{p,i+1} \to \Pi_U^{(p,i+1)} \to G_k \to 1. \tag{3.3}$$

Recall the exact sequence $1 \to \varprojlim_{j\geq 1}((\Delta'_{i+1,j})^U)_{i/p} \to \Delta_U^{p,i+1} \to \Delta_{U,1/p^{i+1}} \to 1$ (cf. loc. cit.). The quotient $\Pi_U \twoheadrightarrow \Pi_U^{(1/p^{i+1}-\mathrm{sol})}$ (cf. 3.1) factorizes through $\Pi_U \twoheadrightarrow \Pi_U^{(p,i+1)}$ (cf. exact sequence (1.7)), and we have a commutative diagram of exact sequences.

(3.4)

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \varprojlim_{j\geq 1}((\Delta'_{i+1,j})^U)_{i/p} & = & \varprojlim_{j\geq 1}((\Delta'_{i+1,j})^U)_{i/p} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Delta_U^{p,i+1} & \longrightarrow & \Pi_U^{(p,i+1)} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_{U,1/p^{i+1}} & \longrightarrow & \Pi_U^{(1/p^{i+1}-\mathrm{sol})} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Similarly, $\mathrm{Ker}(G_{\overline{X}} \twoheadrightarrow G_{\overline{X}}^{p,i+1})$ is a normal subgroup of G_X, and we have an exact sequence

$$1 \to G_{\overline{X}}^{p,i+1} \to G_X^{(p,i+1)} \to G_k \to 1, \tag{3.5}$$

where $G_X^{(p,i+1)} \overset{\mathrm{def}}{=} G_X / \mathrm{Ker}(G_{\overline{X}} \twoheadrightarrow G_{\overline{X}}^{p,i+1})$. The exact sequence (1.8) induces a commutative diagram of exact sequences.

(3.6)

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \varprojlim_{U}(\varprojlim_{j\geq 1}((\Delta'_{i+1,j})^U)_{i/p}) & = & \varprojlim_{U}(\varprojlim_{j\geq 1}((\Delta'_{i+1,j})^U)_{i/p}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & G_{\overline{X}}^{p,i+1} & \longrightarrow & G_X^{(p,i+1)} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G_{\overline{X},1/p^{i+1}} & \longrightarrow & G_X^{(1/p^{i+1}-\mathrm{sol})} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Furthermore, $\Pi_U^{(i+1/p-\mathrm{sol})}$ (resp. $G_X^{(i+1/p-\mathrm{sol})}$) is a quotient of $\Pi_U^{(p,i+1)}$ (resp. of

$G_X^{(p,i+1)}$) (cf. Lemmas 1.2.4 and 1.2.5) and we have commutative diagrams

$$(3.7)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_U^{p,i+1} & \longrightarrow & \Pi_U^{(p,i+1)} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \Delta_{U,i+1/p} & \longrightarrow & \Pi_U^{(i+1/p-\mathrm{sol})} & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

resp.

$$(3.8)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & G_{\overline{X}}^{p,i+1} & \longrightarrow & G_X^{(p,i+1)} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & G_{\overline{X},i+1/p} & \longrightarrow & G_X^{(i+1/p-\mathrm{sol})} & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

where the left and middle vertical maps are surjective.

The lifting problem to sections of $\Pi_U^{(p,i+1)}$, and $G_X^{(p,i+1)}$. *Given a section $s : G_k \to \Pi_X$ of the projection $\Pi_X \twoheadrightarrow G_k$ as in 3.2, is it possible to construct a section $\tilde{s}_{U,i+1} : G_k \to \Pi_U^{(p,i+1)}$ of the projection $\Pi_U^{(p,i+1)} \twoheadrightarrow G_k$ which* **lifts** *the section s, i.e., which fits in a commutative diagram*

$$\begin{array}{ccc} G_k & \xrightarrow{\tilde{s}_{U,i+1}} & \Pi_U^{(p,i+1)} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow \\ G_k & \xrightarrow{s} & \Pi_X \end{array}$$

where the right vertical map is the natural surjection?

Similarly, is it possible to construct a section $\tilde{s}_{i+1} : G_k \to G_X^{(p,i+1)}$ of the projection $G_X^{(p,i+1)} \twoheadrightarrow G_k$ which **lifts** *the section s, i.e., which fits in a commutative diagram*

$$\begin{array}{ccc} G_k & \xrightarrow{\tilde{s}_{i+1}} & G_X^{(p,i+1)} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow \\ G_k & \xrightarrow{s} & \Pi_X \end{array}$$

where the right vertical map is the natural surjection?

Lemma 3.3.1. *A positive answer to the lifting problem posed in 3.3 implies a positive answer to the lifting problem posed in 3.2 in the case $t = 1$.*

Proof. Follows immediately from the above commutative diagrams (3.7) and (3.8). □

3.4. In this section we investigate the lifting problem posed in 3.3 in the case $i = 1$, and draw consequences for the lifting problem posed in 3.2 in the case $t = i = 1$ (the only case we need for applications in §4). Let $s : G_k \to \Pi_X$ be a section of the projection $\Pi_X \twoheadrightarrow G_k$.

3.4.I. First, we investigate the problem of lifting the section s to a section $s'_{U,i+1} : G_k \to \Pi_U^{(1/p^{i+1}-\mathrm{sol})}$ (resp. $s'_{i+1} : G_k \to G_X^{(1/p^{i+1}-\mathrm{sol})}$) of the projection $\Pi_U^{(1/p^{i+1}-\mathrm{sol})} \twoheadrightarrow$

