Modular Forms and Elliptic Curves over Imaginary Quadratic Number Fields

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I certify that all the material in this thesis which is not my own work has been identified and that no material is included for which a degree has previously been conferred upon me.

Elise Whitley October 1990

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CONTENTS

ABSTRACT

The motivation for this thesis is two-fold.

First we investigate the correspondence between elliptic curves with conductor $\mathbf{a}$ and newforms of weight 2 for $\Gamma_0(\mathbf{a})$, where $\mathbf{a}$ is an ideal of ϑ_K and K is one of the 4 non-Euclidean imaginary quadratic number fields with class number 1. In Part I we develop an algorithm for finding rational newforms by calculating the action of the Hecke algebra on the first rational homology group of the hyperbolic upper half-space modulo $\Gamma_0(\mathbf{a})$. This work is an extension of Cremona's work [4] on modular forms over the 5 Euclidean fields.

We give tables of the results of implementing this algorithm on a computer. We list the dimensions of the $+1$ eigenspaces for the action of J on $H_1(\Gamma_0(\mathbf{a})\backslash\mathcal{H}_3^*, \mathbf{Q})$ along with the first few Hecke eigenvalues for each of the rational newforms. In addition we give tables of elliptic curves with small conductor, found via a systematic computer search using Tate's algorithm, and the trace of Frobenius at the first few primes. In all cases agreement was found in the Hecke eigenvalues and trace of Frobenius at the first 15 primes.

Secondly we provide extensive numerical evidence to support the Birch, Swinnerton-Dyer Conjecture. Part II is a description of joint work carried out with Cremona to calculate the quantities involved. We give tables of the results of these calculations over the 9 imaginary quadratic number fields with class number 1. We provide isogeny classes of curves of given conductor along with the order of the group of torsion points defined over K; the $c_{\mathfrak{p}}$ numbers; and the complex period of each curve. For each of the newforms corresponding to a class of elliptic curves without complex multiplication, we calculate the ratio $L(F,1)/\pi(F)$ where $L(F,1)$ is the value of the L-series of the newform, F, at $s = 1$ and $\pi(F)$ is the period. In the cases where $L(F,1)/\pi(F) \neq 0$ we list the values of $L(F,1)$ and $\pi(F)$. In the majority of cases we find agreement in the quantities predicted in the conjecture.

INTRODUCTION

The work in this thesis falls into two distinct parts.

Part I is an extension of work carried out by Cremona in [4] where he considers an analogue of Weil's conjecture, that every elliptic curve defined over $\mathbf{Q}$ is paramatized by modular functions, for complex quadratic number fields. He develops an method for calculating automorphic forms over complex quadratic fields and, in particular, he carries out the calculations for the five Euclidean fields, ie $\mathbf{Q}(\sqrt{-d})$ where $d \in \{1,2,3,7,11\}$. We will be concerned with extending his work further to the four non-Euclidean fields with unique factorisation, ie $\mathbf{Q}(\sqrt{-d})$ where $d \in \{19,43,67,163\}$.

The general method described in [4] extends to the non-Euclidean fields although several stages are not immediate. It is those stages which will concern us primarily in Part I of this thesis.

We will begin with a general introduction to the theory of elliptic curves and modular forms over imaginary quadratic fields. This will be the subject of Chapter 0 and will also be used in Part II of this thesis.

The contents of Part I, then, are as follows:

Chapter 1 will be a synopsis of Cremona's work in the five Euclidean fields and certain aspects of this will be extended in Chapters 2,3 and 4. Specifically, Chapter 2 deals with the problem of determining a fundamental region for the action of $SL(2,\vartheta_{\mathrm{K}})$ on the 3-dimensional hyperbolic upper half space, where K is one of the four non-euclidean, class number 1 fields, using a method due to Swan [17]; these fundamental regions are used in Chapter 3 where we discuss the determination of a tessellation of $\mathcal{H}_3^*$ by hyperbolic polyhedra and this tessellation is used in Chapter 4 where $H_1(G\backslash\mathcal{H}_3^*, \mathbf{Q})$ is calculated for some subgroup, G, of finite index in $SL(2, \vartheta_{\mathrm{K}})$. Chapter 5 consists mainly of tables of the results of computer programs, written to carry out the algorithms described in the previous chapters.

Part II is a description of joint work, carried out with Cremona, to provide numerical evidence to support the Birch, Swinnerton-Dyer Conjecture over the nine imaginary quadratic number fields with class number 1. This is an extension of work carried out by Cremona in [5], where he provides numerical support for the conjecture over $\mathbf{Q}$. Again, not all of his methods extend immediately to the imaginary case and Chapter 1 is a description of the calculations which have been performed. The results of these calculations, implemented on a computer, are listed in Chapter 2.

Chapter 0: Modular Forms and Elliptic Curves over Imaginary Quadratic Number Fields

This chapter will be an introduction to the theory of modular forms and elliptic curves over imaginary quadratic number fields. I will not attempt to provide an exhaustive survey of these subjects; instead I will only give details of those aspects which will be used in the remainder of this thesis.

§0.1 introduces the notion of elliptic curves over imaginary quadratic number fields and certain quantities associated to them. The main references for this section are [9] and [16]. In §0.2 I will introduce the action of $SL(2,\vartheta_{\mathrm{K}})$ on the 3-dimensional upper half space, while some special subgroups of $SL(2,\vartheta_{\mathrm{K}})$ are the subject of §0.3. These subgroups are used in §0.4, where I define cusp forms of weight 2 for them. The Hecke action on these cusp forms is described in §0.5 and homology is introduced in §0.6 along with a new notation. The main reference for §0.2 – §0.6 is [4]. §0.7 is concerned with the connections between these cusp forms and elliptic curves, and outlines the rest of this thesis.

§0.1 Elliptic Curves over Imaginary Quadratic Number Fields

Let K be an imaginary quadratic number field with ring of integers ϑ_{K} and group of units $\vartheta_{\mathrm{K}}^{*}$. Define E, an **elliptic curve** defined over K. That is, an irreducible, non-singular algebraic curve of genus 1, along with a distinguished point, both defined over K. E has the general form:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in \vartheta_K, \tag{0.1.1}$$

where the distinguished point is the point at ∞.

Define

$$\begin{aligned}
b_2 &= a_1^2 + 4a_2, \\
b_4 &= a_1a_3 + 2a_4, \\
b_6 &= a_3^2 + 4a_6, \\
b_8 &= a_1^2a_6 - a_1a_3a_4 + 4a_2a_6 + a_2a_3^2 - a_4^2, \\
c_4 &= b_2^2 - 24b_4, \\
c_6 &= -b_2^3 + 36b_2b_4 - 216b_6,
\end{aligned} \tag{0.1.2}$$

and the **discriminant** of E,

$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6.$$

In the case where the class number, $h(K)$, is 1 the a_i are chosen to ensure that the norm of Δ, $N(\Delta)$, is minimal. The non-singularity of E is equivalent to $\Delta \neq 0$.

