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The bullwhip effect under count time series: The case of first order integer auto-regressive demand processes

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Abstract

The impact of fast moving items, modeled by auto-regressive moving average (ARMA) type processes, on the bullwhip effect is well known. However, slow moving items that can be modeled using integer ARMA processes have received little attention. Herein, we measure the impact of bullwhip effect under a first order integer auto-regressive, INAR(1), demand process. We consider a simple two-stage supply chain consisting of a retailer and a manufacturer. We assume that the retailer employs a base stock inventory policy when the demand is forecasted using a minimum mean squared error method. We investigate the impact of the INAR(1) demand process parameter, α, and the replenishment lead time, L, on the bullwhip effect generated by the order-up-to replenishment policy. We show that the bullwhip effect is increasing with the lead-time L.

Keywords
Minimum Mean Squared Error Forecasting; Count Time Series; Bullwhip Effect; Integer Auto-Regressive Demand Processes.

1. Introduction

The bullwhip effect refers to the tendency for supply chain replenishment decisions to amplify the variability of the demand when ordering products from production systems or suppliers (Lee, et al., 2000). Many of the existing studies on the bullwhip effect (Chen et al., 2000a and Dejonckheere et al., 2003) have assumed that real demand exists. That is to say, demand can take on any number, even fractional values. For some products sold by volume or weight (powders, granules or liquids etc.) this may be appropriate. However, for other products, only integer (or batched) demand makes sense. For example, you can't buy half a car or half a loaf of bread; demand and replenishment orders must be integers. For some situations, with high volume demand, replenishment calculations can ignore integer effects as any rounding that occurs is negligible in the grand scheme of things. However, some low volume demand settings may be more susceptible to integer effects.

Consider, for example, the daily demand for a single product in a single grocery store in Figure 1. This product has low volume, integer, and occasionally zero demand. Continuous-valued series can be modeled using auto-regressive integrated moving average (ARIMA) type processes (Box and Jenkins, 1970). However, standard ARIMA modeling techniques are not suitable for modelling non-negative integer-valued series (Silva and Oliveira, 2004). Therefore, another type of process has been proposed for a stationary sequence of integer-valued time series. In the literature, this family of models is referred to as integer auto-regressive moving average processes, INARMA, Al-Osh and Alzaid (1988).

The bullwhip effect can be measured in different ways, however it is often convenient to quantify it via the ratio of the variance of orders to the variance of demand. Auto-regressive demand of the first order, AR(1), auto-regressive integrated moving average, ARMA(1,1) and integrated moving average (IMA) processes, are among the most frequently used demand processes that are forecasted with minimum mean squared error (MMSE) forecasting methods (Graves, 1999; Chen et al., 2000a; Chen et al. 2000b; Hosoda and Disney 2006; Chen and Disney, 2007; Duc et al., 2008). Wang and Disney (2016) provide a recent comprehensive review of the research conducted on bullwhip effect categorized into empirical, experimental, and analytical approaches.