G_k (resp. $G_X^{(1/p^{i+1}-\mathrm{sol})} \twoheadrightarrow G_k$). Recall the notations in 2.3, and ($\forall U \subseteq X$ open) the sequence of characteristic open subgroups

$$... \subseteq \Delta_X[j+1] \overset{\mathrm{def}}{=} \Delta_{U,0}[j+1] \subseteq \Delta_X[j] \overset{\mathrm{def}}{=} \Delta_{U,0}[j] \subseteq ... \subseteq \Delta_X[1] \overset{\mathrm{def}}{=} \Delta_{U,0}[1] = \Delta_X$$

of $\Delta_{U,0} = \Delta_X$ with $\bigcap_{j\geq 1} \Delta_X[j] = \{1\}$. The sequence of (geometrically characteristic) open subgroups

$$... \subseteq \Pi_X[j+1] \overset{\mathrm{def}}{=} \Pi_{U,0}[j+1] \subseteq \Pi_X[j] \overset{\mathrm{def}}{=} \Pi_{U,0}[j] \subseteq ... \subseteq \Pi_X[1] \overset{\mathrm{def}}{=} \Pi_{U,0}[1] = \Pi_X,$$

where $\Pi_X[j] \overset{\mathrm{def}}{=} \Delta_X[j].s(G_k)$, corresponds to a tower of finite (not necessarily Galois) étale covers

$$... \to X_{j+1} \overset{\mathrm{def}}{=} X^U_{0,j+1} \to X_j \overset{\mathrm{def}}{=} X^U_{0,j} \to ... \to X \overset{\mathrm{def}}{=} X^U_{0,1}.$$

Note that $\Pi_X[j]$ identifies naturally with $\Pi_{X_j} \overset{\mathrm{def}}{=} \pi_1(X_j, \eta_j)^{(\Sigma)}$, where η_j is the base point induced by η. Moreover, the section s restricts to a section $s : G_k \to \Pi_{X_j}$ of the projection $\Pi_{X_j} \twoheadrightarrow G_k$, $\forall j \geq 1$.

Let $i \geq 0$, $j \geq 1$, be integers. Recall the Kummer sequence

$$1 \to \mu_{p^{i+1}} \to \mathbb{G}_m \xrightarrow{p^{i+1}} \mathbb{G}_m \to 1$$

in étale topology, which induces an exact sequence

$$0 \to \mathrm{Pic}(X_j)/p^{i+1}\,\mathrm{Pic}(X_j) \to H^2(X_j, \mu_{p^{i+1}}) \to {}_{p^{i+1}}\,\mathrm{Br}(X_j) \to 0.$$

Here $\mathrm{Pic} \overset{\mathrm{def}}{=} H^1_{\mathrm{et}}(\ ,\mathbb{G}_m)$ is the Picard group, $\mathrm{Br} \overset{\mathrm{def}}{=} H^2_{\mathrm{et}}(\ ,\mathbb{G}_m)$ the Brauer-Grothendieck cohomological group, and ${}_{p^{i+1}}\,\mathrm{Br} \subseteq \mathrm{Br}$ the subgroup of Br which is annihilated by p^{i+1}. We identify $\mathrm{Pic}(X_j)/p^{i+1}\,\mathrm{Pic}(X_j)$ with its image in $H^2(X_j, \mu_{p^{i+1}})$ and refer to it as the Picard part of $H^2(X_j, \mu_{p^{i+1}})$. By pulling back cohomology classes via the section $s : G_k \to \Pi_{X_j}$, and bearing in mind the natural identification $H^2(\Pi_{X_j}, \mu_{p^{i+1}}) \overset{\sim}{\to} H^2(X_j, \mu_{p^{i+1}})$ (cf. [Mochizuki], Proposition 1.1), we obtain a restriction homomorphism $s_j^\star : H^2(X_j, \mu_{p^{i+1}}) \to H^2(G_k, \mu_{p^{i+1}})$.

Observe that if k'/k is a finite extension, and $X_{k'} \overset{\mathrm{def}}{=} X \times_k k'$, then we have a cartesian diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{X_{k'}} & \longrightarrow & \Pi_{X_{k'}} & \longrightarrow & G_{k'} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

and the section s induces a section $s_{k'} : G_{k'} \to \Pi_{X_{k'}}$ of the projection $\Pi_{X_{k'}} \twoheadrightarrow G_{k'}$.

Definition 3.4.1 (Sections with Cycle Classes Orthogonal to Pic **mod-p^{i+1}).** (Compare with [Saïdi], 1.4.)
(i) We say that the section s has a **cycle class orthogonal to** Pic **mod-p^{i+1}** if the homomorphism $s_j^\star : H^2(X_j, \mu_{p^{i+1}}) \to H^2(G_k, \mu_{p^{i+1}})$ annihilates the Picard part $\mathrm{Pic}(X_j)/p^{i+1}\,\mathrm{Pic}(X_j)$ of $H^2(X_j, \mu_{p^{i+1}})$, $\forall j \geq 1$.
(ii) We say that the section s has a **cycle class uniformly orthogonal to** Pic **mod-p^{i+1}** (relative to the system of neighbourhoods $\{X_j\}_{j\geq 1}$ of s) if, for every finite extension k'/k, the induced section $s_{k'} : G_{k'} \to \Pi_{X_{k'}}$ has a cycle class orthogonal to Pic mod-p^{i+1} (relative to the system of neighbourhoods of $s_{k'}$ which is induced by the $\{X_j\}_{j\geq 1}$).

Definition 3.4.2. We say that the field k satisfies the condition $(\mathbf{H_{p^{i+1}}})$ if the following holds. The Galois cohomology groups $H^1(G_k, M)$ are *finite* for every finite G_k-module M annihilated by p^{i+1}.