Define the standard differential:

$$\omega = \frac{dx}{2y + a_1x + a_3} = \frac{dy}{3x^2 + 2a_2x + a_4 - a_1y}.$$

Now fix a prime $\wp$ of ϑ_K. We define $\overline{E} = E_\wp$, the **reduction** of E modulo $\wp$, to be :

$$y^2 + \overline{a_1}xy + \overline{a_3}y = x^3 + \overline{a_2}x^2 + \overline{a_4}x + \overline{a_6}, \tag{0.1.3}$$

where $\overline{a_i}$ is the reduction of a_i modulo $\wp$.

Then $E_\wp$ is defined over $\vartheta_K/\wp$.

$\overline{E}$ is an elliptic curve $\Leftrightarrow \overline{\Delta} \neq 0$.

If $\text{ord}_\wp(\overline{\Delta}) > 0$ then $\overline{E}$ is **singular** and we say that E has **bad reduction** at $\wp$.

The reduction is said to be $\begin{cases} \textbf{multiplicative} & \text{if } \overline{E} \text{ has a node,} \\ \textbf{additive} & \text{if } \overline{E} \text{ has a cusp.} \end{cases}$

If $\overline{E}$ has multiplicative reduction then the reduction is said to be:

$$\begin{cases} \textbf{split} & \text{if the slopes of the tangents at the node are in } \vartheta_K/\wp, \\ \textbf{non-split} & \text{otherwise.} \end{cases}$$

The **conductor** of E, f_E, is the ideal of ϑ_K defined by:

$$f_E := \prod_{\wp \in \vartheta_K} \wp^{f_\wp}, \tag{0.1.4}$$

where

$$f_\wp = \begin{cases} 0 & \text{if } E_\wp \text{ is non-singular,} \\ 1 & \text{if } E_\wp \text{ has multiplicative reduction,} \\ 2+\delta_\wp & \text{if } E_\wp \text{ has additive reduction.} \\ & (\text{where } \delta_\wp = 0 \text{ if } \text{char}(K_\wp) \neq 2,3) \end{cases}$$

An elliptic curve, E, is an abelian group, with zero element = the point at ∞, and group operation defined as follows [9]:

Given 2 points $P, Q \in E$, define the point $PQ \in E$ to be the third point of intersection of E and the line passing through P, Q. This will make sense when $P = Q$ as we consider the line tangent to E at P to pass through P twice. Now define $P + Q$ to be the third point of intersection of E and the vertical line passing through PQ and the point at ∞. This "addition" is our group law with respect to which E is an abelian group.

We can now define the **order** of a point P with respect to the group operation in the usual way.

Denote the set of K-rational points on E by $E(\mathrm{K})$, and the torsion part of $E(\mathrm{K})$ by $E_{tors}(\mathrm{K})$. Then the Mordell-Weil Theorem [16] states that $E(\mathrm{K})$ is finitely generated. From this it can be seen that

$$E(\mathrm{K}) \cong E_{tors}(\mathrm{K}) \times \mathbf{Z}^r,$$

where r is a non-negative integer and is known as the **rank** of E. We will consider $E_{tors}(\mathrm{K})$ further in Part II of this thesis.

A **morphism** between 2 elliptic curves E_1 and E_2, both defined over the same field K, is a map $\phi : E_1 \rightarrow E_2$, also defined over K, which preserves the group structure.

An **isogeny** between E_1 and E_2 is a non-constant morphism

$$\phi : E_1 \longrightarrow E_2.$$

E_1 and E_2 are said to be **isogenous** if there is an isogeny, ϕ, between them. Two isogenous elliptic curves have the same conductor.

So, to each elliptic curve with given conductor there is associated a class of curves, all isogenous to it. We will call this the **isogeny class** of curves of that conductor.

Because elliptic curves are groups, the maps between them also form groups. Thus

$$\mathrm{Hom}(E_1, E_2) = \{\text{isogenies } \phi : E_1 \longrightarrow E_2\}$$

is a group with the addition law $(\phi + \psi)(P) = \phi(P) + \psi(P)$.

If $E_1 = E_2$ we can compose isogenies. Thus

$$\mathrm{End}(E) = \mathrm{Hom}(E, E).$$

End(E) is a ring with addition as above and multiplication given by:

$$(\phi\psi)(P) = \phi(\psi(P)).$$

If End(E) is strictly larger than $\mathbf{Z}$ then E is said to have **complex multiplication.**

For each prime ideal, $\wp$, at which $E_\wp$ is non-singular, we can define a **local L-series** to be:

$$L_\wp(s) := 1 - c(\wp)s + N(\wp)s^2, \tag{0.1.5}$$

where the **trace of Frobenius,** $c(\wp) = N(\wp) + 1 - \#(E_\wp)$, and $\#(E_\wp)$ is the number of points on the reduced curve.

If $E_\wp$ is singular then define:

$$L_\wp(s) := \begin{cases} 1 - s & \text{if the reduction is split multiplicative,} \\ 1 + s & \text{if the reduction is non-split multiplicative,} \\ 1 & \text{if the reduction is additive.} \end{cases} \tag{0.1.5'}$$

Now define the **global zeta function** or **L-series**, obtained by taking the product of the local L-series over all prime ideals, $\wp$, of ϑ_{K}, to be:

$$L(E,s) := \zeta_E(s) := \prod_{\wp \triangleleft \vartheta_{\mathrm{K}}} L_\wp(N(\wp)^{-s}). \tag{0.1.6}$$

Define

$$\xi_E(s) := |D_{\mathrm{K}}|^s (N(f_E))^{\frac{1}{2}s} (2\pi)^{-2s} (\Gamma(s))^2 \zeta_E(s), \tag{0.1.7}$$

where D_{K} is the absolute discriminant of K, and Γ is the Gamma Function. This is conjectured to satisfy the functional equation

$$\xi_E(s) = \pm \xi_E(2 - s).$$

Finally, it will be useful to us to write $L(E,s)$ as a sum:

$$L(E,s) = \sum_{\mathbf{a} \lhd \vartheta_K} c(\mathbf{a}) N(\mathbf{a})^{-s}, \tag{0.1.8}$$

where the summation is over all non-zero ideals $\mathbf{a}$ of ϑ_K, $c(\mathbf{a}) \in \mathbf{Z}$ and, for prime ideals $\wp$ of ϑ_K, $c(\wp)$ is the trace of Frobenius.

§0.2 $\mathcal{H}_3^*$ and some Hyperbolic Geometry.

Define the 3-dimensional upper half space by:

$$\mathcal{H}_3 := \{(z,\zeta) : z \in \mathbf{C}, \zeta \in \mathbf{R}, \zeta > 0\}. \tag{0.2.1}$$

We define the **extended** 3-dimensional upper half space to be

$$\mathcal{H}_3^* := \mathcal{H}_3 \cup \mathrm{K} \cup \{\infty\}.$$

The points $\{(k,0) : k \in \mathrm{K}\}$ and the point at infinity, ∞, are called **cusps**.

