Sliding Mode Observers for a Class of LPV Systems

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Abstract

In this paper, a new framework for the synthesis of a class of sliding mode observers for affine linear parameter varying (LPV) systems is proposed. The sliding mode observer is synthesized by selecting the design freedom via LMIs. Posing the problem from a small gain perspective allows existing numerical techniques from the literature to be used for the purpose of synthesizing the observer gains. In particular, the framework allows affine parameter-dependent Lyapunov functions to be considered for analyzing the stability of the state estimation error dynamics, to help reduce design conservatism. Initially a variable structure observer formulation is proposed, but by imposing further constraints on the LMIs, a stable sliding mode is introduced, which can force and maintain the output estimation error to be zero in finite time. The efficacy of the scheme is demonstrated using an LPV model of the short period dynamics of an aircraft and demonstrates simultaneous asymptotic estimation of the states and disturbances.

Keyword: Sliding mode observer; Linear parameter-varying; Fault detection and diagnosis

I. Introduction

Linear parameter varying (LPV) systems [21], [18], [16] have been the focus of significant research in recent decades. These ideas evolved initially from efforts to add rigour to the industrially popular, but often somewhat ad hoc, gain scheduling techniques. Many engineering systems subject to time varying parameters, and certain classes of nonlinear system can be formulated (or approximated with sufficient accuracy) by LPV system representations. From a synthesis perspective, LPV systems can be viewed as bridging the gap between linear-system based approaches, and offer design frameworks for nonlinear systems. The practicality and intuitively appealing aspects of the LPV methodology have made it popular in applications areas such as aerospace systems and many other fields of engineering. [19]

LPV observer synthesis has become a popular research topic in its own right. A good deal of this research has been motivated and developed within the context of fault detection for (nonlinear) dynamic systems. This has been driven by the requirement for the underpinning observers to give high performance over a wide range of operating conditions. Much of the work (mostly within an $\mathscr{H}_\infty$ framework [3], [26], [22], [25]) extends earlier linear time invariant (LTI) system design ideas into an LPV framework. By scheduling the gains with respect to operating conditions, a high level of performance over the operating envelope can be achieved. LPV based observer designs, which have emerged from within the fault detection literature include, [4], [5] and many have been applied to aerospace problems. [25], [18]

Apart from the early work (in terms of sliding mode control) more than two decades ago,[23], [20] very little sliding mode themed work for LPV systems has appeared in the literature until very recently. LPV observers based on classical sliding mode ideas have been suggested, [1] while LPV based higher order sliding mode observers were proposed to create so-called interval observers. [11] In contrast to linear $\mathscr{H}_\infty$ filtering approaches, the sliding mode observer schemes [1] completely reject the effects of a class of disturbances/faults as opposed to just attenuating them. They also have the capability to simultaneously estimate the states and faults. However the work [1] depends on a simplifying assumption that the disturbance distribution matrix can be factorized into the product of a fixed matrix and a nonsingular parameter-dependent matrix, which was required to establish the coordinate transformation underpinning the synthesis of the observer. Furthermore, these papers deploy a fixed quadratic Lyapunov function when analysing the stability of the error system, and the design problem is posed as a constrained Lyapunov function formulation. [10], [14] In the LTI literature, the necessary and sufficient condition of the existence of linear unknown input observers are well understood, [8], [6] and equivalent conditions have been established for

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the existence of sliding mode observers for the same problem. [9] These conditions have been extended to LPV systems [4] and state affine systems. [15] However the literature in this area is sparse.

The main contribution of this paper is that it proposes for the first time a synthesis procedure for creating a sliding mode observer for a general class of LPV systems. This scheme removes the restriction on the factorisation of the disturbance distribution matrix. [1] To the authors' knowledge, no other methods are available to address this situation. By loop shifting, the problem is posed in terms of a small gain formulation (as opposed to exploiting passivity-like conditions). As a consequence of formulating the problem in this manner, parameter dependent Lyapunov functions can be considered to increase the level of design freedom available. [13] The practicality of the proposed method is demonstrated on a realistic aerospace example involving the simultaneous estimation of the internal states and disturbances.

The notation used in this paper is quite standard. For a symmetric matrix X, $X<0$ indicates that X is negative definite. The notation $\|\cdot\|$ indicates the Euclidean 2-norm of a vector and $\|\cdot\|_2$ represents the signal 2-norm. For an operator or function $\Phi: u \mapsto y$, the induced $\mathscr{L}_2$ gain $\|\Phi\|_2 := \sup \frac{\|y\|_2}{\|u\|_2}, \forall u \in \mathscr{L}_2, u \neq 0$.

II. Variable structure observer

Consider the LPV plant model subject to disturbances (or faults) represented by

$$\begin{aligned} \dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) + D(\rho)\xi(t,x) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $A(\rho) \in \mathbb{R}^{n\times n}$, $B(\rho) \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and the distribution matrix $D(\rho) \in \mathbb{R}^{n\times q}$, where $q<p$. In (1), it is assumed that system inputs $u(t)$ and output $y(t)$ are measurable, but the state $x(t)$ is unknown. The disturbance signal $\xi(t,x)$ is assumed to be unknown but norm bounded satisfying $\|\xi(t,x)\| \leq \alpha(t,y)$ where the bounding scalar function $\alpha(\cdot)$ is known. Throughout the paper, the scheduling parameter ρ is assumed to be known (measured) and satisfies $\rho \in \Omega \subset \mathbb{R}^r$ where Ω is a hyper-rectangle. Without loss of generality it is assumed throughout the paper that

$$C = \begin{bmatrix} 0 & I_p \end{bmatrix} \tag{2}$$

If the state-space representation does not naturally take this form, a fixed linear change of coordinates can always be introduced to bring the assumed structure to the (fixed) output distribution matrix.

Remark 1: In contrast to the earlier work, [1] here there is no assumption that $D(\rho)$ can be factorized as the product of a fixed matrix and a nonsingular parameter dependent matrix. This factorization is very beneficial since it greatly reduces the complexity of the synthesis problem, but it clearly imposes a limitation on the class of systems to which the results are applicable.

