Modular Forms and Modular Symbols over Imaginary Quadratic Fields

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Submitted by

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I certify that all material in this thesis that is not my own work has been identified and that no material is included for which a degree has previously been conferred upon me.

Jeremy S. Bygott, September 1998

Abstract

The aim of this thesis is to contribute to the understanding of cusp forms over number fields, primarily over imaginary quadratic fields, from both a theoretical and a computational point of view.

There is already a deep theory of automorphic forms over general global fields k which arose out of trying to generalise the classical case $k = \mathbb{Q}$. The sophisticated approach (Jacquet-Langlands theory) is via the representations of the adele group $G_{\mathbb{A}}$ of $\mathrm{GL}(2)$; following instead Weil's "elementary" book [Wei71], we define cusp forms of weight two for $\Gamma_0(\mathfrak{n})$ as certain functions on $G_{\mathbb{A}}$. When k has r real embeddings, s pairs of complex embeddings and class number h, the upper half-plane of the classical theory must be replaced by h copies of the product of r upper half-planes and s upper half-spaces; when k is imaginary quadratic, we obtain an especially concrete description, which we work out in detail. In the general theory, Hecke operators are usually introduced via double cosets; in the special case, we can give a "classical" description in terms of lattices.

The main motivation for the work in this thesis comes from the theory of elliptic curves, in which an analogue of the Taniyama-Weil conjecture predicts that every elliptic curve of conductor $\mathfrak{n}$ defined over a number field k should (usually) be attached to a newform at level $\mathfrak{n}$. The existing theory in the classical case is especially rich; in particular, there are good computational techniques for finding newforms and their Hecke eigenvalues, and for determining the associated (strong Weil) curve [Cre97].

Cremona [Cre81] and his student Whitley [Whi90] began the programme of trying to extend these techniques to the case of imaginary quadratic fields, treating the case $h = 1$. This thesis describes an algorithm for determining the space of cusp forms and for computing the eigenforms and eigenvalues for the action of the Hecke algebra on this space in the case $h = 2$. The approach, using modular symbols, closely follows the work of Cremona and Whitley, but new features arise from the presence of a non-trivial class group. The methods presented here suffice for $h = 2$ and probably for an elementary abelian 2-group; the general case remains open but promising. Results from an implementation of the algorithm in the case $k = \mathbb{Q}(\sqrt{-5})$ form part of this thesis.

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