Theorem 3.4.3 (Lifting of Sections to Cuspidally mod-p^{i+1} abelian Arithmetic Fundamental Groups). *Assume that k satisfies the condition $(\mathbf{H_{p^{i+1}}})$ (cf. Definition 3.4.2). Let $s : G_k \to \Pi_X$ be a section of the projection $\Pi_X \twoheadrightarrow G_k$. Assume that s has a* **cycle class uniformly orthogonal to** Pic **mod-**p^{i+1} *(cf. Definition 3.4.1(ii)). Let $U \subseteq X$ be a nonempty open subscheme. Then there exists a section $s'_{U,i+1} : G_k \to \Pi_U^{(1/p^{i+1}-\mathrm{sol})}$ of the projection $\Pi_U^{(1/p^{i+1}-\mathrm{sol})} \twoheadrightarrow G_k$ which* **lifts** *the section s, i.e., which inserts into the following commutative diagram.*

$$\begin{array}{ccc} G_k & \xrightarrow{s'_{U,i+1}} & \Pi_U^{(1/p^{i+1}-\mathrm{sol})} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ G_k & \xrightarrow{s} & \Pi_X \end{array}$$

Proof. Similar to the proof of Theorem 2.3.3 in [Saïdi]. □

Theorem 3.4.4 (Lifting of Sections to Cuspidally mod-p^{i+1} abelian Galois Groups). *Assume that the field k satisfies the condition $(\mathbf{H_{p^{i+1}}})$ (cf. Definition 3.4.2). Let $s : G_k \to \Pi_X$ be a section of the projection $\Pi_X \twoheadrightarrow G_k$. Then s has a* **cycle class uniformly orthogonal to** Pic **mod-**p^{i+1} *(cf. Definition 3.4.1 (ii))* **if and only if** *there exists a section $s'_{i+1} : G_k \to G_X^{(1/p^{i+1}-\mathrm{sol})}$ of the projection $G_X^{(1/p^{i+1}-\mathrm{sol})} \twoheadrightarrow G_k$ which* **lifts** *the section s, i.e., which inserts in the following commutative diagram.*

$$\begin{array}{ccc} G_k & \xrightarrow{s'_{i+1}} & G_X^{(1/p^{i+1}-\mathrm{sol})} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ G_k & \xrightarrow{s} & \Pi_X \end{array}$$

Proof. Similar to the proof of Theorem 2.3.5 in [Saïdi]. □

3.4.II. Next, let $s' : G_k \to G_X^{(1/p^2-\mathrm{sol})}$ be a section of the projection $G_X^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$, which induces for every open subscheme $U \subseteq X$ a section $s'_U : G_k \to \Pi_U^{(1/p^2-\mathrm{sol})}$ of the projection $\Pi_U^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$. We investigate the problem of lifting the section s'_U (resp. s') to a section $\tilde{s}_U : G_k \to \Pi_U^{(p,2)}$ (resp. $\tilde{s} : G_k \to G_X^{(p,2)}$) of the projection $\Pi_U^{(p,2)} \twoheadrightarrow G_k$ (resp. $G_X^{(p,2)} \twoheadrightarrow G_k$) (cf. diagrams (3.4) and (3.6)). Let $U \subseteq X$ be an open subscheme. Consider the following pull-back diagram.

$$(3.9)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \varprojlim_{j\geq 1}((\Delta'_{2,j})^U)_{1/p} & \longrightarrow & \mathcal{H}_U^{(p,2)} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & {\scriptstyle s'_U}\downarrow & & \\ 1 & \longrightarrow & \varprojlim_{j\geq 1}((\Delta'_{2,j})^U)_{1/p} & \longrightarrow & \Pi_U^{(p,2)} & \longrightarrow & \Pi_U^{(1/p^2-\mathrm{sol})} & \longrightarrow & 1 \end{array}$$

Lemma 3.4.5. *The section s'_U lifts to a section $\tilde{s}_U : G_k \to \Pi_U^{(p,2)}$ of the projection $\Pi_U^{(p,2)} \twoheadrightarrow G_k$ if and only if the group extension $\mathcal{H}_U^{(p,2)}$ splits.*

Proof. Follows immediately from diagram (3.9). $\square$

Recall the discussion and notations after Lemma 1.2.2, especially the definition of the $\{\Delta_{U,1/p^{i+1}}[j]\}_{j\geq 1}$. For $j \geq 1$, write $\Pi_{U,1/p^2}[j] \overset{\text{def}}{=} \Delta_{U,1/p^2}[j].s'_U(G_k)$. Thus, $\Pi_{U,1/p^2}[j] \subseteq \Pi_U^{(1/p^2-\text{sol})}$ is an open subgroup corresponding to a (possibly tamely ramified) cover $\widetilde{X}_j^U \to X$ between smooth, proper, and geometrically connected k-curves. The geometric point η determines a geometric point η_j of $\widetilde{X}_j^U$. Write $\Pi_j^U = \Pi_j^U[s'_U] \overset{\text{def}}{=} \Pi_{\widetilde{X}_j^U} \overset{\text{def}}{=} \pi_1(\widetilde{X}_j^U, \eta_j)^{(\Sigma)}$, which inserts in the exact sequence $1 \to \Delta_j^U \to \Pi_j^U \to G_k \to 1$, where $\Delta_j^U \overset{\text{def}}{=} \Delta_{\widetilde{X}_j^U \times_k \overline{k}}$. Further, consider the push-out diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_j^U & \longrightarrow & \Pi_j^U & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & (\Delta_j^U)_{1/p} & \longrightarrow & (\Pi_j^U)^{(1/p-\text{sol})} & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

which defines the geometrically $1/p$-th step solvable quotient $\Pi_j^U \twoheadrightarrow (\Pi_j^U)^{(1/p-\text{sol})}$ of Π_j^U.

Lemma 3.4.6. *There are natural isomorphisms $\varprojlim_{j\geq 1}((\Delta'_{2,j})^U)_{1/p} \xrightarrow{\sim} \varprojlim_{j\geq 1}(\Delta_j^U)_{1/p}$, and $\mathcal{H}_U^{(p,2)} \xrightarrow{\sim} \varprojlim_{j\geq 1}(\Pi_j^U)^{(1/p-\text{sol})}$.*

Proof. Follows from the various definitions. $\square$

Note that $(\Delta'_{2,j})^U = \Delta_j^U$, we will in the sequel write Δ_j^U instead of $(\Delta'_{2,j})^U$.

Lemma 3.4.7. *Assume that k satisfies the condition ($\mathbf{H_p}$) (cf. Definition 3.4.2). Then the group extension $\mathcal{H}_U^{(p,2)}$ splits* **if and only if** *the group extension $(\Pi_j^U)^{(1/p-\text{sol})}$ splits, $\forall j \geq 1$.*

Proof. Similar to the proof of Lemma 2.3.4, using the fact that $H^1(G_k, (\Delta_j^U)_{1/p})$ is finite if k satisfies ($\mathbf{H_p}$). $\square$

For $j \geq 1$, let $J_j[U] \overset{\text{def}}{=} \text{Pic}_k^0(\widetilde{X}_j^U)$ be the jacobian of $\widetilde{X}_j^U$, and $J_j^1[U] \overset{\text{def}}{=} \text{Pic}_k^1(\widetilde{X}_j^U)$.