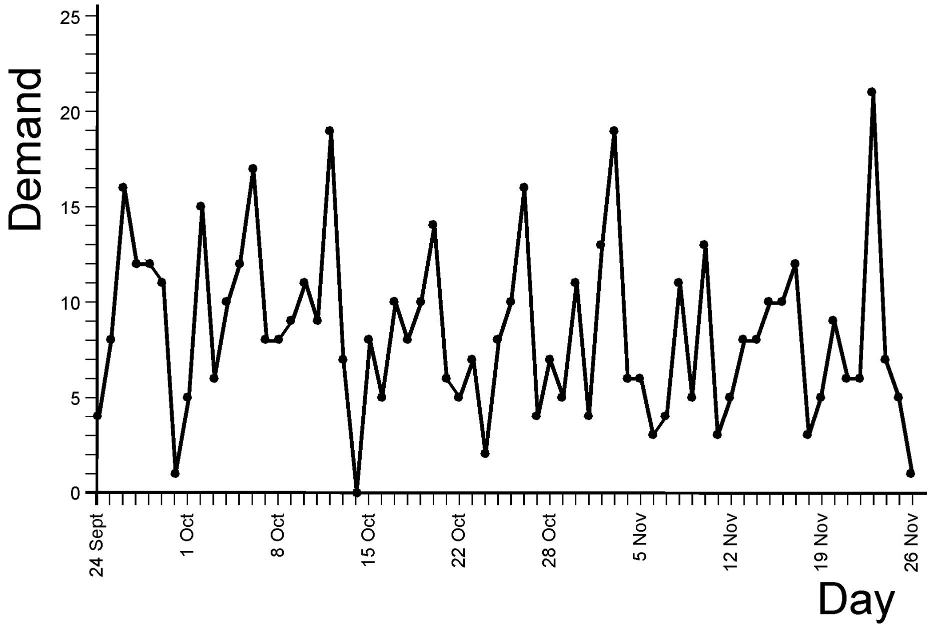

Figure 1. Example count data from a single store of UK grocery retailer

It has been shown that when AR(1) demand exists, bullwhip is generated when auto-regressive parameter is positive (Lee et al., 2000). Moreover, there is an auto-regressive parameter with maximal bullwhip for a given lead time; bullwhip increases with the lead time up to an upper bound. When the demand series follows an ARMA(1,1) process, the bullwhip effect occurs when the auto-regressive parameter is larger than the moving average parameter of the demand process. Moreover, the bullwhip effect does not always increase with lead time. The bullwhip effect increment depends on the values of the auto-regressive and moving average parameters and whether lead time is odd or even (Gaalman and Disney, 2006; Duc et al., 2008). Although the behavior of bullwhip effect for continuous-valued series modeled by ARIMA type process is rather well understood, to the best of our knowledge, there is no analytical study looking at the measure of the bullwhip effect with integer-valued demand modeled by INARMA processes. Therefore, this is the first attempt to fill this gap; we aiming to quantify the bullwhip effect under integer demand.

In this paper, we analytically develop a measure of bullwhip effect in a supply chain where a retailer receives a demand following a first order integer auto-regressive demand process, INAR(1), Silva and Oliveria (2004). A MMSE forecasting mechanism is adopted as it is the optimal forecast for the demand process under consideration (although interestingly, it hardly ever produces integer forecasts). The MMSE forecast is used inside an order-up-to level (OUT) replenishment policy, used to maintain control over a finished goods inventory from which customers are satisfied. Our contribution to the literature is threefold: 1) we develop a measure of bullwhip in supply chains with INAR(1) demand, 2) we investigate the impact of the auto-regressive process parameter and lead time on the bullwhip with integer demand, and 3) we compare and contrast our results to the measure of bullwhip with the real valued AR(1) demand process.

The remainder of our paper is structured as follows. In Section 2 we discuss our model and develop an analytical expression for the bullwhip effect. We first describe the INAR(1) demand process and the OUT replenishment policy and the procedure to determine order quantity. Then, the variance of demand, variance of forecast, and the covariance between demand and its forecast is derived. These variances are then used to determine the variance of the replenishment orders and consequently, the bullwhip effect measure is calculated. Along the way, we also derive an expression for the variance of the inventory levels maintained by the OUT policy under integer demand. In section 3, we discuss and interpret the result. The paper concludes in section 4 with a summary of our main result and a discussion of implications for further research.

2. Model development

In this section, a measure of bullwhip is derived for an OUT policy with MMSE forecasting reacting to an INAR(1) demand process will be obtained.

2.1. The INAR(1) demand process

We assume that the demand follows a first order integer autoregressive process, INAR(1), where demand in period t is given by,

$$d_t = \alpha \circ d_{t-1} + z_t, \tag{1}$$

where, $0 < \alpha < 1$, z_t is a sequence of i.i.d. non-negative integer-valued Poisson distributed random variables, with a mean of λ and a finite variance of λ (Silva et al., 2009), $\alpha \circ d_{t-1}$ is the binomial thinning operation defined by

$$\alpha \circ d_t = \sum_{i=1}^{d_t} X_i , \tag{2}$$

where X_i is a sequence of i.i.d. Bernoulli indicators with parameter α (i.e. with $P(X_i = 1) = \alpha$) for $i = 1, 2, ..., Y,$. A natural interpretation of (1) is that d_t is the number of customers at time t , z_t is the number of new customers, and $d_{t-1} - \alpha \circ d_{t-1}$ is the number of old customers lost between time $t-1$ and time t, (Janjić, et al., 2014). Silva and Oliveira (2004) provide a number of useful relations and properties of the INAR(1) model that we will exploit throughout this paper. Notably, the relations

$$\left. \begin{aligned} &\mathrm{E}(\alpha \circ d_{t-1}) = \alpha \mathrm{E}(d_{t-1}) \\ &\mathrm{E}(\alpha \circ d_{t-1})^2 = \alpha^2 \mathrm{E}(d_{t-1}{}^2) + \alpha(1-\alpha)\mathrm{E}(d_{t-1}) \end{aligned} \right\}, \tag{3}$$

are very useful.

Proposition 1: When the demand follows an INAR(1) process, the auto-covariance of demand is given by

$$\gamma_k = \begin{cases} \dfrac{\lambda}{1-\alpha}, & k = 0, \\ \alpha^k \gamma_0, & k \geq 1. \end{cases} \tag{4}$$

Proof. The proof of Proposition 1 is given in Appendix 1. ■

Remark. The auto-covariance function at lag $k = 0$ gives the demand variance.

2.2. The order-up-to replenishment policy

At the end of period t, the retailer places an order of quantity q_t to the manufacturer;

$$q_t = S_t - S_{t-1} + d_t. \tag{5}$$

If the base stock policy is employed, the order-up-to level S_t can be determined by

$$S_t = f_{t,L} + z\hat{\sigma}_{t,L}, \tag{6}$$

in which z is a safety factor determined to achieve a given target inventory availability. We note that this factor will not affect the analytical evaluation if $\hat{\sigma}_{t,L} = \hat{\sigma}_{t-1,L}$ as the expression $z\hat{\sigma}_{t,L}$ will disappear when the order quantity is calculated with (5). The bullwhip calculation requires the determination of: the variance of demand over lead time, the variance of forecast and the covariance between demand and its forecast over lead time. In the following sections, we analytically develop these expressions.