The geometry of $\mathcal{H}_3^*$ is hyperbolic. So geodesic lines are vertical half-lines and semi-circles centred on the plane $\{(z,\zeta) : \zeta = 0\}$ while geodesic surfaces are vertical half-planes and hemispheres centred on $\{(z,\zeta) : \zeta = 0\}$.

There is a hyperbolic structure defined on $\mathcal{H}_3$ by

$$ds^2 = \frac{dx^2 + dy^2 + d\zeta^2}{\zeta^2},$$

where $z = x + iy$.

We will be interested in the action of the group $GL(2,\vartheta_K)$, and certain subgroups of it, on $\mathcal{H}_3^\star$.

$GL(2,\mathrm{C})$ acts on $\mathcal{H}_3^\star$ according to:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z,\zeta) \mapsto (z',\zeta'),$$

where

$$z' = \frac{(az+b)(\overline{cz+d}) + (a\zeta)(\overline{c\zeta})}{|cz+d|^2 + |c\zeta|^2},$$

$$\zeta' = \frac{|ad-bc|\zeta}{|cz+d|^2 + |c\zeta|^2}. \tag{0.2.2}$$

Alternatively, if we restrict our attention to $SL(2,\vartheta_K)$, we can write $q = z + \zeta j$ as a quaternion and then (0.2.2) becomes:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : q \mapsto (aq+b)(cq+d)^{-1}. \tag{0.2.2'}$$

(0.2.2) obviously makes sense when $\zeta = 0$, as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z,0) \mapsto (z',0),$$

where

$$z' = \frac{az+b}{cz+d}. \tag{0.2.3}$$

The elements of $GL(2,\vartheta_K)$ map cusps to cusps; in particular:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{cases} \infty & \mapsto \frac{a}{c}, \\ -\frac{d}{c} & \mapsto \infty. \end{cases} \tag{0.2.4}$$

We will define here, the action of 4 particular matrices in $GL(2,\vartheta_K)$ which we will use later. If $\mathrm{K} = \mathrm{Q}(\sqrt{-d})$ then define

$$\omega := \begin{cases} \sqrt{-d} & \text{if } d \equiv 1,2 \pmod 4, \\ \frac{\sqrt{-d}+1}{2} & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

so that $\vartheta_K = \mathbf{Z} + \mathbf{Z}\omega$.

Then we have

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : (z, \zeta) \mapsto (z + 1, \zeta); \tag{0.2.5}$$

$$U := \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} : (z, \zeta) \mapsto (z + \omega, \zeta); \tag{0.2.6}$$

$$J := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} : (z, \zeta) \mapsto (-z, \zeta); \tag{0.2.7}$$

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : (z, \zeta) \mapsto (\frac{-\bar{z}}{|z|^2 + |\zeta|^2}, \frac{-\zeta}{|z|^2 + |\zeta|^2}). \tag{0.2.8}$$

By abuse of notation we can write $U = T^{\omega}$. More generally, for $\alpha \in \vartheta_K$, we write

$$T^{\alpha} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : (z, \zeta) \mapsto (z + \alpha, \zeta).$$

The topology of $\mathcal{H}_3$ is induced by the invariant metric and is identical to the Euclidean topology of $\mathcal{H}_3$ as a subset of $\mathbf{R}^3$. We can extend this to $\mathcal{H}^3 \cup \mathbf{C} \cup \{\infty\}$ as follows:

Given a point $\alpha \in \mathbf{C}$, define a basis of open neighbourhoods for it to be the set of all $S \cup \{\infty\}$, where S is an open sphere with boundary tangent to $\mathbf{C}$ at α. Also, the collection of sets $\{(z, \zeta) : \zeta > \zeta_0\} \cup \{\infty\}$, $\forall \zeta_0 > 0$ is a basis of open neighbourhoods at ∞.

§0.3 Congruence Subgroups

In this section we define a particular type of subgroup of $SL(2,\vartheta_K)$ which we will be interested in later. Let $\Gamma := SL(2, \vartheta_K)$ for some fixed number field K and let $\mathbf{a}$ be some non-zero ideal of ϑ_K. Then we define the **principal congruence subgroup of Γ, of level a,** to be:

$$\Gamma(\mathbf{a}) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a-1, b, c, d-1 \in \mathbf{a}\}. \tag{0.3.1}$$

Thus $\Gamma(\mathbf{a})$ is the group of matrices $\gamma \in \Gamma$ such that γ is "congruent to I modulo $\mathbf{a}$".

The index of $\Gamma(\mathbf{a})$ in Γ is:

$$[\Gamma : \Gamma(\mathbf{a})] = N(\mathbf{a})^3 \prod_{\wp | \mathbf{a}} (1 - N(\wp)^{-2}). \tag{0.3.2}$$

where $N(\mathbf{a})$ = norm of $\mathbf{a}$ and the product is over all prime ideals, $\wp$, of ϑ_K which divide $\mathbf{a}$. (See §2.5, [4])

Next we define:

$$\Gamma_0(\mathbf{a}) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \in \mathbf{a}\}. \tag{0.3.3}$$

The index of $\Gamma_0(\mathbf{a})$ in Γ is (§2.5, [4]):

$$[\Gamma : \Gamma_0(\mathbf{a})] = N(\mathbf{a}) \prod_{\wp | \mathbf{a}} (1 + N(\wp)^{-1}). \tag{0.3.4}$$

$\Gamma_0(\mathbf{a})$ is a subgroup of Γ which contains $\Gamma(\mathbf{a})$ normally. The index of $\Gamma(\mathbf{a})$ in $\Gamma_0(\mathbf{a})$ is (§2.5, [4]):

$$[\Gamma_0(\mathbf{a}) : \Gamma(\mathbf{a})] = N(\mathbf{a})^2 \prod_{\wp | \mathbf{a}} (1 - N(\wp)^{-1}). \tag{0.3.5}$$

Generally, a **congruence subgroup** of Γ is one which contains $\Gamma(\mathbf{a})$ for some non-zero ideal $\mathbf{a}$ of ϑ_K.

We will want to know when 2 cusps are equivalent under the action of $\Gamma_0(\mathbf{a})$, some non-zero ideal $\mathbf{a} \in \vartheta_K$. We do this according to the following:

Lemma 0.3.1: ([4])

Let $\frac{p_1}{q_1}$ *and* $\frac{p_2}{q_2}$ *be elements of* K *written in their lowest terms,*

ie $(p_1, q_1) = (p_2, q_2) = \vartheta_{\mathrm{K}}$.

Then the following are equivalent:

(i) $\exists \gamma \in \Gamma_0(\mathbf{a})$ *such that* $\gamma(\frac{p_1}{q_1}) = \frac{p_2}{q_2}$

(ii) $\exists u \in \vartheta_{\mathrm{K}}^{\star}$ *such that* $s_1 q_2 \equiv u^2 s_2 q_1 \pmod{(q_1 q_2) + \mathbf{a}}$,

where $p_k s_k \equiv 1 \ (mod\ q_k), k = 1, 2$.