For the developments which follow the plant in (1) is rewritten as

$$\begin{aligned} \dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) + \tilde{D}(\rho)\tilde{\xi}(t,x) \\ y(t) &= Cx(t) \end{aligned} \tag{3}$$

where $\tilde{D}(\rho) \in \mathbb{R}^{n\times p}$ is defined as

$$\tilde{D}(\rho) = \begin{bmatrix} D(\rho) & M(\rho) \end{bmatrix} \tag{4}$$

and $\tilde{\xi}(t) = [\xi(t)\ 0]^T \in \mathbb{R}^p$. Clearly this can be done without loss of generality (w.l.o.g). In (4), the design freedom $M(\rho) \in \mathbb{R}^{n\times(p-q)}$ has no physical meaning. However since $\tilde{D}(\rho)$ will become one of the observer gains (associated with the nonlinear injection signal), $M(\rho)$ may be viewed as part of the available observer design freedom. Here it is assumed $A(\rho)$ and $\tilde{D}(\rho)$ are affine with respect to the scheduling parameter ρ, for example, $A(\rho) = A_0 + \sum_{i=1}^r A_i\rho_i$. The input distribution matrix $B(\rho)$ can be polynomially or rationally dependent on ρ, since its structure is not directly exploited in this paper. It is assumed w.l.o.g that by scaling ρ_i satisfies $\rho_i \in [-1 \quad 1]$, and $\dot{\rho}_i \in [\underline{\dot{\rho}_i} \quad \overline{\dot{\rho}}_i]$ for all $i = 1,\ldots,r$ where $\underline{\dot{\rho}_i}$ and $\overline{\dot{\rho}}_i$ denote the lower and the upper bounds of $\dot{\rho}_i$. As in cite [13], define

$$\mathscr{V} := \{(\omega_1,\cdots,\omega_r) : \omega_i \in [-1 \quad 1]\} \tag{5}$$

$$\mathscr{R} := \{(\tau_1,\cdots,\tau_r) : \tau_i \in [\underline{\dot{\rho}}_i \quad \overline{\dot{\rho}}_i]\} \tag{6}$$

to represent the 2^r vertices of the hyper-rectangles associated with ρ and $\dot{\rho}$. *The objective is to design a sliding mode observer to robustly estimate the states $x(t)$ despite the unknown input $\xi(t,x)$, and to simultaneously estimate* *the unknown input, based only on knowledge of the inputs and outputs.* This problem formulation (exploiting the design freedom $M(\rho)$) is quite different from the strategies for designing observers for LTI/LPV systems.[1], [2]

The proposed sliding mode LPV observer is:

$$\begin{aligned} \dot{\hat{x}}(t) &= A(\rho)\hat{x}(t) + B(\rho)u(t) + \tilde{D}(\rho)\nu + G_l(\rho)e_y \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \tag{7}$$

where $e_y(t) = C(x(t) - \hat{x}(t))$. The parameter dependent matrix $G_l(\rho) \in \mathbb{R}^{n\times p}$ represents the observer gain to be calculated – however as discussed above, embedded in $\tilde{D}(\rho)$ there is also design freedom associated with $M(\rho)$. In (7) the nonlinear injection term is given by

$$\nu = \begin{cases} \eta(t,y)\dfrac{e_y}{\|e_y\|} & \text{if} \quad e_y \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

where $\eta(t,y)$ is a positive scalar function satisfying $\eta(t,y) > \alpha(t,y)$. Subtracting (1) from (7) yields the state estimation error system

$$\dot{e}(t) = (A(\rho) - G_l(\rho)C)e(t) - \tilde{D}(\rho)(\nu - \tilde{\xi}) \tag{9}$$

The design freedom $G_l(\rho)$ and $M(\rho)$ in (9) will be synthesized to ensure the origin is asymptotically stable.

Assumption 1: For any fixed $\rho \in \Omega$, it is assumed that

1) $(A(\rho), D(\rho), C)$ is minimum phase.
2) rank$(CD(\rho))$=rank$(D(\rho)) = q$.

Remark 2: The two conditions above are necessary for the existence of an observer of the structure given in (7) to act as an unknown input observer. To see this consider the special case when $\rho_0 \in \Omega$ is arbitrary but fixed. In this situation, the system in (1) becomes an LTI system (A_0, D_0, C). For LTI systems it is well known that the two conditions in Assumption 1 are necessary (and sufficient) for the existence of an unknown input observer (see for example [9]). Whilst the second condition can be tested fairly efficiently numerically, the first condition is less amenable to testing.

As a consequence of the augmentation used to create the gain $\tilde{D}(\rho) \in \mathbb{R}^{n\times p}$, the error dynamics in (9) can be written in the form of the 'square' feedback system in Figure 1.

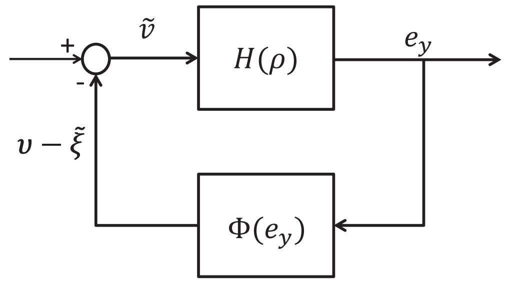

Fig. 1. Plant-observer error dynamic in feedback form

In particular in Figure 1, the nonlinearity in the feedback path is

$$\Phi(e_y) = \eta(t,y)\frac{e_y}{\|e_y\|} - \tilde{\xi}(t,x) \tag{10}$$

and

$$H(\rho): \begin{cases} \dot{e} = (A(\rho) - G_l(\rho)C)e + \tilde{D}(\rho)\tilde{\nu} \\ e_y = Ce \end{cases} \tag{11}$$

where $\tilde{\nu} = \nu - \tilde{\xi}$. In what follows the operator $H(\rho)$ from (11) will be written in compact form [19] as

$$H(\rho) = \left[\begin{array}{c|c} A(\rho) - G_l(\rho)C & \tilde{D}(\rho) \\ \hline C & 0 \end{array}\right] \tag{12}$$

Since $\tilde{\xi}$ is bounded by $\alpha(\cdot)$ it follows $\|e_y^T\tilde{\xi}\| < \|e_y\|\alpha(t,y)$. Then using the Cauchy-Schwarz inequality, the modulation gain $\eta(t,y)$ in (8) can always be selected so that

$$e_y^T\Phi(e_y) = \eta(t,y)\|e_y\| - e_y^T\tilde{\xi} \geq 0 \tag{13}$$

Since $e_y^T\Phi(e_y) \geq 0$, the memoryless function $\Phi(\cdot)$ belongs to the sector $[0,\infty)$.

III. Synthesis of the LPV observers

In this section, a framework is proposed for synthesising the observer gains in (7). In developing the framework the error dynamics in (9) will be first subjected to a loop transformation. [17] As a consequence of the loop transformation, the design problem becomes one of $\mathscr{L}_2$ gain minimization, and the small gain theorem will be exploited [17] to prove stability.