Lemma 3.4.8. *The group extension $(\Pi_j^U)^{(1/p-\text{sol})}$ splits* **if and only if** *the class of $J_j^1[U]$ in $H^1(G_k, J_j[U])$ is divisible by p.*

Proof. This fact is well-known, see [Harari-Szamuely] for instance. Strictly speaking loc. cit. treats the splittings of the group extension $(\Pi_j^U)^{(\text{ab})}$= the geometrically abelian quotient of Π_j^U, but a similar argument leads to a mod-p variant as above for any prime $p \in \Sigma$. $\square$

Theorem 3.4.9. *With the above notations, assume that k satisfies the condition* ($\mathbf{H_p}$) *(cf. Definition 3.4.2). Then the section s'_U lifts to a section $\tilde{s}_U : G_k \to \Pi_U^{(p,2)}$ of the projection $\Pi_U^{(p,2)} \twoheadrightarrow G_k$* **if and only if** *the class of $J_j^1[U]$ in $H^1(G_k, J_j[U])$ is divisible by p, $\forall j \geq 1$.*

Proof. Follows from Lemmas 3.4.5, 3.4.7, and 3.4.8. □

Theorem 3.4.10. *With the above notations, assume that k satisfies the condition* ($\mathbf{H_p}$) *(cf. Definition 3.4.2). Then the section s' lifts to a section $\tilde{s} : G_k \to G_X^{(p,2)}$ of the projection $G_X^{(p,2)} \twoheadrightarrow G_k$* **if and only if** *the class of $J_j^1[U]$ in $H^1(G_k, J_j[U])$ is divisible by p, $\forall j \geq 1$, and $\forall U \subseteq X$ nonempty open subscheme as in the above discussion.*

Proof. Similar to the proof of Theorem 3.4.9. □

The following is our main result in this section.

Theorem 3.4.11. *With the above notations, assume that the field k satisfies the condition* ($\mathbf{H_{p^2}}$) *(cf. Definition 3.4.2). Let $s : G_k \to \Pi_X$ be a section of the projection $\Pi_X \twoheadrightarrow G_k$. Then s lifts to a section $\tilde{s} : G_k \to G_X^{(p,2)}$ (resp. $s' : G_k \to G_X^{(2/p)}$) of the projection $G_X^{(p,2)} \twoheadrightarrow G_k$ (resp. $G_X^{(2/p)} \twoheadrightarrow G_k$)* **if and only if** *(resp.* **if***) the following two conditions occur.*

(i) The section s has a **cycle class uniformly orthogonal to** Pic **mod-**p^2 *(cf. Definition 3.4.1 (ii))*

(ii) There exists a section $s' : G_k \to G_X^{(1/p^2-\mathrm{sol})}$ of the projection $G_X^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ which **lifts** *the section s (this holds if (i) holds by Theorem 3.4.4) such that the class of $J_j^1[U]$ in $H^1(G_k, J_j[U])$ is* **divisible** *by p, $\forall j \geq 1$, and $\forall U \subseteq X$ nonempty open subscheme.*

Proof. Follows from Theorems 3.4.4 and 3.4.10. The resp. assertion follows from diagram (3.8) (cf. Lemma 3.3.1). □

§4. Geometric sections of arithmetic fundamental groups of p-adic curves.

In this section, applying the results in §3, we provide a characterisation of sections of (geometrically pro-Σ, $p \in \Sigma$) arithmetic fundamental groups of p-adic curves which arise from rational points. We use the notations in §2 and §3.

Let p be a prime integer. In this section k is a p-adic local field, i.e., $k/\mathbb{Q}_p$ is a finite extension, and we assume $p \in \Sigma$. Let $s : G_k \to \Pi_X$ be a section of the projection $\Pi_X \twoheadrightarrow G_k$.

Definition 4.1. We say that the section s is *geometric* if the image $s(G_k)$ of s is contained (hence equal to) in the decomposition group $D_x \subset \Pi_X$ associated to a rational point $x \in X(k)$.

Recall the tower of finite étale covers $... \to X_{t+1} \to X_t \to ... \to X_1 = X$ in 3.4.I, and the section $s : G_k \to \Pi_{X_t}$ of the projection $\Pi_{X_t} \twoheadrightarrow G_k$ induced by s, $\forall t \geq 1$. Assume that the section $s : G_k \to \Pi_X$ has a cycle class uniformly orthogonal to Pic mod-p^2 (cf. Definition 3.4.1). In particular, the induced section $s : G_k \to \Pi_{X_t}$ also has a cycle class uniformly orthogonal to Pic mod-p^2 (cf. loc. cit.). There exists, $\forall t \geq 1$, a section

$$s'_t : G_k \to G_{X_t}^{(1/p^2-\mathrm{sol})}$$

of the projection $G_{X_t}^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ which lifts the section s (cf. Theorem 3.4.4). (Note that k satisfies the condition $(\mathbf{H_{p^2}})$.) Given integers $t_1 \geq t_2 \geq 1$, we have a commutative diagram

$$\begin{array}{ccc} G_{X_{t_1}}^{(1/p^2-\mathrm{sol})} & \longrightarrow & G_k \\ \downarrow & & \downarrow \mathrm{id} \\ G_{X_{t_2}}^{(1/p^2-\mathrm{sol})} & \longrightarrow & G_k \end{array}$$

where the left vertical map is induced by the scheme morphism $X_{t_1} \to X_{t_2}$. We say that the above sections $\{s'_t\}_{t\geq 1}$ are **compatible** if $\forall t_1 \geq t_2 \geq 1$ we have a commutative diagram.