§0.4 Cusp Forms over K

We can express the hyperbolic upper half space, described in §0.2, as

$$\mathcal{H}_3 = GL(2, \mathrm{C})/\mathcal{Z}.SU(2),$$

where $\mathcal{Z}$ is the group of scalar matrices in $GL(2,\mathbf{C})$. A complete set of coset representatives for $\mathcal{Z}.SU(2)$ in $GL(2, \mathbf{C})$ is given by the subgroup

$$B := \{\begin{pmatrix} \zeta & z \\ 0 & 1 \end{pmatrix} : z \in \mathbf{C}, \zeta \in \mathbf{R}, \zeta > 0\}.$$

We will identify B with $\mathcal{H}_3$ via:

$$\begin{pmatrix} \zeta & z \\ 0 & 1 \end{pmatrix} \leftrightarrow (z, \zeta).$$

Denote the projection from $G = GL(2, \mathbf{C})$ to $\mathcal{H}_3$ by π. Then the action of G on B is given by

$$g : b \mapsto \pi(gb).$$

$\mathcal{H}_3$ has the structure of a Riemannian metric symmetric space with metric

$$(ds)^2 = \frac{(dzd\bar{z} + (d\zeta)^2)}{\zeta^2}.$$

A basis for the 3-dimensional space of left-invariant differential forms on $\mathcal{H}_3$ is given by

$$\beta = (\beta_0, \beta_1, \beta_2) = (\frac{-dz}{\zeta}, \frac{d\zeta}{\zeta}, \frac{d\bar{z}}{\zeta}). \qquad (0.4.1)$$

It will be convenient to consider not only differential forms on $\mathcal{H}_3$, but also their pullback to G. For $i = 0, 1, 2$ let ω_i be the differential form on G which coincides with $\pi^*\beta_i$ (the pullback of β_i to G) at the identity. Right translations by elements of $\mathcal{Z}.SU(2)$ operate on the ω_i by means of a 3-dimensional polynomial representation ρ of $SU(2)$ which is trivial on $\mathcal{Z}$.

When $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$, say, where

$$|u|^2 + |v|^2 = 1.$$

Define

$$\rho(u, v) = \begin{pmatrix} u^2 & 2uv & v^2 \\ -\bar{u}v & u\bar{u} - v\bar{v} & \bar{u}v \\ \bar{v}^2 & -2\overline{uv} & \bar{u}^2 \end{pmatrix}.$$

Then, a differential form on G is the inverse image of one on $\mathcal{H}_3 \Leftrightarrow$ it can be written as $\sum_{i=0}^{2} \phi_i\omega_i$ where $\Phi = (\phi_0, \phi_1, \phi_2)$ satisfies:

$$\Phi(g\kappa\zeta) = \Phi(g)\rho(\kappa\zeta), \forall g \in G, \kappa \in SU(2), \zeta \in \mathcal{Z}.$$

Let f_i be the function induced on B by ϕ_i, $i = 0, 1, 2$ and write $F = (f_0, f_1, f_2)$. Let $\omega = F.\beta = \sum_{i=0}^{2} f_i\beta_i$ be a 1-form on $\mathcal{H}_3$. Recall that, on a Riemannian manifold V of dimension m, there exists a linear map from r-forms to $(m-r)$-forms, called the **adjoint** or $*$ - operator, with the following properties (due to de Rham [14]):

(i) $**\alpha = (-1)^{r(m+1)}\alpha$,

(ii) $\alpha \wedge *\beta = \beta \wedge *\alpha$,

(iii) $\alpha \wedge *\alpha = f dx_1 \wedge dx_2 \wedge \ldots \wedge dx_m$,

where $f \geq 0$, and $f = 0$ at exactly those points of V where α is zero,

(iv) $(\alpha, \beta) := \int_V \alpha \wedge *\beta$ is a scalar product.

Let d be the standard differentiation operator from r-forms to $(r+1)$-forms; then its transpose δ, with respect to the above inner product, is an operator of degree -1 given by $\delta = (-1)^{r-1} d*$. We say that a differential, β, is **closed** if $d\beta = 0$.

Definition 0.4.1:

The 1-form ω, defined above, is **harmonic** $\Leftrightarrow$ *ω and $*\omega$ are closed forms.*

Definition 0.4.2:

We say that F is **slowly increasing** *if $\exists N \geq 0$ such that*

$$|F(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}(z, \zeta))| = O(|x|^N)$$

as $x \to \infty$ uniformly over compact sets in $\mathcal{H}_3$, $x \in \mathbf{R}$.

We make the following important

Definition 0.4.3:

A function $F : \mathcal{H}_3 \to \mathbf{C}^3$ is said to be **harmonic** *if:*

(i) $F.\beta$ is a harmonic differential form,

(ii) F is slowly increasing.

Proposition 0.4.4: (Proposition 3.1.6, [4])

Let $\Phi : G \to \mathbf{C}^3$ satisfy

$$\Phi(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} g) = e^{-2\pi i(z+\bar{z})}\Phi g, \ \forall g \in G,$$

and suppose that it induces $F : \mathcal{H}_3 \to \mathbf{C}^3$ which is harmonic. Then

$$F(z,\zeta) = ce^{-2\pi i(z+\bar{z})}H(\zeta),$$

where c is a constant and

$$H(\zeta) = (-\frac{1}{2}i\zeta^2 K_1(4\pi\zeta), \zeta^2 K_0(4\pi\zeta), \frac{1}{2}i\zeta^2 K_1(4\pi\zeta)). \tag{0.4.2}$$

(K_0 and $K_1(s) = -\frac{d}{ds}(K_0(s))$ are Hankel's functions, [1]).

Given $F : \mathcal{H}_3 \to \mathbf{C}^3$, a harmonic function, and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we define a new function $F|g$ as follows:

Set $\Delta = ad - bc, \ r = \overline{cz + d}, \ s = \bar{c}\zeta$ and

$$J(g;(z,\zeta)) = \frac{1}{|\Delta|(|r|^2 + |s|^2)} \begin{pmatrix} r^2\Delta & -2rs\Delta & s^2\Delta \\ r\bar{s}|\Delta| & (r\bar{r} - s\bar{s})|\Delta| & -\bar{r}s|\Delta| \\ \bar{s}^2\overline{\Delta} & 2\overline{rs\Delta} & \overline{r^2\Delta} \end{pmatrix}.$$

Then $(F|g)(z,\zeta) = F(g(z,\zeta))J(g;(z,\zeta))$.

Note: The differential $F.\beta$ is invariant under $g \Leftrightarrow F|g = F$.

Now let K be a complex quadratic field with ring of integers ϑ_{K}, and Γ a discrete subgroup of $GL(2,\mathrm{K})$ containing all the translations $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \alpha \in \vartheta_{\mathrm{K}}$, eg. $\Gamma = \Gamma_0(\mathbf{a})$. Then, if the harmonic function F is invariant under Γ, we have $F(z+\alpha,\zeta) = F(z,\zeta), \forall \alpha \in \vartheta_{\mathrm{K}}$.