Lemma 1: The interconnected system in Figure 1 is asymptotically stable if $\|\tilde{H}(\rho)\|_2 \leq 1$ where

$$\tilde{H}(\rho) = \left[\begin{array}{c|c} A(\rho) - G_l(\rho)C - \tilde{D}(\rho)C & \tilde{D}(\rho) \\ \hline 2C & -I \end{array}\right] \tag{14}$$

Proof 1: The proof, based on the concept of loop shifting, is given in the Appendix.

The main result of this section will now be presented.

Theorem 1: Suppose there exist a nonsingular matrix $Y \in \mathbb{R}^{n\times n}$, matrices $S(\omega) \in \mathbb{R}^{n\times p}$, $R(\omega) \in \mathbb{R}^{n\times(p-q)}$, and symmetric positive definite (s.p.d) matrices $P(\omega) \in \mathbb{R}^{n\times n}$, $P(\tau) \in \mathbb{R}^{n\times n}$ where

$$\begin{aligned} S(\omega) &= S_0 + \sum_{i=1}^{r}\omega_i S_i, \quad R(\omega) = R_0 + \sum_{i=1}^{r}\omega_i R_i \\ P(\omega) &= P_0 + \sum_{i=1}^{r}\omega_i P_i, \quad P(\tau) = P_0 + \sum_{i=1}^{r}\tau_i P_i \end{aligned} \tag{15}$$

such that the following LMIs

$$\begin{bmatrix} -Y-Y^T & Y^TA(\omega)-S(\omega)C-U(\omega)C+P(\omega) & Y^T & U(\omega) & 0 \\ * & -P(\omega)+P(\tau)-P_0 & 0 & 0 & -2C^T \\ * & * & -P(\omega) & 0 & 0 \\ * & * & * & -I_p & I_p \\ * & * & * & * & -I_p \end{bmatrix} \leq 0, \quad Y+Y^T > 0 \tag{16}$$

are feasible for all $(\omega,\tau) \in \mathscr{V}\times\mathscr{R}$ with respect to the decision variables Y, S_i, R_i and P_i where

$$U(\omega) = \begin{bmatrix} Y^TD(\omega) & R(\omega) \end{bmatrix} \tag{17}$$

Then, for a given feasible solution to (16), choosing

$$G_l(\rho) = (Y^T)^{-1}S(\rho) \quad \text{and} \quad M(\rho) = (Y^T)^{-1}R(\rho) \tag{18}$$

guarantees the error system in (9) is quadratically stable subject to choosing ν as in (8), where by choice $\eta(t,y) \geq \alpha(t,y)$.

Proof 2: Let a parameter dependent candidate Lyapunov function $V(e) = \tilde{e}^TP(\rho)e$ where $\tilde{e}$ represents the error states associated with the operator $\tilde{H}(\rho)$ in (14) and $P(\rho) = P_0 + \sum_{i=1}^{r}\rho_iP_i$ subject to $P(\rho) > 0$, and define

$$\mathscr{L}(\rho,\dot{\rho}) = \tilde{A}(\rho)^TP(\rho) + P(\rho)\tilde{A}(\rho) + P(\dot{\rho}) - P_0 \tag{19}$$

where $\tilde{A}(\rho) = A(\rho) - G_l(\rho)C - \tilde{D}(\rho)C$. Then as argued in Lemma 3.1 from [27], if

$$\begin{bmatrix} \mathscr{L}(\rho,\dot{\rho}) & P(\rho)\tilde{D}(\rho) & -2C^T \\ \tilde{D}(\rho)^TP(\rho) & -I_p & I_p \\ -2C & I_p & -I_p \end{bmatrix} \leq 0 \tag{20}$$

it follows $\|\bar{H}(\rho)\|_2 \leq 1$. Using the Schur Complement, inequality (20) is equivalent to

$$\begin{bmatrix} \mathscr{L}(\rho,\dot{\rho})-P(\rho) & P(\rho) & P(\rho)\tilde{D}(\rho) & -2C^T \\ P(\rho) & -P(\rho) & 0 & 0 \\ \tilde{D}(\rho)^TP(\rho) & 0 & -I_p & I_p \\ -2C & 0 & I_p & -I_p \end{bmatrix} \leq 0 \tag{21}$$

Let $N(\rho) = \begin{bmatrix}\tilde{A}(\rho) & I_n & \tilde{D}(\rho) & 0_{n\times p}\end{bmatrix}$ and define two ρ dependent matrices as

$$\Psi(\rho) = \left[\begin{array}{c|cccc} \tilde{A}(\rho)^T & I_n & 0 & 0 & 0 \\ I_n & 0 & I_n & 0 & 0 \\ \tilde{D}(\rho)^T & 0 & 0 & I_p & 0 \\ 0_{p\times n} & 0 & 0 & 0 & I_p \end{array}\right] := \left[\begin{array}{c|c} N(\rho)^T & I_{2n+2p} \end{array}\right] \tag{22}$$

and

$$\Pi(\rho) = \begin{bmatrix} 0 & P(\rho) & 0 & 0 & 0 \\ P(\rho) & Q(\rho) & 0 & 0 & -2C^T \\ 0 & 0 & -P(\rho) & 0 & 0 \\ 0 & 0 & 0 & -I_p & I_p \\ 0 & -2C & 0 & I_p & -I_p \end{bmatrix} \tag{23}$$

where in (23) the term

$$Q(\rho) = -P(\rho) + P(\dot{\rho}) - P_0 \tag{24}$$

From (24) a sufficient condition for $Q(\rho) \leq 0$ is

$$\begin{bmatrix} Q(\rho) & 0 & 0 & -2C^T \\ 0 & -P(\rho) & 0 & 0 \\ 0 & 0 & -I_p & I_p \\ -2C & 0 & I_p & -I_p \end{bmatrix} \leq 0 \tag{25}$$

Define

$$\Xi = \begin{bmatrix} 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & 0 & I_p \end{bmatrix}$$

the inequalities in (21) and (25) can be written in the form

$$\begin{aligned} \Psi(\rho)\Pi(\rho)\Psi(\rho)^T &\leq 0 \\ \Xi\Pi(\rho)\Xi^T &\leq 0 \end{aligned} \tag{26}$$

Note that in (26), $\Psi(\rho)$ and Ξ can be interpreted as kernels of suitably defined matrices. Specifically,

$$\begin{aligned} \Psi(\rho)^T &= \ker([-I_n \; N(\rho)]) \\ \Xi^T &= \ker([I_n \; 0_{n\times(2n+2p)}]) \end{aligned} \tag{27}$$