2.3. Variance of demand over lead time L

The lead-time demand can be expressed as

$$d_{t,L} = \sum_{i=1}^{L} d_{t+i} = d_{t+1} + d_{t+2} + \ldots + d_{t+L} \,. \tag{7}$$

The variance of demand over lead time is calculated from

$$\begin{aligned} Var\left(d_{t,L}\right) &= Var\left(d_{t+1} + d_{t+2} + \ldots + d_{t+L}\right) \\ &= \sum_{i=1}^{L} \begin{pmatrix} Var\left(d_{t+i}\right) + 2Cov(d_{t+1}, d_{t+2}) + \ldots + 2Cov(d_{t+1}, d_{t+L}) + \\ 2Cov(d_{t+2}, d_{t+3}) + \ldots + 2Cov(d_{t+2}, d_{t+L}) + \ldots + 2Cov(d_{t+L-1}, d_{t+L}) \end{pmatrix}. \end{aligned} \tag{8}$$

Eq. (4) shows that $Var(d_{t-k}) = \gamma_0$ for $k \geq 0$ and $Cov(d_t, d_{t-k}) = \gamma_k$ for $k \geq 1$. Substituting these relations into (8) yields

$$\begin{aligned} Var\left(d_{t,L}\right) &= L\gamma_0 + 2\left(\gamma_1 + \gamma_2 + \ldots \gamma_{L-1}\right) + 2\left(\gamma_1 + \gamma_2 + \ldots \gamma_{L-2}\right) + \ldots + 2\gamma_1 \\ &= L\gamma_0 + 2\alpha\gamma_0 \sum_{i=1}^{L-1} \frac{1-\alpha^i}{1-\alpha} = \frac{\lambda\left(\left(\alpha^2 - 1\right)L - 2\alpha\left(\alpha^L - 1\right)\right)}{(\alpha - 1)^3}. \end{aligned} \tag{9}$$

2.4. Variance of forecast over lead-time

We assume that the demand forecast is carried out in such a way that the forecast minimizes the mean square forecast error. Therefore, the forecast at period $t+i$ is defined as

$$f_{t+i} = E[d_{t+i} \mid d_t, d_{t-1}, \ldots]. \tag{10}$$

The forecast over lead time is

$$f_{t,L} = \sum_{i=1}^{L} f_{t+i} \,. \tag{11}$$

To derived exact expressions of f_{t+i} we note

$$d_{t+i} = \alpha \circ d_{t+i-1} + z_{t+i} \tag{12}$$

and

$$d_{t+i-1} = \alpha \circ d_{t+i-2} + z_{t+i-1} \,. \tag{13}$$

Then by substituting (13) into (12) and applying this procedure for d_{t+i-2}, d_{t+i-3},…, recursively yields

$$d_{t+i} = \alpha^i \circ d_t + \alpha^{i-1} \circ z_{t+1} + \ldots + \alpha^2 \circ z_{t+i-2} + \alpha \circ z_{t+i-1} + z_{t+i} \,. \tag{14}$$

By substituting (14) into (10), we have,

$$\begin{aligned} f_{t+i} &= E[\alpha^i \circ d_t + \alpha^{i-1} \circ z_{t+1} + ... + \alpha^2 \circ z_{t+i-2} + \alpha \circ z_{t+i-1} + z_{t+i} \mid d_{t-1}, d_{t-2}, ...] \\ &= \alpha^i \left(d_t - \frac{\lambda}{1-\alpha} \right) + \frac{\lambda}{1-\alpha}. \end{aligned} \tag{15}$$

Finally, the forecast over L periods is obtained by substituting (15) into (11),

$$f_{t,L} = \sum_{i=1}^{L} \left(\alpha^i \left(d_t - \frac{\lambda}{1-\alpha} \right) + \frac{\lambda}{1-\alpha} \right) = \frac{L\lambda}{1-\alpha} + \alpha \left(d_t - \frac{\lambda}{1-\alpha} \right) \left(\frac{1-\alpha^L}{1-\alpha} \right). \tag{16}$$

As $\gamma_0 = Var(d_t)$, the variance of forecast over lead time is

$$Var(f_{t,L}) = Var\left(\frac{L\lambda}{1-\alpha} + \alpha \left(d_t - \frac{\lambda}{1-\alpha} \right) \left(\frac{1-\alpha^L}{1-\alpha} \right) \right) = Var\left(\alpha \left(\frac{1-\alpha^L}{1-\alpha} \right) d_t \right) = \gamma_0 \left(\frac{\alpha(1-\alpha^L)}{1-\alpha} \right)^2. \tag{17}$$

2.5. Covariance of demand and its forecast over *L*

The covariance of demand over lead time and its forecast is calculated as,

$$Cov(d_{t,L}, f_{t,L}) = Cov(d_{t+1} + d_{t+2} + ... + d_{t+L}, f_{t,L}). \tag{18}$$