$$\begin{array}{ccc} G_k & \xrightarrow{s'_{t_1}} & G_{X_{t_1}}^{(1/p^2-\mathrm{sol})} \\ \mathrm{id}\downarrow & & \downarrow \\ G_k & \xrightarrow{s'_{t_2}} & G_{X_{t_2}}^{(1/p^2-\mathrm{sol})} \end{array}$$

Lemma 4.2. *With the above notations, let $s' = s'_1 : G_k \to G_X^{(1/p^2-\mathrm{sol})}$ be a section of the projection $G_X^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ which lifts the section s. Then s' induces naturally compatible sections $s'_t : G_k \to G_{X_t}^{(1/p^2-\mathrm{sol})}$ of the projections $G_{X_t}^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ which lift the section s, $\forall t \geq 1$.*

Proof. Follows from the fact that we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{I}_{X_t} & \longrightarrow & G_{X_t} & \longrightarrow & \Pi_{X_t} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{I}_X & \longrightarrow & G_X & \longrightarrow & \Pi_X & \longrightarrow & 1 \end{array}$$

where the right square is cartesian, and $\mathcal{I}_{X_t} = \mathcal{I}_X$. In particular, $\mathcal{I}_{X_t,1/p^2} = \mathcal{I}_{X,1/p^2}$ and $G_{X_t}^{(1/p^2-\mathrm{sol})}$ is the pull back of the group extension $1 \to \mathcal{I}_{X,1/p^2} \to G_X^{(1/p^2-\mathrm{sol})} \to \Pi_X \to 1$ via the natural inclusion $\Pi_{X_t} \hookrightarrow \Pi_X$, $\forall t \geq 1$. $\square$

Next, recall the exact sequence (cf. diagram (3.6), the case $i = 1$)

$$1 \to \mathcal{I}_X[p,2] \overset{\mathrm{def}}{=} \varprojlim_U (\varprojlim_{j\geq 1} ((\Delta'_{2,j})^U)_{1/p}) \to G_X^{(p,2)} \to G_X^{(1/p^2-\mathrm{sol})} \to 1.$$

We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{I}_{X_t}[p,2] & \longrightarrow & G_{X_t}^{(p,2)} & \longrightarrow & G_{X_t}^{(1/p^2-\mathrm{sol})} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{I}_X[p,2] & \longrightarrow & G_X^{(p,2)} & \longrightarrow & G_X^{(1/p^2-\mathrm{sol})} & \longrightarrow & 1 \end{array}$$

where the right square is cartesian (as one easily verifies). In particular, $\mathcal{I}_{X_t}[p,2] = \mathcal{I}_X[p,2]$. Let $s' : G_k \to G_X^{(1/p^2-\mathrm{sol})}$ be a section of the projection $G_X^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ which lifts the section s, and $\{s'_t : G_k \to G_{X_t}^{(1/p^2-\mathrm{sol})}\}_{t\geq 1}$ the induced compatible sections as in Lemma 4.2 which lift the section s. Let $\tilde{s} : G_k \to G_X^{(p,2)}$ be a section of the projection $G_X^{(p,2)} \twoheadrightarrow G_k$ which lifts the section s'. Then $\tilde{s}$ induces sections $\tilde{s}_t : G_k \to G_{X_t}^{(p,2)}$ of the projections $G_{X_t}^{(p,2)} \twoheadrightarrow G_k$ which lift the section s'_t, $\forall t \geq 1$, and which are compatible in the sense that $\forall t_1 \geq t_2 \geq 1$ integers we have a commutative diagram (cf. above diagram whose right square is cartesian).

$$\begin{array}{ccc} G_k & \xrightarrow{s'_{t_1}} & G_{X_{t_1}}^{(p,2)} \\ \mathrm{id}\downarrow & & \downarrow \\ G_k & \xrightarrow{s'_{t_2}} & G_{X_{t_2}}^{(p,2)} \end{array}$$

Also, recall the notations and definitions in 3.4.II, the case $i = 1$, relative to the sections $s'_U : G_k \to \Pi_U^{(1/p^2-\mathrm{sol})}$ induced by s', $\forall$ nonempty open subscheme $U \subseteq X$. Thus, the $\{\widetilde{X}_j^U\}_{j\geq 1}$ are defined in this case $\forall U \subseteq X$ open (cf. loc. cit.); they form a system of neighbourhoods of the section s'_U. Suppose that the group extension $1 \to \pi_1(\widetilde{X}_j^U \times_k \overline{k}, \overline{\eta}_j)^{1/p} \to \pi_1(\widetilde{X}_j^U, \eta_j)^{(1/p)} \to G_k \to 1$ splits, or equivalently that the class of $J_j^1[U]$ in $H^1(G_k, J_j[U])$ is divisible by p, $\forall j \geq 1$, and $\forall U \subseteq X$ as above (cf. Lemma 3.4.8). Then the section $s' : G_k \to G_X^{(1/p^2-\mathrm{sol})}$ lifts to a section $\tilde{s} : G_k \to G_X^{(p,2)}$ of the projection $G_X^{(p,2)} \twoheadrightarrow G_k$ (cf. Theorem 3.4.10). Moreover, $\tilde{s}$ induces compatible sections $\tilde{s}_t : G_k \to G_{X_t}^{(p,2)}$ of the projections $G_{X_t}^{(p,2)} \twoheadrightarrow G_k$ which lift the section s'_t, $\forall t \geq 1$ (cf. above discussion). In particular, the above sections $\tilde{s}_t$ induce naturally sections

$$\rho_t : G_k \to G_{X_t}^{(2/p-\mathrm{sol})}$$

of the projections $G_{X_t}^{(2/p-\mathrm{sol})} \twoheadrightarrow G_k$ which lift the sections s, $\forall t \geq 1$ (cf. Lemma 3.3.1, as well as the diagrams (3.7) and (3.8)).