From the Projection Lemma [12], satisfying (21) and (25) is equivalent to satisfying the following LMI

$$\Pi(\rho) + \begin{bmatrix} -I_n \\ N(\rho)^T \end{bmatrix} Y \begin{bmatrix} I_n & 0_{n\times(2n+2p)} \end{bmatrix} + \begin{bmatrix} I_n \\ 0_{(2n+2p)\times n} \end{bmatrix} Y^T \begin{bmatrix} -I_n & N(\rho) \end{bmatrix} \leq 0 \tag{28}$$

which in turn is equivalent to the LMIs

$$\begin{bmatrix} -Y-Y^T & Y^T\tilde{A}(\rho)+P(\rho) & Y^T & Y^T\tilde{D}(\rho) & 0 \\ * & -P(\rho)+P(\dot{\rho})-P_0 & 0 & 0 & -2C^T \\ * & * & -P(\rho) & 0 & 0 \\ * & * & * & -I_p & I_p \\ * & * & * & * & -I_p \end{bmatrix} \leq 0 \tag{29}$$

Note that a feasible solution to (16) guarantees that Y is non-singular. Finally, let $S(\rho) = Y^T G_l(\rho)$, $R(\rho) = Y^T M(\rho)$ and $U(\rho) = [Y^T D(\rho) \; R(\rho)]$. Then substituting back for $\tilde{A}(\rho) = A(\rho) - G_l(\rho)C - \tilde{D}(\rho)C$, inequality (29) is equivalent to

$$\begin{bmatrix} -Y-Y^T & Y^T A(\rho) - S(\rho)C - U(\rho)C + P(\rho) & Y^T & U(\rho) & 0 \\ * & -P(\rho)+P(\dot{\rho})-P_0 & 0 & 0 & -2C^T \\ * & * & -P(\rho) & 0 & 0 \\ * & * & * & -I_p & I_p \\ * & * & * & * & -I_p \end{bmatrix} \leq 0 \tag{30}$$

Clearly (30) is affine with respect to ρ and $\dot{\rho}$. By replacing the variables ρ and $\dot{\rho}$ in (30) with ω and τ, respectively, (16) is obtained.

Remark 3: The motivation underlying Theorem 1 is to formulate the observer design problem in terms of a finite number of LMIs (using the Projection Lemma) as shown in (30), which are affine with respect to ρ and $\dot{\rho}$.

Additional assumptions will now be imposed to guarantee a sliding motion takes place in the state estimation error space in finite time.

A condition for the existence of an unique equivalent control to maintain a first order sliding mode forcing the output estimation error to zero is that $\det(C\tilde{D}(\rho)) \neq 0$ for all $\rho \in \Omega$ which in turn requires that $\mathrm{rank}(CD(\rho)) = q$ for all $\rho \in \Omega$. Rather than including rank conditions in the matrix inequalities (a well researched problem) which destroys convexity, a different approach will be followed. Instead, in the sequel, the condition will be imposed that for each $\rho \in \Omega$ all the eigenvalues of $FC\tilde{D}(\rho)$ belong in the right hand plane (RHP), where $F \in \mathbb{R}^{p\times p}$ is a nonsingular matrix. An approach for selecting F is outlined below and the eigenvalue condition will be enforced by incorporating an additional LMI to those in Theorem 1.

Remark 4: Let (A_0, D_0, C) denote the central LTI system (i.e. the system in (1) when $\rho = 0$) then as a specific case of Assumption 1, this LTI system is minimum phase and relative degree one. Since

$$\mathrm{rank}(CD_0) = q \tag{31}$$

create a nonsingular output scaling matrix

$$F = \begin{bmatrix} (CD_0)^T \\ (CD_0)^{\perp} \end{bmatrix} \tag{32}$$

where $(CD_0)^{\perp} \in \mathbb{R}^{(p-q)\times p}$ is a full rank annihilator of CD_0, i.e. $(CD_0)^{\perp} CD_0 = 0$. Note that $\det(F) \neq 0$. Then scale the output of (1) so that $y \mapsto Fy = \tilde{y}$ and consider the scaled LPV system $(A(\rho), D(\rho), \tilde{C})$ where $\tilde{C} = FC$. Now consider the problem of designing an observer for the scaled system $(A(\rho), D(\rho), \tilde{C})$, involving the scaled outputs $\tilde{y}$, of the form

$$\dot{\hat{x}} = A(\rho)\hat{x} + B(\rho)u + \tilde{D}(\rho)\nu_F + G_l(\rho)\tilde{e}_y \tag{33}$$

where $\tilde{e}_y = Fy - \tilde{C}\hat{x}$ and

$$\nu_F = \begin{cases} \eta(t,y)\dfrac{\tilde{e}_y}{\|\tilde{e}_y\|} & \text{if } \tilde{e}_y \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

It follows that, by construction,

$$\tilde{C}D(\rho)|_{\rho=0} = \begin{bmatrix} (CD_0)^T(CD_0) \\ 0_{(p-q)\times p} \end{bmatrix} \tag{35}$$

and therefore

$$\lambda((CD_0)^T CD_0) \subset \lambda(FC\tilde{D}(\rho))|_{\rho=0}$$

Note of course, by construction, the eigenvalues of the matrix $(CD_0)^T CD_0$ lie in the RHP. Furthermore, since according to (2) $C = [0 \;\; I_p]$, it follows $\tilde{C} = [0 \;\; I_p]$.

Theorem 2: Suppose there exist matrices $S(\omega) \in \mathbb{R}^{n\times p}$, $R(\omega) \in \mathbb{R}^{n\times(p-q)}$, and s.p.d matrices $P(\omega) \in \mathbb{R}^{n\times n}$, $P(\tau) \in \mathbb{R}^{n\times n}$ with structure given in (15), and a nonsingular matrix $Y \in \mathbb{R}^{n\times n}$ with the structure

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ 0 & Y_3 \end{bmatrix} \tag{36}$$

where $Y_1 \in \mathbb{R}^{(n-p)\times(n-p)}$ and $Y_3 \in \mathbb{R}^{p\times p}$ is s.p.d, such that the LMIs

$$\begin{bmatrix} -Y - Y^T & Y^T A(\omega) - S(\omega)\tilde{C} - U(\omega)\tilde{C} + P(\omega) & Y^T & U(\omega) & 0 \\ * & -P(\omega) + P(\tau) - P_0 & 0 & 0 & -2\tilde{C}^T \\ * & * & -P(\omega) & 0 & 0 \\ * & * & * & -I_p & I_p \\ * & * & * & * & -I_p \end{bmatrix} \leq 0, \quad Y + Y^T > 0 \tag{37}$$

and the LMIs

$$\tilde{C}U(\rho) + (\tilde{C}U(\rho))^T - \varepsilon_0 I > 0 \;\; \forall \rho \in \Omega \tag{38}$$

are feasible for all $(\omega,\tau) \in \mathscr{V} \times \mathscr{R}$, where ε_0 is a positive design scalar. Then, for a feasible solution, choosing

$$G_l(\rho) = (Y^T)^{-1}S(\rho) \quad \text{and} \quad M(\rho) = (Y^T)^{-1}R(\rho) \tag{39}$$

results in the observer in (33) exhibiting a stable sliding motion in which the output estimation error $\tilde{e}_y$ is forced to zero in finite time, and remaining at zero for all subsequent time.