By substituting (16) into (18) and considering that $\forall k,\ Cov(d_t, d_{t-k}) = \gamma_k$ (from (4)), we have,

$$\begin{aligned} Cov(d_{t,L}, f_{t,L}) &= Cov\left(d_{t+1} + ... + d_{t+L}, \frac{L\lambda}{1-\alpha} + \alpha \left(d_t - \frac{\lambda}{1-\alpha} \right) \left(\frac{1-\alpha^L}{1-\alpha} \right) \right) \\ &= \frac{\alpha(1-\alpha^L)}{1-\alpha} (\gamma_1 + \gamma_2 + ... + \gamma_L) = \gamma_0 \left(\frac{\alpha(1-\alpha^L)}{1-\alpha} \right)^2. \end{aligned} \tag{19}$$

The variance of forecast error over lead time is calculated from

$$Var(d_{t,L} - f_{t,L}) = Var(d_{t,L}) + Var(f_{t,L}) - 2Cov(d_{t,L}, f_{t,L}). \tag{20}$$

Substituting (9), (17) and (19) into (20) yields,

$$\begin{aligned} Var(d_{t,L} - f_{t,L}) &= \frac{\lambda}{1-\alpha} \left(L + 2\alpha \sum_{i=1}^{L-1} \frac{1-\alpha^i}{1-\alpha} \right) + \gamma_0 \left(\frac{\alpha(1-\alpha^L)}{1-\alpha} \right)^2 - 2\gamma_0 \left(\frac{\alpha(1-\alpha^L)}{1-\alpha} \right)^2 \\ &= \frac{\lambda}{1-\alpha} \left(L + 2\alpha \left(\frac{\alpha^L - \alpha L + L - 1}{(\alpha-1)^2} \right) \right) - \gamma_0 \left(\frac{\alpha(1-\alpha^L)}{1-\alpha} \right)^2. \end{aligned} \tag{21}$$

Finally, by substituting (4) into (21) we have,

$$\hat{\sigma}_{t,L}^2 = Var\left(d_{t,L} - f_{t,L}\right) = \frac{\lambda}{1-\alpha}\left(L + 2\alpha\left(\frac{\alpha^L - \alpha L + L - 1}{(\alpha-1)^2}\right) - \left(\frac{\alpha(1-\alpha^L)}{1-\alpha}\right)^2\right). \quad (22)$$

Remark. The variance of the forecast error over the lead-time is also the variance of the inventory levels maintained by the OUT policy. It is interesting to note the similarity of $NSAmp = Var\left(d_{t,L} - f_{t,L}\right) / Var\left(d_t\right)$ and the NSAmp metric in Disney and Lambrecht (2008) for the real valued AR(1) demand. The NSAmp measure for the OUT policy under INAR(1) demand is the same as the NSAmp measure under AR(1) demand, albeit with a small change in notation; $\phi \to \alpha$ and $T_p + 1 \to L$. The NSAmp measure has been plotted in Figure 2.

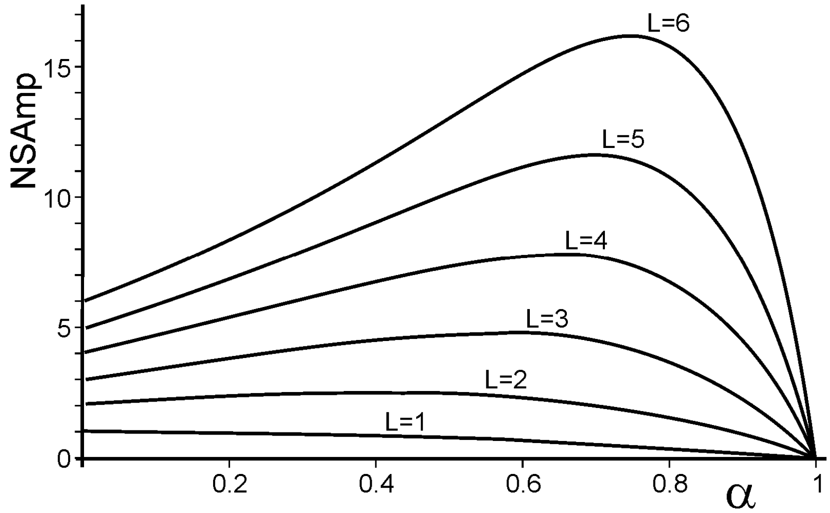

Figure 2. The NSAmp maintained by the OUT policy under INAR(1) demand

2.4. Variance of the order quantity

In order to calculate the variance of order quantity, we need to first calculate the order quantity. Substituting S_t and S_{t-1} from (6) into (5) yields

$$q_t = f_{t,L} + z\hat{\sigma}_{t,L}, - f_{t-1,L} - z\hat{\sigma}_{t-1,L} + d_t, \quad (23)$$