Definition 4.3. With the above notations, we say that the section s is **admissible** if the following two conditions hold.
A1) The section s has a **cycle class uniformly orthogonal to** Pic **mod-**p^2.
A2) There **exists** a section $s' : G_k \to G_X^{(1/p^2-\mathrm{sol})}$ of the projection $G_X^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ which **lifts** the section s (this holds if condition A1 is satisfied by Theorem 3.4.4) such that the following holds. The class of $J_j^1[U]$ in $H^1(G_k, J_j[U])$ is **divisible** by p, $\forall U \subseteq X$ nonempty open subscheme, and $\forall j \geq 1$. Or, equivalently, the section s' **lifts** to a section $\tilde{s} : G_k \to G_X^{(p,2)}$ of the projection $G_X^{(p,2)} \twoheadrightarrow G_k$ (cf. Theorem 3.4.10).

Lemma 4.4. *Let k'/k be a finite extension, $X_{k'} \overset{\mathrm{def}}{=} X \times_k k'$, and $s_{k'} : G_{k'} \to \Pi_{X_{k'}}$ the section of the projection $\Pi_{X_{k'}} \twoheadrightarrow G_{k'}$ which is induced by s. Assume s is admissible then $s_{k'}$ is admissible.*

Proof. First, if s has a cycle class uniformly orthogonal to Pic mod-p^2 then so does $s_{k'}$ (cf. Definition 3.4.1(ii)). The second assertion follows from the various definitions. □

The following is our main result in this section; it provides a characterisation of sections of (geometrically pro-Σ, $p \in \Sigma$) arithmetic fundamental groups of p-adic curves which are geometric.

Theorem 4.5. *We use the above notations. The section s is* **admissible** *(cf. Definition 4.3)* **if and only if** *s is* **geometric** *(cf. Definition 4.1).*

Proof. The if part follows easily from the various Definitions. We prove the only if part. Assume that s is admissible, and k contains a primitive p-th root ζ_p of 1. The section $s : G_k \to \Pi_{X_t}$ lifts to a section $\rho_t : G_k \to G_{X_t}^{(2/p-\mathrm{sol})}$ of the projection $G_{X_t}^{(2/p-\mathrm{sol})} \twoheadrightarrow G_k$ (cf. discussion before Definition 4.3), $\forall t \geq 1$. The section ρ_t induces a section $\tilde{\rho}_t : (G_k)_{2/p} \to (G_{X_t})_{2/p}$ of the projection $(G_{X_t})_{2/p} \twoheadrightarrow (G_k)_{2/p}$, where the $(\ \ )_{2/p}$ of the various profinite groups are the second quotients of the $\mathbb{Z}/p\mathbb{Z}$-derived series (cf. 1.2). The section $\tilde{\rho}_t$ is geometric and arises from a rational point $x_t \in X_t(k)$ by a result of Pop (cf. [Pop]). (Here one uses the fact that $\zeta_p \in k$.) In particular, $X_t(k) \neq \varnothing$, $\forall t \geq 1$. A well-known limit argument shows that s is geometric (cf. [Tamagawa], Proposition 2.1, (iv), see also the details of the proof of Theorem A in [Saïdi1]). In case $\zeta_p \notin k$, let $k' \overset{\mathrm{def}}{=} k(\zeta_p)$. The section $s_{k'}$ is admissible (cf. Lemma 4.4), hence is geometric by the above discussion. One then verifies easily that s is geometric (cf. [Saïdi2], proof of Theorem B). $\square$

In the course of proving Theorem 4.5 we proved the following (cf. discussion before Definition 4.3, and the proof of Theorem 4.5).

Proposition 4.6. *Let $\tilde{s} : G_k \to G_X^{(p,2)}$ be a section of the projection $G_X^{(p,2)} \twoheadrightarrow G_k$, and $s : G_k \to \Pi_X$ the section of the projection $\Pi_X \twoheadrightarrow G_k$ which is induced by $\tilde{s}$. Then s is geometric.*

Remarks 4.7. 1) Theorem 4.5 above is stronger and more precise than Theorem A in [Saïdi1].
2) There are examples of sections $s : G_k \to \Pi_X$ as above, where $\Sigma = \{p\}$, which are *not* geometric (cf. [Hoshi]). These provide examples of sections s as above which are *not* admissible by Theorem 4.5 (where $\Sigma = \{p\}$). It would be interesting to know which of the conditions A1 and A2 in the definition of admissible sections fail to hold in Hoshi's example. In [Saïdi3] we observe that the section in Hoshi's example is orthogonal to Pic^0 in the sense that the map $s^\star : H^2(\Pi_X, \mathbb{Z}_p) \to H^2(G_k, \mathbb{Z}_p)$ annihilates the image of $\mathrm{Pic}^0(X)$.

The following is an application of our results to the absolute anabelian geometry of p-adic curves.

Theorem 4.8. *Let $p_X, p_Y \in \mathfrak{Primes}$, and X (resp. Y) a proper smooth and geometrically connected hyperbolic curve over a p_X-adic local field k_X (respectively, p_Y-adic local field k_Y). Let $p_X \in \Sigma_X$ (resp. $p_Y \in \Sigma_Y$) be a non-empty set of prime integers of cardinality ≥ 2, Π_X (resp. Π_Y) the geometrically pro-Σ_X (resp. pro-Σ_Y) arithmetic fundamental group of X (resp. Y), and $\varphi : \Pi_X \to \Pi_Y$ an isomorphism of profinite groups which fits in the following commutative diagram*

$$\begin{array}{ccc} G_X^{(p_X,2)} & \xrightarrow{\widetilde{\varphi}} & G_Y^{(p_Y,2)} \\ \downarrow & & \downarrow \\ \Pi_X & \xrightarrow{\varphi} & \Pi_Y \end{array}$$

where $\widetilde{\varphi}$ is an isomorphism of profinite groups, and the vertical maps are the natural projections. Then φ is geometric, i.e., arises from a uniquely determined isomorphism of schemes $X \tilde{\to} Y$.

Proof. The existence of the lifting $\widetilde{\varphi}$ of φ implies, by Proposition 4.6, that φ preserves the decomposition groups at closed points. The statement follows then from [Mochizuki1], Corollary 2.9. $\square$

§5. Local sections of arithmetic fundamental groups of p-adic curves. We prove that a certain class of sections of arithmetic fundamental groups of p-adic curves are (uniformly) orthogonal to $\mathrm{Pic}^\wedge$. We use the notations in §4.