Fix ζ and consider F as a function of z alone. Then F has a Fourier expansion with respect to the characters of $\mathbf{C}^+$, the additive group, which are trivial on ϑ_{K}.

If ψ is any non-trivial character of $\mathbf{C}^+$ then, for any fixed $\omega \in \mathbf{C}$, the function $z \mapsto \psi(\omega z)$ is also a character and, in fact, all characters of $\mathbf{C}^+$ have this form.

Thus, $\mathbf{C}^+$ may be identified with its character group. To fix this identification we need to use a particular character ψ, namely,

$$\psi(z) := e^{2\pi i(z+\bar{z})}.$$

Write $\eta = \sqrt{D}$ where D is the discriminant of K. Then we have:

Proposition 0.4.5: (Proposition 3.2.12, [4])

Let $F : \mathcal{H}_3 \to \mathbf{C}^3$ *be a harmonic function, invariant under all matrices of the form* $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \alpha \in \vartheta_{\mathrm{K}}$.

Then F *has a Fourier expansion of the form :*

$$F(z,\zeta) = c_0(\zeta) + \sum_{\alpha \in \vartheta_{\mathrm{K}}} c(\alpha) H(\eta^{-1}\alpha\zeta)\psi(\eta^{-1}\alpha z), \tag{0.4.3}$$

where $c_0(\zeta) = \int_{\vartheta_{\mathrm{K}} \backslash \mathbf{C}} F(z,\zeta)dz$,

$c(\alpha)$ is a coefficient which depends on α,

$H(\zeta)$ is as defined in (0.4.2)

We now come to the main definition of this section:

Definition 0.4.6:

Let K *be a complex quadratic number field with* $h(\mathrm{K}) = 1$ *and ring of integers* ϑ_{K}.

Let Γ *be a subgroup of* $SL(2,\vartheta_{\mathrm{K}})$ *of finite index.*

Then an **automorphic form of weight 2 for Γ** *is a function* $F : \mathcal{H}_3 \to \mathbf{C}^3$ *satisfying:*

(i) F *is harmonic,*

(ii) $F|\gamma = F, \forall \gamma \in \Gamma$.

If, in addition, $\forall \sigma \in SL(2,\vartheta_{\mathrm{K}})$ *and all* $\zeta \geq 0, F$ *satisfies*

(iii) $\int_{\vartheta_{\mathrm{K}} \backslash \mathbf{C}} (F|\sigma)(z, \zeta) dz = 0$,

then F *is called a* **cusp form of weight 2 for Γ**.

Notes:

(1) We will only be concerned with forms of weight 2 so, from now on, we will omit the qualification.

(2) Cusp forms F for Γ correspond to harmonic differentials $F.\beta$ on the quotient space which are zero at the cusps of $\Gamma \backslash \mathcal{H}_3^\star$.

(3) The only part of Definition 0.4.6 which depends on Γ itself is (ii), the invariance condition. So if F is an automorphic form, or cusp form, for Γ, and F is also invariant under another subgroup Γ' of $SL(2,\vartheta_{\mathrm{K}})$, then F is a form for Γ'.

For some non-zero ideal $\mathbf{a}$ of ϑ_{K}, we denote the space of all cusp forms for $\Gamma_0(\mathbf{a})$ by $S(\mathbf{a})$.

§0.5 Hecke Operators

We now consider the action of certain operators on the space of cusp forms, $S(\mathbf{a})$, for a given ideal $\mathbf{a}$ in ϑ_K. These are known as **Hecke operators.** Let ε be a unit of ϑ_K and set $I_\varepsilon := \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$.

Clearly

$$I_\varepsilon : (z,\zeta) \mapsto (\varepsilon z, \zeta), \forall (z,\zeta) \in \mathcal{H}_3^\star, \tag{0.5.1}$$

since $|\varepsilon| = 1$.

Now, let F be a cusp form for $\Gamma_0(\mathbf{a})$. It has a Fourier expansion given by (0.4.3), so we can calculate the action of I_ε on F via the action on the Fourier coefficients $c(\alpha)$. Thus:

$$\begin{aligned}(F|I_\varepsilon)(z,\zeta) &= F(\varepsilon z, \varepsilon\zeta) \\ &= c_0(\varepsilon\zeta) + \sum_{\alpha\in\vartheta_K} c(\alpha\varepsilon^{-1}) H(\eta^{-1}\alpha\zeta)\psi(\eta^{-1}\alpha z).\end{aligned} \tag{0.5.2}$$

So F is invariant under $I_\varepsilon \Leftrightarrow$ the coefficients of F satisfy $c(\alpha) = c(\alpha\varepsilon), \forall \alpha \in \vartheta_K$.

Notice that the matrices $\begin{pmatrix} \varepsilon^2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$ are projectively equivalent and therefore give the same transformation of $\mathcal{H}_3^\star$; however, the latter is in $\Gamma_0(\mathbf{a})$, $\forall$ ideals $\mathbf{a}$ of ϑ_K and so we must have $F|I_\varepsilon^2 = F$. So the coefficients of F satisfy $c(\alpha) = c(\varepsilon^2\alpha), \forall$ units $\varepsilon \in \vartheta_K^\star$. Moreover, I_ε normalises $\Gamma_0(\mathbf{a})$, so $F|I_\varepsilon$ is invariant under $\Gamma_0(\mathbf{a}) \Leftrightarrow F$ is invariant under $\Gamma_0(\mathbf{a})$.

If ε_0 is a generator of $\vartheta_K^\star$ then I_{ε_0} induces an involution of $S(\mathbf{a})$, denoted J and called the **main involution** of $S(\mathbf{a})$.

J splits $S(\mathbf{a})$ up as $S(\mathbf{a}) = S(\mathbf{a})^+ \oplus S(\mathbf{a})^-$, where J acts as $+1$ on $S(\mathbf{a})^+$ and -1 on $S(\mathbf{a})^-$. In $S(\mathbf{a})^+$ the Fourier coefficients satisfy $c(\alpha) = c(\varepsilon_0\alpha)$ while in $S(\mathbf{a})^-$ they satisfy $c(\alpha) = -c(\varepsilon_0\alpha)$.

Now suppose that $h(\mathrm{K}) = 1$. Then ϑ_{K} is a principal ideal domain. Let $\wp$ be a prime ideal of ϑ_{K}, generated by π, where $\pi \not| \mathbf{a}$. Then we define the **Hecke operator:**

$$T_\pi := \sum_{\alpha \bmod \pi} \begin{pmatrix} 1 & \alpha \\ 0 & \pi \end{pmatrix} + \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}. \qquad (0.5.3)$$

Note: For a unit $\varepsilon \in \vartheta_{\mathrm{K}}^\star$, $T_{\varepsilon\pi} = JT_\pi \neq T_\pi$, so we cannot define a general operator $T_\wp$, for prime ideal $\wp$, as the definition depends on the choice of generator π.

Lemma 0.5.1: (Lemma 3.3.7, [4])

Let $F \in S(\mathbf{a})$ and $\wp$ a prime ideal of ϑ_{K} generated by π where $\pi \not| \mathbf{a}$.