It is clear from the RHS of (22) that the variance of forecast error does not depend on *t*, consequently we have $\hat{\sigma}_{t,L} = \hat{\sigma}_{t-1,L}$ and (23) becomes:

$$q_t = f_{t,L} - f_{t-1,L} + d_t, \quad (24)$$

By substituting $f_{t,L}$ and $f_{t-1,L}$ from (16) into (24), we obtain:

$$\begin{aligned} q_t &= \frac{L\lambda}{1-\alpha} + \alpha\left(d_t - \frac{\lambda}{1-\alpha}\right)\left(\frac{1-\alpha^L}{1-\alpha}\right) - \frac{L\lambda}{1-\alpha} - \alpha\left(d_{t-1} - \frac{\lambda}{1-\alpha}\right)\left(\frac{1-\alpha^L}{1-\alpha}\right) + d_t \\ &= \alpha d_t\left(\frac{1-\alpha^L}{1-\alpha}\right) - \alpha d_{t-1}\left(\frac{1-\alpha^L}{1-\alpha}\right) + d_t, \end{aligned} \quad (25)$$

Finally the variance of order quantity is calculated as follows:

$$\begin{aligned} Var\left(q_t\right) &= Var\left(\frac{\alpha d_t(1-\alpha^L)}{1-\alpha} - \frac{\alpha d_{t-1}(1-\alpha^L)}{1-\alpha} + d_t\right) \\ &= Var\left(\frac{\alpha d_t(1-\alpha^L)}{1-\alpha}\right) + Var\left(\frac{\alpha d_{t-1}(1-\alpha^L)}{1-\alpha}\right) + Var\left(d_t\right) \\ &-2Cov\left(\frac{\alpha d_t(1-\alpha^L)}{1-\alpha}, \frac{\alpha d_{t-1}(1-\alpha^L)}{1-\alpha}\right) + 2Cov\left(\frac{\alpha d_t(1-\alpha^L)}{1-\alpha}, d_t\right) \\ &-2Cov\left(\frac{\alpha d_{t-1}(1-\alpha^L)}{1-\alpha}, d_t\right) \end{aligned} \tag{26}$$

Knowing that $Var\left(d_{t-k}\right) = \gamma_0$ and $\forall\ k \geq 1,\ Cov\left(d_t, d_{t-k}\right) = \gamma_k$, (26) reduces to

$$Var\left(q_t\right) = \gamma_0\left(1 + 2\alpha(1-\alpha^L)\left(1 + \frac{\alpha(1-\alpha^L)}{1-\alpha}\right)\right). \tag{27}$$

The bullwhip effect is calculated based on the ratio of the variance of order quantity experienced by the manufacturer to the actual variance of the demand.

$$B = \frac{Var\left(q_t\right)}{Var\left(d_t\right)} \tag{28}$$

A ratio of $B > 1$ indicates the existence of the bullwhip effect. By substituting (4) and (27) into (28) we obtain the following expression for the bullwhip effect,

$$B = 1 + 2\alpha\left(1-\alpha^L\right)\left(1 + \frac{\alpha(1-\alpha^L)}{1-\alpha}\right). \tag{29}$$

Equation (29) is interesting. First the influence of the Poisson distribution, λ, has no influence on the bullwhip effect. Second, (29) has the same structural form as the bullwhip generated by the OUT policy with MMSE forecasting under AR(1) demand, albeit a change in notation $\phi \to \alpha$ and $T_p + 1 \to L$, Disney and Lambrecht (2008). Furthermore, (29) is increasing in *L*.

3. Results and discussion

Using the measure of bullwhip in (29) we can investigate the existence of the bullwhip effects and the impact of α and L on its magnitude.

Proposition 2: The bullwhip effect exists, i.e. $B > 1$, regardless the values of the auto-regressive parameter, α and the lead time, *L*.

Proof: As $0 \leq \alpha \leq 1$, it is clear that $B > 1$ as all the terms involving α are positive. Appendix 2 also provides an alternative proof of Proposition 2. ■

From the above proposition we see that *B* is always greater than unity highlighting that bullwhip always exists under an INAR(1) demand process.

Figure 3 illustrates the impact of the demand process parameter and lead time on the bullwhip effect confirming that the bullwhip effect ratio is always greater than one and increases in the lead time *L*.

Proposition 3: The upper bound of the bullwhip effect is a function of α and equals $(1+\alpha)/(1-\alpha)$.

Proof: The proof of Proposition 3 is provided in Appendix 2. ■

Remark. From the above proposition it can be seen that an upper bound exists for the bullwhip effect measure of the INAR(1) demand. This has also been plotted in Figure 3.

From (4) we can determine the autocorrelation function (ACF) of the demand process; $ACF = \alpha^k$. This is exactly the same form of the ACF for the AR(1) process. Therefore, the auto-regressive parameter α controls the form of the demand series and its auto-correlation. Moreover, the auto-regressive parameter, α, also controls the presence of zero observations in a demand series. Lower values of α and λ generate a time series with high frequency of zero demands but the bullwhip effect will always exist regardless the presence of zero demand values.