Proof 3: The state estimation error system describing the evolution of $\tilde{e} = x - \hat{x}$ associated with the plant in (3) and the observer in (33) is

$$\dot{\tilde{e}} = (A(\rho) - G_l(\rho)\tilde{C})\tilde{e} - \tilde{D}(\rho)(\nu_F - \tilde{\xi}) \tag{40}$$

where ν_F is defined in (34) and $\tilde{e}_y = \tilde{C}\tilde{e}$. Arguing as in the proof of Theorem 1, a feasible solution to (36)-(37) for the decision variables Y, S_i, R_i, P_i yields a variable structure observer of the form in (33) which guarantees the state estimation error in (40) tends to zero asymptotically. Define

$$V_s(\tilde{e}_y) = \tilde{e}_y^T Y_3 \tilde{e}_y \tag{41}$$

then it will be shown $V_s \equiv 0$ in finite time and therefore a sliding mode on $\tilde{e}_y \equiv 0$ exists in finite time. Note since Y_3 is s.p.d, the candidate Lyapunov function $V_s(\tilde{e}_y)$ is positive definite. Multiplying (40) on the left by $\tilde{C}$ yields

$$\dot{\tilde{e}}_y = \tilde{C}(A(\rho) - G_l(\rho)\tilde{C})\tilde{e} - \tilde{C}\tilde{D}(\rho)(\nu_F - \tilde{\xi}) \tag{42}$$

Since $\tilde{e}(t) \to 0$, there exists a (finite) time t_0 such that

$$\|\tilde{C}(A(\rho) - G_l(\rho)\tilde{C})\tilde{e}(t)\| < \kappa_1 \quad \forall t \geq t_0 \tag{43}$$

for any given positive scalar κ_1. Using the special structure of Y in (36), it follows

$$\tilde{C}Y = \begin{bmatrix} 0 & Y_3 \end{bmatrix} = Y_3\tilde{C} \tag{44}$$

because $\tilde{C} = [0 \;\; I_p]$. Then it is clear from (44) that (38) is equivalent to

$$Y_3\tilde{C}\tilde{D}(\rho) + (\tilde{C}\tilde{D}(\rho))^T Y_3 > \varepsilon_0 I \quad \forall \rho \in \Omega \tag{45}$$

(and therefore all the eigenvalues of $\tilde{C}\tilde{D}(\rho)$ for each ρ are in the RHP). Therefore for all $t \geq t_0$, from (42)

$$\begin{aligned} \dot{V}_s &= 2\tilde{e}_y^T Y_3 \tilde{C}(A(\rho) - G_l(\rho)\tilde{C})\tilde{e} - 2\tilde{e}_y^T Y_3 \tilde{C}\tilde{D}(\rho)(\nu_F - \tilde{\xi}) \\ &= 2\tilde{e}_y^T Y_3 \tilde{C}(A(\rho) - G_l(\rho)\tilde{C})\tilde{e} - 2\|\tilde{e}_y\| Y_3 \tilde{C}\tilde{D}(\rho)\eta(t) + 2\tilde{e}_y^T Y_3 \tilde{C}\tilde{D}(\rho)\tilde{\xi} \end{aligned} \tag{46}$$

From (43) and (45), it follows

$$\dot{V}_s \leq 2\|\tilde{e}_y\|\|Y_3\|\kappa_1 - \varepsilon_0\eta(t)\|\tilde{e}_y\| + 2\|\tilde{e}_y\|\|\tilde{C}\tilde{D}(\rho)\|\|Y_3\|\|\tilde{\xi}\| \tag{47}$$

Provided the modulation gain in (8) satisfies

$$\eta(t) \geq \frac{1}{\varepsilon_0}(\eta_0 + 2\|Y_3\|\kappa_1 + 2\|\tilde{C}\tilde{D}(\rho)\|\|Y_3\|\alpha(t,y)) \tag{48}$$

for some positive scalar η_0, from (47)

$$\dot{V}_s \leq -\eta_0\|\tilde{e}_y\| \leq -\tilde{\eta}_0\sqrt{V_s} \tag{49}$$

where $\tilde{\eta}_0 = \eta_0/\sqrt{\lambda_{\max}(Y_3)}$. From (49), it follows $V_s \equiv 0$ in finite time and therefore sliding on $\tilde{e}_y \equiv 0$ takes place in finite time.

Remark 5: The most significant difference between Theorem 1 and Theorem 2 is the structure imposed in (36) and the LMI in (38). These additional constraints not only guarantee asymptotic stability of the state estimation error $\tilde{e}(t)$ but also the existence of a sliding mode on $\tilde{e}_y \equiv 0$. The existence of a sliding motion allows the disturbance ξ to be estimated.

Remark 6: Notice that, if the modulation gain $\eta(t)$ in (48) is chosen as a sufficiently large constant value, sliding on $\tilde{e}_y \equiv 0$ can still take place in finite time.

During sliding, $\dot{\tilde{e}}_y = \tilde{e}_y = 0$. Consequently from (42) the sliding motion is governed by

$$0 = \tilde{C}(A(\rho) - G_l(\rho)\tilde{C})\tilde{e} - \tilde{C}\tilde{D}(\rho)(\nu_{eq} - \tilde{\xi}) \tag{50}$$

where ν_{eq} is the equivalent injection required to maintain sliding [24]. However $\tilde{e}(t) \to 0$ as $t \to \infty$ and therefore

$$\tilde{C}\tilde{D}(\rho)(\nu_{eq} - \tilde{\xi}) = 0 \tag{51}$$

Since from (45) the square matrix $\tilde{C}\tilde{D}(\rho)$ has the positive eigenvalues for all $\rho \in \Omega$, $\det(\tilde{C}\tilde{D}(\rho)) \neq 0$ and therefore during sliding

$$v_{eq} \to \tilde{\xi} \tag{52}$$

Partition $v_{eq} = \mathrm{col}(v_{eq,1}, v_{eq,2})$ where $v_{eq,1} \in \mathbb{R}^q$ and $v_{eq,2} \in \mathbb{R}^{p-q}$. It follows from (4) that

$$v_{eq,1} \to \xi \tag{53}$$

i.e. $v_{eq,1}$ is an asymptotic estimate of the disturbance.