Then $F|T_\pi \in S(\mathbf{a})$.

If F is an eigenform for some T_π then $F|T_\pi = \lambda_\pi F$, for some constant λ_π. So the Fourier coefficients, $c'(\alpha)$, of $F|T_\pi$ satisfy $c'(\alpha) = \lambda_\pi c(\alpha), \forall \alpha \in \vartheta_{\mathrm{K}}$.

If F is an eigenform for **all** of the $T_\pi, \pi \not| \mathbf{a}$ then the Fourier coefficients of F can be derived in terms of the eigenvalues of T_π via the recursive relation

$$c(\pi^{r+1}) = \begin{cases} \lambda_\pi c(\pi^r) - N(\pi)c(\pi^{r-1}), & \pi \not| \mathbf{a} \\ \lambda_\pi c(\pi^r), & \pi | \mathbf{a} \end{cases} \qquad (0.5.4)$$

for prime powers. F is normalised so that $c(1) = 1$, and, for composite α, the function $c(\alpha)$ is multiplicative.

Let F be a cusp form in $S^+(\mathbf{a})$. It has a Fourier expansion given by (0.4.3).

If $\mathbf{a}$ is an ideal of ϑ_{K}, generated by α, we can write $c(\alpha)$, the coefficient associated with it. Then we can attach a formal Dirichlet series to F:

$$L(F, s) := \sum_{\alpha \in \vartheta_{\mathrm{K}}} c(\alpha) N(\alpha)^{-s}.$$

The multiplicative property of the Fourier coefficients, $c(\alpha)$, implies that $L(F,s)$ has an Euler product expansion:

$$L(F,s) = \prod_{\wp}(1 - c(\wp)N(\wp)^{-s} + \chi(\wp)N(\wp)^{1-2s})^{-1},$$

where $\chi(\wp) = \begin{cases} 0 & \text{if } \wp|\mathbf{a}, \\ 1 & \text{if } \wp \not|\mathbf{a}. \end{cases}$

Now consider a prime ideal $\wp$ of ϑ_K, generated by π, where $\pi|$ $\mathbf{a}$. Instead of defining a Hecke operator, T_π, for $\wp$, we define the involution W_π, which is easier to compute and has simpler properties. We do this as follows:

Let π^r be the highest power of π which divides $\mathbf{a}$. Let α be a generator of the ideal $\mathbf{a}$ and choose $x, y, z, w \in \vartheta_K$ to satisfy $\pi^{2r}xw - \alpha zy = \pi^r$. Then

$$W_\pi := \begin{pmatrix} \pi^r x & y \\ \alpha z & \pi^r w \end{pmatrix} \tag{0.5.5}$$

which has determinant π^r.

Lemma 0.5.2: ([4])

Let $F \in S(\mathbf{a})$ and $\wp$ a prime ideal of ϑ_K, generated by π where $\pi|$ $\mathbf{a}$.

Then $f|W_\pi \in S(\mathbf{a})$.

There are several useful properties of these T_π and W_π operators which we list here:

(i) $\pi_1, \pi_2 \not|\mathrm{a} \Rightarrow T_{\pi_1}T_{\pi_2} = T_{\pi_2}T_{\pi_1}$.

(ii) $\pi_1, \pi_2|\mathrm{a} \Rightarrow W_{\pi_1}W_{\pi_2} = W_{\pi_2}W_{\pi_1}$.

(iii) $\pi_1|$ $\mathbf{a}$ and $\pi_2 \not|$ $\mathbf{a} \Rightarrow T_{\pi_2}W_{\pi_1} = W_{\pi_1}T_{\pi_2}$.

(iv) $\exists$ an inner product on $S(\mathbf{a})$ such that every T_{π_i}, where $\pi_i \not| \mathbf{a}$, and every W_{π_j}, where $\pi_j | \mathbf{a}$, is self-adjoint with respect to it.

So, there is a basis of $S(\mathbf{a})$ which consists of those forms which are eigenvectors, or **eigenforms**, for all the T_{π_i} where $\pi_i \not| \mathbf{a}$ and all the W_{π_j} where $\pi_j | \mathbf{a}$. Elements of such a basis have leading coefficient $c(1) \neq 0$; we shall normalise them so that $c(1) = 1$.

We now turn our attention to two particular classes of eigenforms, known as **oldforms** and **newforms**.

If $F \in S(\mathbf{b})$, where $\mathbf{b}$ is an ideal of ϑ_{K} such that $\mathbf{b}|\mathbf{a}$, then $\forall\ k|\mathbf{ab}^{-1}, F|\gamma_k \in S(\mathbf{a})$, where $\gamma_k = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$.

Denote the subspace generated by $\{F|\gamma_k : k|\mathbf{ab}^{-1}, \forall\ \mathbf{b}|\mathbf{a}\}$ by $S^{\mathrm{old}}(\mathbf{a})$. This space is spanned by eigenforms which we will call "oldforms". $S^{\mathrm{old}}(\mathbf{a})$ is mapped to itself by all the T_{π_i}, where $\pi_i \not| \mathbf{a}$, and the W_{π_j}, where $\pi_j|\mathbf{a}$.

Now consider the orthogonal compliment of $S^{\mathrm{old}}(\mathbf{a})$ with respect to the inner product of (iv) above. This new space is spanned by eigenforms, which are **not** oldforms, and are known as "newforms". We will denote the space of newforms by $S^{\mathrm{new}}(\mathbf{a})$. These newforms are eigenforms for all the $T_{\pi_i}, \pi_i \not|\mathbf{a}$, and all the $W_{\pi_j}, \pi_j|\mathbf{a}$, and all have leading coefficient 1. The algebra generated by all the T_{π_i}, restricted to $S^{\mathrm{new}}(\mathbf{a})$, is commutative, semi-simple, and has rank $= \dim(S^{\mathrm{new}}(\mathbf{a}))$.

Theorem 0.5.3: ([13])

If F and G are newforms for $\Gamma_0(\mathbf{a})$ and $\Gamma_0(\mathbf{b})$ respectively, then **either** $F = G$ *and* $\mathbf{a} = \mathbf{b}$ **or** F *and* G *have different eigenvalues for infinitely many* T_π*, where* $\pi \not| \mathbf{ab}$.

§0.6 Homology and Modular Symbols

We will not calculate the space of cusp forms for $\Gamma_0(\mathbf{a})$ directly. Instead we will consider the homology $H_1(\Gamma_0(\mathrm{a})\backslash\mathcal{H}_3^*, \mathbf{Q})$. The reasoning behind this is as follows (for details see [4]):

Let $\Gamma_0(\mathbf{a})$ be the congruence subgroup defined in (0.3.3). Denote the quotient topological space, $\Gamma_0(\mathbf{a})\ \backslash\mathcal{H}_3$, by $X_0(\mathbf{a}) = X_{\Gamma_0(\mathbf{a})}$ and its closure, $\Gamma_0(\mathbf{a})\backslash\mathcal{H}_3^*$, by $\overline{X_0(\mathbf{a})}$.