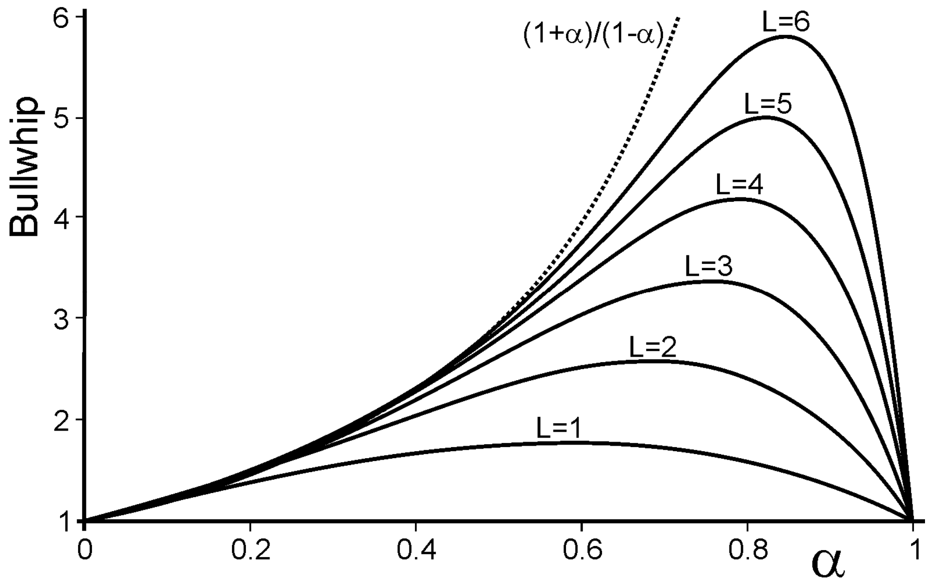

Figure 3. The Bullwhip effect in the OUT policy under INAR(1) demand

4. Conclusions

In this research, we have analytically examined the behavior of the bullwhip measure for an integer-valued INAR(1) demand series in a simple supply chain consisting of an OUT policy with MMSE forecasts. The bullwhip effect measure is first derived and then the conditions under which bullwhip effect exists are discussed. The impact of the auto-regressive demand parameter and lead time are analyzed. The main findings can be summarized as follows:

1. The bullwhip effect exists regardless of the autoregressive parameter and the lead time. There exists a lower bound of $B = 1$ obtained at $\alpha = 0$ and $\alpha = 1$. The amount of bullwhip generated depends on autoregressive and lead time values. For a given lead time L, and $0 < \alpha < 1$, the bullwhip measure first increases with α, it then reaches a maximum value and decreases towards unity at $\alpha = 1$.
2. There exists an upper bound for the bullwhip which is a function of the autoregressive parameter, α. For a given value of α, the upper bound represents the maximum value of the bullwhip effect regardless of the large lead time. The upper bound is tight when α is small.
3. The value of α also controls the intermittency of the demand series. Low values of α (and λ) lead to intermittent demand series that may contain a high proportion of zero values. Results show that the bullwhip effect measure of a highly intermittent series is lower compare to a normal demand series. i.e. the upper bound of the bullwhip effect for $\alpha = 0.05$ equals to $B = 1.1$ while for $\alpha = 0.95$ equals to $B = 39$. However, bullwhip will always exist for intermittent demand.

4. The structure of OUT policies response to the INAR(1) demand and its properties is similar to those of an AR(1) demand. The bullwhip measures take the same form, with $\alpha = \phi > 0$, where ϕ is the auto-regressive parameter in the AR(1) process.

Finally, we note that in our model the forecasts and orders are no longer integer, even though the demand is. While there is no need for the forecasts to be integer (it is internal calculation within the replenishment decision), presumably, if we can only sell integer numbers of products, we can only make integer numbers of products. Thus, there is a need to either a) round the orders to integers in some way or b) to forecast and place orders in integers. For option a) there are at least four options for rounding could be considered: rounding down, rounding up, rounding to the nearest integer, and stochastic rounding. For option b) perhaps by using the thinning operator, or Markov Chains, we can find alternative OUT policy formulations that generate only integer orders. The exploration of these issues are left for further work.

5. References

Al-Osh, M., Alzaid, A., 1988. Integer-valued moving average (INMA) process. *Statistical Papers* 29, 281–300.

Box, G., Jenkins, G., 1970. *Time Series Analysis: Forecasting and Control.* San Francisco: Holden-Day.

Chen, Y.F., Disney, S.M., 2007. The myopic order-up-to policy with a feedback controller. *International Journal of Production Research* 45 (2), 351–368.

Chen, Y.F., Drezner, Z., Ryan, J.K., Simchi-Levi, D., 2000a. Quantifying the Bullwhip effect in a simple supply chain: The impact of forecasting, lead times, and information. *Management Science* 46 (3), 436–443.

Chen, Y.F., Ryan, J.K., Simchi-Levi, D., 2000b. The impact of exponential smoothing forecasts on the bullwhip effect. *Naval Research Logistics* 47 (4), 269–286.

Dejonckheere, J., Disney, S.M., Lambrecht, M., Towill, D.R., 2003. Measuring and avoiding the bullwhip effect: A control theoretic approach. *European Journal of Operational Research* 147 (3), 567–590.