IV. Design example

The short period dynamics of a civilian aircraft is considered in this section to demonstrate the previous theoretical developments [7]. For an LPV perspective, the scheduling parameters are weight, center of gravity, calibrated airspeed and Mach number. In this section all the scheduling parameters have been scaled to belong to the interval $[-1\ \ 1]$. The variation rates of the scheduling parameters $(\dot{\rho})$ are assumed to belong to the interval $[-0.1\ \ 0.1]$. In what follows, the model states are angle of attack, pitch rate and pitch angle, and the measured outputs are pitch rate and pitch angle. Therefore,

$$C = \begin{bmatrix} 0_{2\times 1} & I_2 \end{bmatrix} \tag{54}$$

In the simulations, the external disturbance is considered to be a sinusoidal head wind with an amplitude of 16kts. The disturbance begins 30sec after the start of the simulation and persists until the end of the simulation. The system matrix $A(\rho) = A_0 + \sum_{i=1}^{4} A_i \rho_i$ and the disturbance distribution matrix $D(\rho) = D_0 + \sum_{i=1}^{4} D_i \rho_i$ where A_i and D_i for $i = 1, \ldots, 4$ are given in [7].

$$A_0 = \begin{bmatrix} -0.8094 & 0.9531 & 0 \\ -1.2527 & -1.3151 & 0 \\ 1.0000 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0.1778 & 0.0086 & 0 \\ 0.0999 & 0.1948 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.0160 & 0.0015 & 0 \\ 0.3214 & 0.0068 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} -0.1716 & -0.0059 & 0 \\ -0.1329 & -0.2576 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0.1497 & 0.0157 & 0 \\ -0.0958 & 0.2331 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_0 = \begin{bmatrix} 0.0467 \\ -0.0266 \\ 0 \end{bmatrix}, D_1 = \begin{bmatrix} 0.0009 \\ -0.0019 \\ 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0.0002 \\ 0.0038 \\ 0 \end{bmatrix}, D_3 = \begin{bmatrix} -0.0044 \\ -0.0021 \\ 0 \end{bmatrix}, D_4 = \begin{bmatrix} -0.0144 \\ 0.0103 \\ 0 \end{bmatrix}$$

The matrix F from (32) is selected as $F = \mathrm{diag}([-0.027, 1])$. Applying Theorem 2 to above system yields a feasible solution to (36)-(38). The individual components associated with the gain $G_l(\rho)$ are

$$G_{l0} = \begin{bmatrix} -5.793 & -0.403 \\ 200.622 & -57.157 \\ -61.443 & 36.323 \end{bmatrix}, G_{l1} = \begin{bmatrix} -0.331 & 0.180 \\ 0.143 & -1.155 \\ 0.146 & 0.462 \end{bmatrix}, G_{l2} = \begin{bmatrix} -0.178 & 0.123 \\ -0.336 & -0.441 \\ 0.090 & 0.202 \end{bmatrix}$$

$$G_{l3} = \begin{bmatrix} 0.760 & -0.197 \\ -0.239 & 0.958 \\ -0.072 & -0.361 \end{bmatrix}, G_{l4} = \begin{bmatrix} 2.164 & 0.328 \\ -1.916 & 0.465 \\ 0.078 & -0.221 \end{bmatrix}$$

The components of $M(\rho)$ are

$$M_0 = \begin{bmatrix} -0.004 \\ -0.049 \\ 0.044 \end{bmatrix}, M_1 = \begin{bmatrix} -0.002 \\ 0.003 \\ -0.000 \end{bmatrix}, M_2 = \begin{bmatrix} -0.002 \\ 0.007 \\ -0.001 \end{bmatrix}, M_3 = \begin{bmatrix} 0.001 \\ -0.007 \\ 0.001 \end{bmatrix}, M_4 = \begin{bmatrix} -0.000 \\ -0.007 \\ 0.001 \end{bmatrix}$$

and the components of the parameter dependent Lyapunov function are

$$P_0 = \begin{bmatrix} 4.709 & 2.086 & -5.037 \\ 2.086 & 308.366 & 6.577 \\ -5.037 & 6.577 & 344.467 \end{bmatrix}, P_1 = \begin{bmatrix} -0.698 & -1.167 & -1.269 \\ -1.167 & -1.419 & -0.582 \\ -1.269 & -0.582 & 2.112 \end{bmatrix}, P_2 = \begin{bmatrix} 0.055 & -1.001 & -1.334 \\ -1.001 & 0.424 & 1.457 \\ -1.334 & 1.457 & 2.960 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0.675 & 1.136 & 1.179 \\ 1.136 & -0.386 & -1.306 \\ 1.179 & -1.306 & -3.423 \end{bmatrix}, P_4 = \begin{bmatrix} 0.073 & 0.256 & -0.015 \\ 0.256 & -1.246 & -1.556 \\ -0.015 & -1.556 & -2.500 \end{bmatrix}$$

where the slack variable

$$Y = \begin{bmatrix} 2.304 & 3.196 & 3.565 \\ 0 & 2.992 & 4.665 \\ 0 & 4.665 & 14.984 \end{bmatrix}$$

Note the structure of Y above corresponds to the constraint imposed in (36). It can be further verified that $Y_3 > 0$. Then from Theorem 2 it follows the observer in (3) exhibits a sliding motion in finite time and robustly estimates the states, rejecting the effect of the wind.