In fact, instead of working with $\Gamma_0(\mathbf{a}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\vartheta_{\mathrm{K}}) : c \in \mathbf{a}\}$, we can, and will, work with the larger group $\Gamma_0^+(\mathbf{a}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\vartheta_{\mathrm{K}}) : c \in \mathbf{a}\}$.

In the case where K has class number 1, the corresponding projective groups satisfy [4]:

$$[\overline{\Gamma_0^+(\mathbf{a})} : \overline{\Gamma_0(\mathbf{a})}] = 2,$$

since

$$\overline{\Gamma_0^+(\mathbf{a})} = \overline{\Gamma_0(\mathbf{a})} \cup \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix} \overline{\Gamma_0(\mathbf{a})}.$$

In [10], Kurčanov proves that the map

$$S(\mathbf{a}) \to H^1(\overline{X_0(\mathbf{a})}, \mathbf{C}), \tag{0.6.1}$$

given by

$$F \mapsto F.\beta,$$

is an isomorphism.

There is an exact duality:

$$H^1(\overline{X}_\Gamma, \mathbf{C}) \times H_1(\overline{X}_\Gamma, \mathbf{C}) \to \mathbf{C}, \tag{0.6.2}$$

given by

$$(\omega, \gamma) \mapsto \int_\gamma \omega.$$

In the case $\Gamma = \Gamma_0^+(\mathbf{a})$ we can use Kurčanov's isomorphism to induce a further isomorphism:

$$S(\mathbf{a}) \rightarrow H_1(\overline{X_0(\mathbf{a})}, \mathbf{C}). \tag{0.6.3}$$

For each non-zero ideal $\mathbf{a}$ of ϑ_K we set $V(\mathbf{a}) = H_1(\overline{X_0(\mathbf{a})}, \mathbf{Q})$.

Note: There is a Hecke action on $V(\mathbf{a})$ which we will define at the end of this section. It is important that the isomorphism (0.6.3) respects the Hecke action on both $S(\mathbf{a})$ and $V(\mathbf{a})$, as we will find newforms in $S(\mathbf{a})$ by computing the action of a number of Hecke operators on $V(\mathbf{a})$ and finding one-dimensional eigenspaces with rational eigenvalues common to both.

Part I of this thesis will be concerned with the computation of $V(\mathbf{a})$, where $\mathbf{a}$ is a non-zero ideal of ϑ_K with small norm and K is an imaginary quadratic number field with $h(\mathrm{K}) = 1$. The case where K is one of the 5 Euclidean fields is the subject of [4] and I will give a summary of Cremona's method in Chapter 1. Chapters 2, 3 and 4 will be concerned with the extension of this work to the 4 non-Euclidean fields with class number 1. In all cases we will be able to compute:

the main involution of J;

the W_π involution for $\pi|\mathbf{a}$;

the T_π for any $\pi \nmid \mathbf{a}$.

From the results of §0.5, it follows that there is a basis for $V(\mathbf{a})$ with respect to which each of the above operators acts as a diagonal matrix. This basis will be explicitly represented in terms of cycles on $\overline{X_0(\mathbf{a})}$. The coefficients of newforms can be computed from their Hecke eigenvalues while oldforms can be recognised as they have occurred previously (at lower levels) as newforms.

It seems appropriate, at this point, to introduce some notation which will be useful in the calculation of homology. Called **modular symbols**, these were originally invented by Birch and Swinnerton-Dyer for use in work relating to the Birch, Swinnerton-Dyer Conjecture. They are discussed over the rationals in [11] and [12] and over imaginary quadratic fields in [10]. We define them as follows:

Take 2 points $A, B \in \mathcal{H}_3^*$ such that $\exists \gamma \in \Gamma$, with $\gamma(A) = B$. Then any smooth path from A to B in $\mathcal{H}_3$ projects to a closed path in the quotient space $\overline{X}_\Gamma = \Gamma \backslash \mathcal{H}_3^*$. Because $\mathcal{H}_3^*$ is simply connected, the homology class of this path in $H_1(\overline{X}_\Gamma, \mathbf{Z})$ depends only on A and B and not on the chosen path between them. We denote this homology class by $\{A, B\}_\Gamma$, or simply $\{A, B\}$, if Γ is clear from the context. If we identify homology classes with functionals on the space of differentials then we may extend this definition to points A and B which are **not** equivalent under Γ:

We denote the real homology class identified with the functional $\omega \mapsto \int_A^B \phi\omega$, where ω is a differential on $\overline{X}_\Gamma$ and $\phi : \mathcal{H}_3^* \rightarrow \overline{X}_\Gamma$ is the natural projection, by $\{A, B\}$.

Modular symbols $\{A, B\}$ have the following properties:

(i) $\{A, A\} = 0$;

(ii) $\{A, B\} + \{B, A\} = 0$;

(iii) $\{A,B\}+\{B,C\}+\{C,A\}=0$;

(iv) $\{\gamma(A),\gamma(B)\}_\Gamma=\{A,B\}_\Gamma, \forall\gamma\in\Gamma$;

(v) $\{A,\gamma(A)\}_\Gamma=\{B,\gamma(B)\}_\Gamma, \forall\gamma\in\Gamma$ and $\forall A,B\in\mathcal{H}_3^*$;

(vi) $\{A,\gamma(A)\}_\Gamma\in H_1(\overline{X}_\Gamma,\mathbf{Z})$ if $\gamma\in\Gamma$.

An elementary geometrical argument shows that any element of $H_1(\overline{X}_\Gamma,\mathbf{Z})$ can, in fact, be written as $\{A,\gamma(A)\}$, for some $\gamma\in\Gamma$ and $A\in \mathrm{K}\cup\{\infty\}$, (the cusps).

Now we can define the Hecke action on $V(\mathbf{a})$ in terms of modular symbols.

T_π acts according to:

$$T_\pi:\{A,B\}\mapsto\sum_{x \bmod \pi}\{\frac{A+x}{\pi},\frac{B+x}{\pi}\}+\{\pi A,\pi B\}. \tag{0.6.4}$$

and W_π acts according to:

$$W_\pi:\{A,B\}\mapsto\{\frac{\pi^r xA+y}{\alpha zA+\pi^r w},\frac{\pi^r xB+y}{\alpha zB+\pi^r w}\}, \tag{0.6.5}$$

where $\mathbf{a}=(\alpha), r$ is the highest power of π dividing α and x,y,z,w are chosen so that $\pi^{2r}xw-\alpha zy=\pi^r$.

§0.7 Connections

There is a conjecture due to Weil, Taniyama and Shimura [21] which states that every elliptic curve, defined over $\mathbf{Q}$, is isogenous to a factor of the Jacobian of one of the modular curves $X_0(N)$, where $N\in\mathbf{Z}$. This has been vastly generalised, in particular to elliptic curves over arbitrary number fields. In [4], Cremona studies the conjecture over imaginary quadratic number fields, namely that there is a correspondence between:

(i) 1-dimensional eigenspaces of $V^+(\mathbf{a})$ with rational eigenvalues, and,

(ii) Isogeny classes of elliptic curves $E(\mathrm{K})$, with conductor $\mathbf{a}$, which do not have complex multiplication by an order in K.