Disney, S.M., Lambrecht, M.R., 2008, On replenishment rules, forecasting and the bullwhip effect in supply chains. *Foundations and Trends in Technology, Information and Operations Management,* 2 (1), 1–80.

Duc, T.T.H., Luong, H.T., Kim, Y.-D., 2008. A measure of bullwhip effect in supply chains with a mixed autoregressive-moving average demand process. *European Journal of Operational Research* 187 (1), 243–256.

Gaalman, G., Disney, S.M., 2006. State space investigation of the bullwhip problem with ARMA(1,1) demand processes. *International Journal of Production Economics,* 104 (2), 327–339.

Graves, S.C., 1999. A single-item inventory model for a non-stationary demand process. *Manufacturing and Service Operations Management*, 1 (1), 50–61.

Hosoda, T., Disney, S.M., 2006. The governing dynamics of supply chains: The impact of altruistic behavior. *Automatica* 42 (4), 1301–1309.

Janjić, A.D.,. Ristić, M.M., Nastić, A.S., 2014. Mixed thinning INAR(1) model. Lecture notes available for download at https://web.math.pmf.unizg.hr/cqd/files/Mixed%20thinning.pdf. Verified 29 May 2017.

Lee, H.L, So, K.C., Tang, C.S., 2000. The value of information sharing in a two-level supply chain. *Management Science* 46 (5), 626–643.

Silva, M.E. and Oliveira, V.L., 2004. Difference equations for the higher-order moments and cumulants of the INAR(1) model. *Journal of Time Series Analysis* 25 (3), 317–333.

Silva, N., Pereira, I., Silva, M.E., 2009. Forecasting in INAR(1) model. *Revstat Statistical Journal* 7 (1), 119–134.

Wang, X., Disney, S.M., 2016. The bullwhip effect: Progress, trends, and directions. *European Journal of Operational Research* 250 (3), 691–701.

Appendix 1: The auto-covariance function of INAR(1) demand

We can show that the mean of d_t is given by,

$$E(d_t) = E(\alpha \circ d_{t-1} + z_t) = E(\alpha \circ d_{t-1}) + E(z_t) = \alpha E(d_{t-1}) + \lambda \,. \tag{30}$$

As an INAR(1) process is stationary, $E(d_t) = E(d_{t-1})$, and (30) reduces to

$$E(d_t) = \frac{\lambda}{1-\alpha} \,. \tag{31}$$

The variance of demand at period t is defined as

$$\begin{aligned} Var(d_t) &= Var(\alpha \circ d_{t-1} + z_t) \\ &= Var(\alpha \circ d_{t-1}) + Var(z_t) + 2Cov(d_{t-1}, z_t) \\ &= Var(\alpha \circ d_{t-1}) + \lambda \\ &= E(\alpha \circ d_{t-1})^2 - (E(\alpha \circ d_{t-1}))^2 + \lambda. \end{aligned} \tag{32}$$

Using the properties of the thinning operator, (3) we can simplify (32) as follows,

$$\begin{aligned} Var(d_t) &= \alpha^2 E(d_{t-1})^2 + \alpha(1-\alpha)E(d_{t-1}) - \alpha^2 (E(d_{t-1}))^2 + \lambda \\ &= \alpha^2 \left(E(d_{t-1})^2 - (E(d_{t-1}))^2 \right) + \alpha\lambda + \lambda = \alpha^2 Var(d_{t-1}) + \lambda(\alpha+1). \end{aligned} \tag{33}$$

because of stationarity, we have $Var(d_t) = Var(d_{t-1})$, yielding,

$$Var(d_t) - \alpha^2 Var(d_t) = \lambda(\alpha+1). \tag{34}$$

Collecting together terms provides

$$Var(d_t) = \frac{\lambda(\alpha+1)}{(1-\alpha^2)} = \frac{\lambda(\alpha+1)}{(1-\alpha)(1+\alpha)} = \frac{\lambda}{(1-\alpha)}. \tag{35}$$

By recursive substitutions of d_{t-k} for $k \geq 1$, (1) can be written as

$$d_t = \alpha^k \circ d_{t-k} + \sum_{j=0}^{k-1} \alpha^j \circ z_{t-j}. \tag{36}$$

The auto-covariance of lag $k \geq 1$ can then be calculated as

$$\gamma_k = Cov(d_t, d_{t-k}) = Cov\left(\alpha^k \circ d_{t-k} + \sum_{j=0}^{k-1} \alpha^j \circ z_{t-j}, d_{t-k} \right) = Cov(\alpha^k \circ d_{t-k}, d_{t-k}) + Cov\left(\sum_{j=0}^{k-1} \alpha^j \circ z_{t-j}, d_{t-k} \right). \tag{37}$$

As $Cov(\alpha \circ d_{t-k}, d_{t-k}) = \alpha Cov(d_{t-k}, d_{t-k})$,

$$\gamma_k = \alpha^k Cov(d_{t-k}, d_{t-k}) + Cov\left(\sum_{j=0}^{k-1} \alpha^j \circ z_{t-j}, d_{t-k} \right). \tag{38}$$