5.1. Arithmetic fundamental groups of formal fibres of p-adic curves. Let $\mathcal{O}_k$ be the valuation ring of k, and $\widetilde{X} \to \operatorname{Spec} \mathcal{O}_k$ a flat and proper model of X over $\mathcal{O}_k$ with $\widetilde{X}$ normal. Let $x \in \widetilde{X}^{\mathrm{cl}}$ be a closed point, and $\hat{\mathcal{O}}_{\widetilde{X},x}$ the completion of the local ring of $\widetilde{X}$ at x. We will refer to $\mathcal{X} \overset{\text{def}}{=} \operatorname{Spec}(\hat{\mathcal{O}}_{\widetilde{X},x} \otimes_{\mathcal{O}_k} k)$ as the *formal fibre* of $\widetilde{X}$ at x (or simply a formal fibre). Assume $\mathcal{X}$ is geometrically connected, and write $\overline{\mathcal{X}} \overset{\text{def}}{=} \mathcal{X} \times_k \overline{k}$. Let $\bar{\beta}$ be a geometric point of $\overline{\mathcal{X}}$, which determines a geometric point β of $\mathcal{X}$. Write $\Delta_{\mathcal{X}} \overset{\text{def}}{=} \pi_1(\overline{\mathcal{X}}, \overline{\beta})^{\Sigma}$ for the maximal pro-Σ quotient of $\pi_1(\overline{\mathcal{X}}, \overline{\beta})$, and $\Pi_{\mathcal{X}} \overset{\text{def}}{=} \pi_1(\mathcal{X}, \beta) / \operatorname{Ker}(\pi_1(\overline{\mathcal{X}}, \overline{\beta}) \twoheadrightarrow \pi_1(\overline{\mathcal{X}}, \overline{\beta})^{\Sigma})$. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{\mathcal{X}} & \longrightarrow & \Pi_{\mathcal{X}} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \\ 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & G_k & \longrightarrow & 1 \end{array} \tag{5.1}$$

where the middle vertical map (defined up to inner conjugation) is induced by the scheme morphism $\mathcal{X} \to X$.

For the rest of this section we assume that $\mathcal{X}$ is a formal fibre as in 5.1, which is geometrically connected.

Definition 5.2. A section $\tilde{s} : G_k \to \Pi_{\mathcal{X}}$ of the projection $\Pi_{\mathcal{X}} \twoheadrightarrow G_k$ induces a section $s : G_k \to \Pi_X$ of the projection $\Pi_X \twoheadrightarrow G_k$ (cf. diagram (5.1)). We will refer to such a section s as a **local section** of the projection $\Pi_X \twoheadrightarrow G_k$.

Note that a geometric section (cf. Definition 4.1) is a local section in the above sense, as one easily verifies. Our main result is the following.

Theorem 5.3. *Let $s : G_k \to \Pi_X$ be a* **local** *section of the projection $\Pi_X \twoheadrightarrow G_k$. Then s has* **a cycle class uniformly orthogonal to** $\mathrm{Pic}^\wedge$ *in the sense of [Saïdi], Definition 1.4.1(i).*

Proof. Let $\mathcal{X} = \operatorname{Spec}(\hat{\mathcal{O}}_{\widetilde{X},x} \otimes_{\mathcal{O}_k} k)$ be as in 5.1, and $\tilde{s} : G_k \to \Pi_{\mathcal{X}}$ a section of the projection $\Pi_{\mathcal{X}} \twoheadrightarrow G_k$ which induces the section $s : G_k \to \Pi_X$. Let $\{X_t\}_{t \geq 1}$ be as in §4, $s_t : G_k \to \Pi_{X_t}$ the section of the projection $\Pi_{X_t} \twoheadrightarrow G_k$ which is induced by s, and $s_t^\star : H^2(X_t, \hat{\mathbb{Z}}(1)^{\Sigma}) \to H^2(G_k, \hat{\mathbb{Z}}(1)^{\Sigma})$ the retraction map induced by s_t, $\forall t \geq 1$. We show $s_t^\star(\mathrm{Pic}(X_t)^\wedge) = 0$ (cf. loc. cit. for the definition of $\mathrm{Pic}(X_t)^\wedge$). Recall the continuous homomorphism $\phi : \Pi_{\mathcal{X}} \to \Pi_X$ (cf. the middle vertical map in diagram (5.1)). Then $\Pi_{\mathcal{X}_t} \overset{\text{def}}{=} \phi^{-1}(\Pi_{X_t})$ is an open subgroup containing $\tilde{s}(G_k)$, and corresponds to an étale cover $\mathcal{X}_t \to \mathcal{X}$ with $\mathcal{X}_t$ geometrically connected (as $\Pi_{\mathcal{X}_t}$ projects onto G_k via the projection $\Pi_{\mathcal{X}} \twoheadrightarrow G_k$). Moreover, the section $\tilde{s}$ induces a retraction $\tilde{s}_t^\star : H^2(\Pi_{\mathcal{X}_t}, \hat{\mathbb{Z}}(1)^{\Sigma}) \to H^2(G_k, \hat{\mathbb{Z}}(1)^{\Sigma})$ of the natural map $H^2(G_k, \hat{\mathbb{Z}}(1)^{\Sigma}) \to H^2(\Pi_{\mathcal{X}_t}, \hat{\mathbb{Z}}(1)^{\Sigma})$ induced by the projection $\Pi_{\mathcal{X}_t} \twoheadrightarrow G_k$. Let