In the simulations which follow a pilot sidestick input excitation is applied as shown in Figure 2. The performance

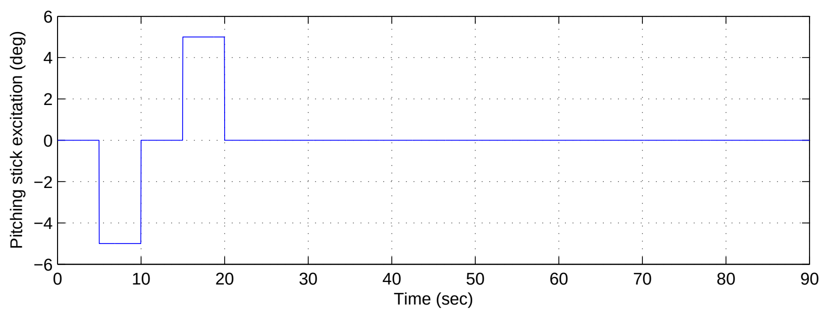

Fig. 2. Sidestick pilot input excitation

of the observer has been evaluated at one corner of the valid LPV region (i.e. ‘A’), as well at the centre ‘E’. Specific details are provided in Table I. In the simulations, the sampling rate is 100Hz and the discontinuity in the injection

TABLE I
TEST POINTS

flight point	weight	centre of gravity	calibrated airspeed	Mach number
A	-1	-1	-1	-1
E	-0.172	-1	0	-0.071

term in (8) has been smoothed by using a sigmoidal approximation $\eta(t,y)\frac{e_y}{\|e_y\|+\delta}$. Here, the modulation gain $\eta(t,y)$ has been chosen as 50 and the smoothing factor $\delta = 10^{-6}$. The observer initial conditions are deliberately chosen to differ from the plant initial conditions. In order to demonstrate the performance in the presence of measurement noise, it is assumed that Lebesgue measurable noise $0.005sin(50t)$ is injected into the pitch rate and pitch angle measurements from 60sec onwards. As shown in Figure 3, $\|\tilde{e}_y\|$ converges to zero in finite time (approx 5sec), and sliding is maintained at all the flight points tested, despite the presence of the head wind disturbance. Although noise is being injected from 60sec onwards, sliding can still be maintained and $\|\tilde{e}_y\|$ is close to zero. The system states (the pitch rate and the angle of attack) are shown in Figure 4 for the different flight points. The external disturbance and the estimates, at the various flight points are shown in Figure 5. It can be seen from Figure 5 that, after 60sec, the estimation performance is degraded due to the existence of measurement noise. As the observer initial conditions are different from the plant, small deviations appear at the beginning of the simulation and the observer converges in approx 5sec thereafter. The state estimation errors are shown in Figure 6 which shows estimation errors are close to zero and indicates good estimates of the states despite measurement noise.

V. CONCLUSION

This paper has considered the problem of numerically synthesising an LPV sliding mode observer. Whilst the structure of the observer is familiar, a new framework exploiting a small gain condition allows a wider class of LPV plants to be considered and, furthermore, enables parameter-dependent Lyapunov functions to be employed. As a consequence, the formulation in this paper is more generic than the comparable existing work in the literature. The effectiveness and practicality of the design scheme has been demonstrated on an LPV model of a commercial aircraft and good results have been achieved in terms of simultaneous estimation of the states and disturbances.