As the correlation between d_{t-k} and z_{t-j} for all $j \leq k-1$ is equal to zero, $Cov\left(d_{t-k}, \sum_{j=0}^{k-1} \alpha^j \circ z_{t-j}\right) = 0$, therefore the auto-covariance function of lag $k \geq 1$ for an INAR(1) process is

$$\gamma_k = \alpha^k Cov(d_{t-k}, d_{t-k}) = \alpha^k \gamma_0. \tag{39}$$

Appendix 2: Bullwhip effect bounds

Recall, from (29), the bullwhip ratio,

$$B = 1 + \left(2\alpha\left(1-\alpha^L\right) + \frac{2\left(\alpha\left(1-\alpha^L\right)\right)^2}{1-\alpha} \right). \tag{40}$$

Given $0 < \alpha < 1$ and $L \geq 1$, we have

$$0 < \alpha^L < 1 \tag{41}$$

Multiplying (40) by minus one (-1) and adding plus one $(+1)$ to both sides, we have

$$0 < 1 - \alpha^L < 1 \tag{42}$$

given $\alpha > 0, 1-\alpha > 0$ are positive parameters, we get

$$0 < \alpha\left(1-\alpha^L\right) < \alpha. \tag{43}$$

Given (43) and that $\alpha > 0$, we have

$$0 < 2\alpha\left(1-\alpha^L\right) < 2\alpha. \tag{44}$$

Squaring (44) leads to

$$0 < 2\left(\alpha\left(1-\alpha^L\right)\right)^2 < 2\alpha^2. \tag{45}$$

Knowing that $\frac{1}{1-\alpha} > 1$, we get

$$0 < \frac{2\left(\alpha\left(1-\alpha^L\right)\right)^2}{1-\alpha} < \frac{2\alpha^2}{1-\alpha}. \tag{46}$$

By adding (44) and (46)

$$0 < 2\alpha\left(1-\alpha^L\right) + \frac{2\left(\alpha\left(1-\alpha^L\right)\right)^2}{1-\alpha} < 2\alpha + \frac{2\alpha^2}{1-\alpha}. \tag{47}$$

By simplify (47)

$$0 < 2\alpha\left(1-\alpha^L\right) + \frac{2\left(\alpha\left(1-\alpha^L\right)\right)^2}{1-\alpha} < \frac{2\alpha}{1-\alpha}. \tag{48}$$

To obtain the bullwhip expression, we add plus one (+1) to (48)

$$1<1+\left(2\alpha\left(1-\alpha^{L}\right)+\frac{2\left(\alpha\left(1-\alpha^{L}\right)\right)^{2}}{1-\alpha}\right)<1+\frac{2\alpha}{1-\alpha}. \tag{49}$$

Therefore the upper and lower bound of the bullwhip effect measure of an INAR(1) is

$$1<B<\frac{1+\alpha}{1-\alpha}, \tag{50}$$

revealing that B is always greater than one and less than $(1+\alpha)/(1-\alpha)$. It means that bullwhip exists regardless the lead time and the auto-regressive parameter.

Biography

Bahman Rostami-Tabar. Dr. Bahman Rostami-Tabar is a Lecturer in Logistics and Operations Management at Cardiff Business School, Cardiff University. Bahman was previously Lecturer in Supply Chain Management at Coventry University (Aug. 2015-Sep. 2016). Bahman worked as a researcher in LGI lab at Ecole Centrale Paris (Sep. 2013-Aug. 2015) with Faurecia (the world's 6th-largest automotive equipment supplier) where he conducted research on logistics and information sharing. He was a Research Assistant between January 2011 and August 2013 at Kedge Business School, Bordeaux, France.

Bahman holds a Ph.D. in Production Engineering from the University of Bordeaux. He moved to France in 2009 and received an M.Sc. in Information Systems from ECE Paris in 2010. He received the MIM best paper award (IFAC, 2013) and was awarded a Campus France Scholarship. Bahman's research goals are directed towards the use of Operations Research (OR) techniques to improve decision making in supply chains and operations management, and as such positively contribute to both industrial and societal advancements. Bahman' research to date reflects collaboration with more than eight universities around the globe.

Stephen M. Disney. Professor Stephen Disney, Ph.D., is a member of, and former Department Chair of the Logistics and Operations Management Department at Cardiff Business School. He currently leads the Logistics Systems Dynamics Group. He has recently returned from research sabbatical at the University of California, Los Angeles. He has previously held visiting positions at the Chinese University of Hong Kong and Boston University. Professor Disney lectures Operations Management to MBA and MSc students at Cardiff Business School. He has also taught Supply Chain Modelling to the Mathematics Department of Cardiff University. He has extensive experience of teaching in-class, on-line and on-site to both postgraduate and executive audiences.

Professor Disney's current research interests involve the application of control theory and statistical techniques to operations management and supply chain scenarios to investigate their dynamic, stochastic and economic performance. Stephen has a particular interest in the bullwhip effect. He has advised several of the world's largest corporations on this problem. He has worked with many companies in the UK (including Tesco), US (including Lexmark) and Europe (including P&G) and on supply chains that operate globally.