$A \overset{\text{def}}{=} \hat{\mathcal{O}}_{\widetilde{X},x}$, and $A_k \overset{\text{def}}{=} \hat{\mathcal{O}}_{\widetilde{X},x} \otimes_{\mathcal{O}_k} k$. The open subgroup $\Pi_{\mathcal{X}_t}$ corresponds to an étale cover $\mathcal{X}_t = \operatorname{Spec} B_k \to \mathcal{X} = \operatorname{Spec} A_k$, where B_k/A_k is an étale extension. Let B be the integral closure of A in B_k. Thus, B is a complete local ring of dimension 2, which dominates $\mathcal{O}_k$, and the residue field of B is finite. We have a scheme theoretic morphism $\mathcal{X}_t \to X_t$. Further, we have an injective homomorphism $H^2(\Pi_{\mathcal{X}_t}, \hat{\mathbb{Z}}(1)^{\Sigma}) \hookrightarrow H^2_{\text{et}}(\mathcal{X}_t, \hat{\mathbb{Z}}(1)^{\Sigma})$ arising from the Cartan-Leray spectral sequence (cf. [Serre], proof of Proposition 1), as well as an injective Kummer homomorphism $\operatorname{Pic}(\mathcal{X}_t)^{\wedge} \hookrightarrow H^2_{\text{et}}(\mathcal{X}_t, \hat{\mathbb{Z}}(1)^{\Sigma})$, where $\operatorname{Pic}^{\wedge} \overset{\text{def}}{=} \operatorname{Pic} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Sigma}$. On the other hand we have a commutative diagram of homomorphisms

$$\begin{array}{ccc}
H^2(\Pi_{\mathcal{X}_t}, \hat{\mathbb{Z}}(1)^{\Sigma}) & \xrightarrow{\tilde{s}_t^{\star}} & H^2(G_k, \hat{\mathbb{Z}}(1)^{\Sigma}) \\
{\scriptstyle \psi_t}\Big\uparrow & & {\scriptstyle \text{id}}\Big\uparrow \\
H^2(\Pi_{X_t}, \hat{\mathbb{Z}}(1)^{\Sigma}) & \xrightarrow{s_t^{\star}} & H^2(G_k, \hat{\mathbb{Z}}(1)^{\Sigma})
\end{array}$$

where ψ_t is induced by the map $\phi : \Pi_{\mathcal{X}_t} \to \Pi_{X_t}$.

We claim that $\psi_t(\operatorname{Pic}(X_t)^{\wedge})$ is *torsion*, from this it follows that $s_t^{\star}(\operatorname{Pic}(X_t)^{\wedge}) = \tilde{s}_t^{\star}(\psi_t(\operatorname{Pic}(X_t)^{\wedge})) = 0$, since $H^2(G_k, \hat{\mathbb{Z}}(1)^{\Sigma}) \xrightarrow{\sim} \hat{\mathbb{Z}}^{\Sigma}$ is torsion free. Indeed, we have a pull-back morphism $\operatorname{Pic}(X_t)^{\wedge} \to \operatorname{Pic}(\mathcal{X}_t)^{\wedge}$, which fits in the commutative diagram

$$\begin{array}{ccc}
\operatorname{Pic}(\mathcal{X}_t)^{\wedge} & \longrightarrow & H^2_{\text{et}}(\mathcal{X}_t, \hat{\mathbb{Z}}(1)^{\Sigma}) \\
{\scriptstyle \text{id}}\Big\uparrow & & \Big\uparrow \\
\operatorname{Pic}(\mathcal{X}_t)^{\wedge} & & H^2(\Pi_{\mathcal{X}_t}, \hat{\mathbb{Z}}(1)^{\Sigma}) \\
\Big\uparrow & & {\scriptstyle \psi_t}\Big\uparrow \\
\operatorname{Pic}(X_t)^{\wedge} & \longrightarrow & H^2(\Pi_{X_t}, \hat{\mathbb{Z}}(1)^{\Sigma})
\end{array}$$

where the horizontal maps are injective Kummer maps (recall the identification $H^2(\Pi_{X_t}, \hat{\mathbb{Z}}(1)^{\Sigma}) \xrightarrow{\sim} H^2_{\text{et}}(X_t, \hat{\mathbb{Z}}(1)^{\Sigma})$), and the upper right vertical map is the injective map discussed above. Our claim follows then from the following.

Proposition 5.4. *With the notations above* $\operatorname{Pic}(\mathcal{X}_t)$*, and a fortiori* $\operatorname{Pic}(\mathcal{X}_t)^{\wedge}$*, is* **finite**.

Proof of Proposition 5.4. This follows from the fact, proven by Shuji Saito, that $\operatorname{Pic}(\operatorname{Spec} B \setminus \{m_B\})$ is finite, where B is as in the above discussion and m_B is its maximal ideal (cf. [Saito], Theorem 0.11). □

This finishes the proof of Theorem 5.3. □

Finally, we provide the following characterisation of local sections which are geometric. We use the above notations.

Theorem 5.5. *Let* $s : G_k \to \Pi_X$ *be a* **local** *section of the projection* $\Pi_X \twoheadrightarrow G_k$ *(cf. Definition 5.2). Then* s **lifts** *to a section* $\rho : G_k \to G_X^{(1-\text{sol})}$ *(resp.* $\rho_n : G_k \to G_X^{(1/p^n-\text{sol})}$*) of the projection* $G_X^{(1-\text{sol})} \twoheadrightarrow G_k$ *(resp.* $G_X^{(1/p^n-\text{sol})} \twoheadrightarrow G_k$*,* $\forall n \geq 1$*). Moreover, the section* s *is* **geometric if and only if** *there exists a lifting of* s *to a section* $\rho_2 : G_k \to G_X^{(1/p^2-\text{sol})}$ *as above such that one of the following equivalent conditions hold.*

(i) With the notations in §4 (cf. the discussion before Definition 4.3), the class of $J_j^1[U]$ *in* $H^1(G_k, J_j[U])$ *is* **divisible** *by p,* $\forall U \subseteq X$ *nonempty open subscheme, and* $\forall j \geq 1$.

(ii) The section ρ_2 **lifts** *to a section* $\tilde{\rho}_2 : G_k \to G_X^{(p,2)}$ *of the projection* $G_X^{(p,2)} \twoheadrightarrow G_k$.

Proof. The first assertion follows from Theorem 5.3, Theorem 3.4.4, and Theorem 2.3.5 in [Saïdi]. The second assertion follows from Theorem 3.4.10, and Theorem 4.5. □

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