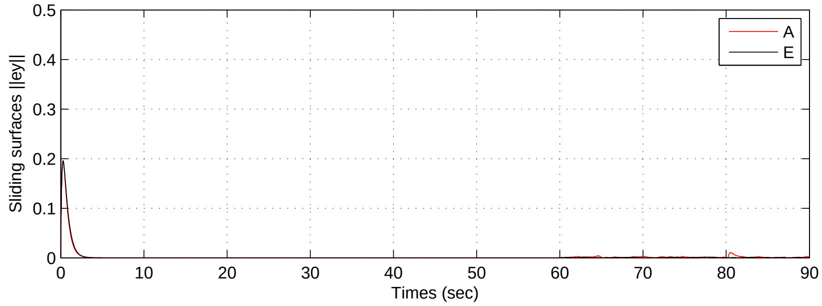

Fig. 3. Sliding surfaces on various flight points

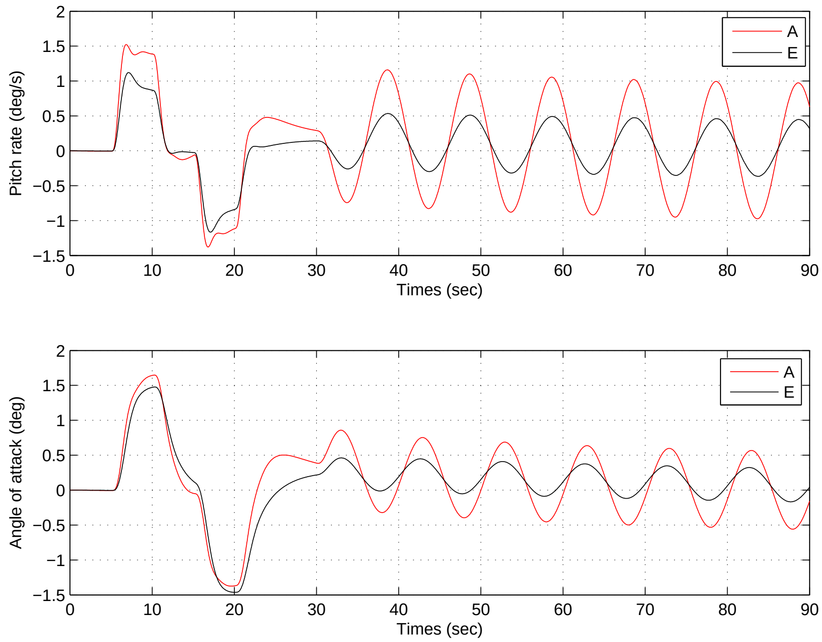

Fig. 4. Short period system states

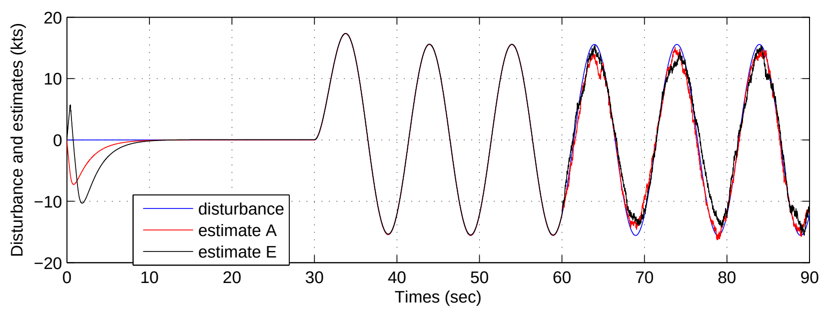

Fig. 5. Disturbance and their estimates

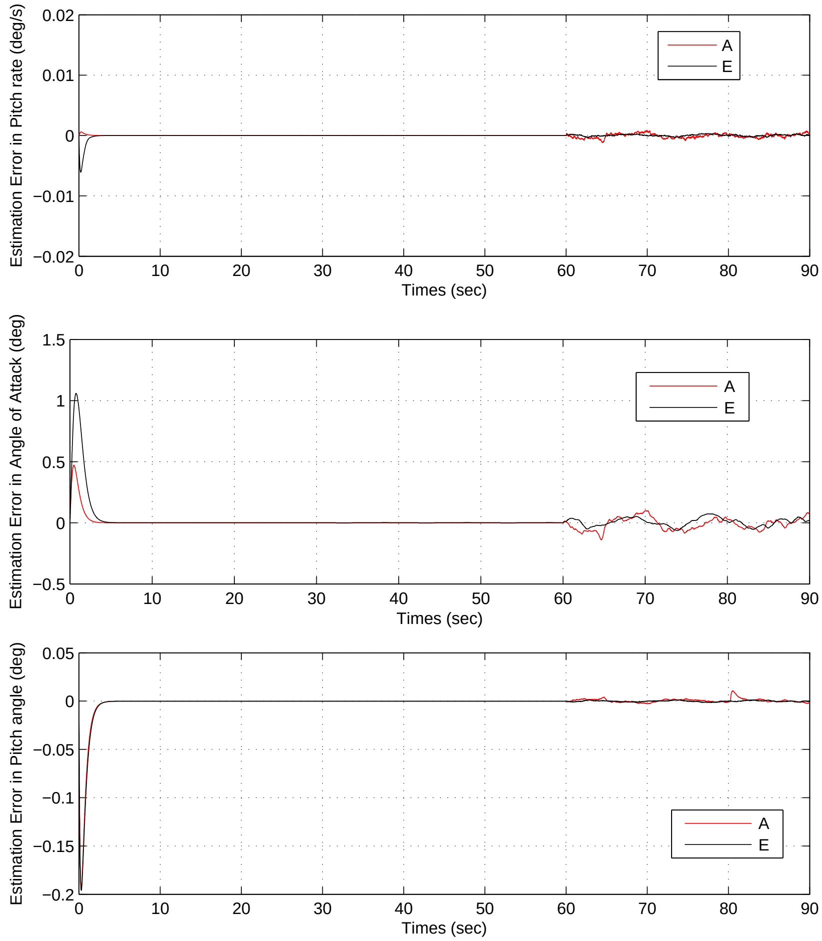

Fig. 6. State estimation errors

Appendix

Applying a loop transformation to Figure 1 as shown in Figure 7 creates a new memoryless function $\bar{\Phi}(\cdot) = \Phi(\cdot) - I$ and the new operator

$$\bar{H}(\rho) = \left[\begin{array}{c|c} A(\rho) - G_l(\rho)C - \tilde{D}(\rho)C & \tilde{D}(\rho) \\ \hline C & 0 \end{array}\right] \tag{55}$$

Applying a further loop transformation to the system representation in Figure 7 generates the feedback system shown in Figure 8 where

$$\tilde{H}(\rho) = \left[\begin{array}{c|c} A(\rho) - G_l(\rho)C - \tilde{D}(\rho)C & \tilde{D}(\rho) \\ \hline 2C & -I \end{array}\right] \tag{56}$$

From Figure 8, it follows the new memoryless function $\tilde{\Phi}(\cdot)$ satisfies

$$z = -\frac{1}{2}\tilde{\Phi}(z) + \frac{1}{2}\tilde{e}_y \tag{57}$$

Since by construction $\tilde{\Phi}(z) = \Phi(z) - z$, it follows by substituting for z from equation (57) that

$$\tilde{\Phi}(z) = \Phi(z) + \frac{1}{2}\tilde{\Phi} - \frac{1}{2}\tilde{e}_y \tag{58}$$

and then from equation (58)

$$\Phi(z) = \frac{1}{2}\tilde{\Phi}(z) + \frac{1}{2}\tilde{e}_y \tag{59}$$

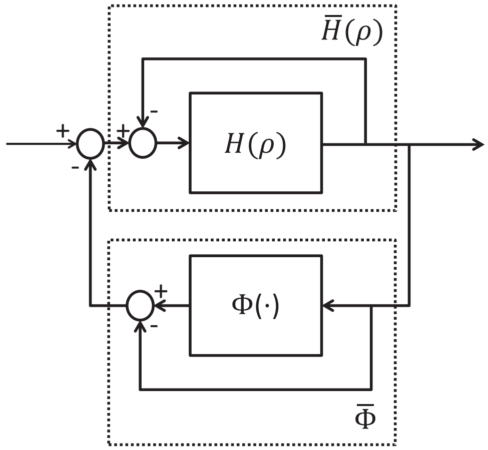

Fig. 7. $\Phi(\cdot)$ is transformed into a memoryless function $\bar{\Phi}(\cdot)$

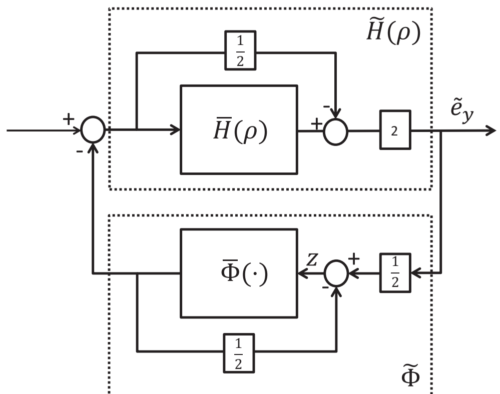

Fig. 8. Function $\bar{\Phi}(\cdot)$ is transformed into a memoryless function $\tilde{\Phi}(\cdot)$ where the L_2 gain of $\tilde{\Phi}$ is smaller or equal to 1

Since it has been demonstrated that $\Phi(\cdot)$ belongs to the sector $[0,\infty)$, $z^T\Phi(z) \geq 0$. By substituting for z from (57) and $\Phi(z)$ from (59), it follows that

$$z^T\Phi(z) = -\frac{1}{2}(\tilde{\Phi} - \tilde{e}_y)^T\frac{1}{2}(\tilde{\Phi} + \tilde{e}_y) \geq 0 \tag{60}$$

Expanding the right hand side of (60) means

$$\tilde{\Phi}^T\tilde{\Phi} - \tilde{e}_y^T\tilde{e}_y \leq 0 \tag{61}$$

and hence

$$\|\tilde{\Phi}(\tilde{e}_y)\|_2 \leq \|\tilde{e}_y\|_2 \tag{62}$$

It follows from (62) that the $\mathscr{L}_2$ gain of $\tilde{\Phi}$ with respect to $\tilde{e}_y$ is less than or equal to 1. As a consequence, using the small gain theorem [17], if the gains $G_l(\rho)$ and $M(\rho)$ can be synthesized to ensure $\|\tilde{H}(\rho)\|_2 \leq 1$, the feedback interconnection system in Figure 8 is asymptotically stable, and hence the original system in Figure 1 is also asymptotically stable (and the error $e(t)$ is asymptotically stable).

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