(Here $V(\mathbf{a}) = H_1(\Gamma_0(\mathbf{a})\backslash\mathcal{H}_3^*, \mathbf{Q})$, as in §0.6, and $V^+(\mathbf{a})$ is the $+1$ eigenspace for J). He then goes on to provide numerical evidence in the case where K is one of the 5 Euclidean fields, ie $\mathrm{K} = \mathbf{Q}(\sqrt{-d})$ where $d \in \{1, 2, 3, 7, 11\}$.

More precisely, he gives support to:

Conjecture 0.7.1: ([6])

If F is a newform of weight 2 for $\Gamma_0(\mathbf{a})$ with rational integer coefficients, then there corresponds **either** *an isogeny class of elliptic curves defined over* K *with conductor* $\mathbf{a}$ **or** *a quadratic character ε of* $\mathrm{Gal}(\overline{\mathrm{K}}\backslash\mathrm{K})$ *such that $F \otimes \varepsilon$ is the lift of a form over* $\mathbf{Q}$.

In the case where F corresponds to an elliptic curve E we have the following:

(i) For prime ideals $\wp = (\pi)$ not dividing $\mathbf{a}$, *the trace of Frobenius of the curve at $\wp$ is equal to the eigenvalue of T_π acting on the space generated by the newform.*

(ii) For prime ideals $\wp$ dividing $\mathbf{a}$: *if $\wp^2$ divides* $\mathbf{a}$ *then the trace of Frobenius of the curve at $\wp$ is 0; otherwise (if $\wp$ divides* $\mathbf{a}$ *exactly) it is minus the corresponding eigenvalue of W_π.*

In Part I of this thesis I will give an overview of Cremona's method for dealing with the Euclidean cases and will then extend this to the non-Euclidean, class number 1 cases to provide similar evidence there.

Part II will be a description of joint work, with Cremona, to provide numerical evidence for the Birch, Swinnerton-Dyer Conjecture (originally defined for curves over $\mathbf{Q}$) for fields, K, with $h(\mathrm{K}) = 1$.

We give a statement of the conjecture here; further details will be given in Part II.

Conjecture 0.7.2: [7]

Let E be an elliptic curve defined over K with L*-series* $L(E, s)$. *Then*

(i) $L(E,1) = 0 \Leftrightarrow \mathrm{rank}(E(\mathrm{K})) > 0$;

(ii) If $\mathrm{rank}(E(\mathrm{K})) = 0$ *then*

$$L(E,1) = \frac{\alpha(E) \prod c_{\wp} |\text{Ш}|}{|E_{tors}|^2},$$

where

Ш *is the Tate-Shafarevitch group of E over* K,

$c_{\wp}$ *is the local index* $[E(\mathrm{K}_{\wp}) : E^0(\mathrm{K}_{\wp})]$,

and $\alpha(E)$ *is the "complex period" of* E.

PART I

MODULAR FORMS OVER IMAGINARY QUADRATIC NUMBER FIELDS

Chapter 1: Calculation of Homology over Euclidean Fields

In this chapter I will introduce an algorithm, originally due to Manin [12], and developed by Cremona in [4], for calculating homology. Manin uses it to calculate $H_1(G\backslash\mathcal{H}_2^*,\mathbf{Q})$, where G is a subgroup of finite index in $SL(2,\mathbf{Z})$. Cremona expands this idea in his work on modular forms and elliptic curves defined over general number fields, K, and extends the algorithm to calculate $H_1(G\backslash\mathcal{H}_3^*,\mathbf{Q})$, where G is a subgroup of finite index in $SL(2,\vartheta_{\mathrm{K}})$ and K is one of the 5 Euclidean fields.

In Chapters 2,3 and 4 I will show how to extend Cremona's methods further, to subgroups G of $SL(2,\vartheta_{\mathrm{K}})$ where K is no longer Euclidean but still has $h(\mathrm{K}) = 1$, ie $\mathrm{K} = \mathbf{Q}\,(\sqrt{-d})$ where $d \in \{19, 43, 67, 163\}$.

The exposition in this chapter is by no means complete; I will describe, in general terms, those steps of the algorithm which are common to all 5 Euclidean fields, and which will be used in extending the algorithm in Chapters 2,3 and 4. For details of the Euclidean case, and for specific calculations in each of the 5 fields, see [4]

In §1.1 I will describe a fundamental region for the action of $SL(2,\vartheta_{\mathrm{K}})$, and subgroups of it, on $\mathcal{H}_3^*$. This will play an important part in establishing a tessellation of $\mathcal{H}_3^*$ by hyperbolic polyhedra in §1.2. In §1.3, I will describe how to calculate homology from this tessellation, although I will not give any details of specific calculations. Again, details can be found in [4]. A new notation will be introduced in §1.5 and this will be used in §1.6 in the calculation of the Hecke algebra.

§1.1 A Fundamental Region for $SL(2,\vartheta_K)$

We wish to find a fundamental region for the action of $SL(2,\vartheta_K)$ on $\mathcal{H}_3$. That is, a subset D of $\mathcal{H}_3$ with the following properties [4]:

(i) D is open in $\mathcal{H}_3$,

(ii) Each orbit of $SL(2,\vartheta_K)$ in $\mathcal{H}_3$ meets D at **most** once, and meets the closure, $\overline{D}$, of D at **least** once.

Recall the definitions (0.2.5), (0.2.6) and (0.2.8) of matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and their action on $\mathcal{H}_3^\star$. Using suitable powers of T and U, which take (z,ζ) to $(z+1,\zeta)$ and $(z+\omega,\zeta)$ respectively, we can translate any point (z,ζ) until $|z| < 1$. In fact, we can bring any $z \in \mathbf{C}$ into the region

$$F := \{z \in \mathbf{C} : |z| \le |z - z_0|, \forall z_0 \in \vartheta_K\}. \tag{1.1.1}$$

If $d \equiv 1, 2 \pmod 4$ then F is a rectangle; if $d \equiv 3 \pmod 4$ it is a hexagon.

If we also consider the action of S, we can bring any point of $\mathcal{H}_3^\star$ into the region

$$F' := \{(z,\zeta) : z \in F, |z|^2 + \zeta^2 \ge 1\}. \tag{1.1.2}$$

F' is a rectangular/hexagonal "chimney", with a curved base which is part of the unit hemisphere centred at the origin.

So, given any point (z, ζ), we can bring it into F' by repeatedly translating z, using powers of T and U, to bring it within F and then, if the result is inside the unit hemisphere, by applying S. S multiplies the ζ co-ordinate of (z, ζ) by $(|z|^2 + \zeta^2)^{-1}$ and so if (z, ζ) is inside the unit sphere, applying S raises the point. There are only finitely many $\zeta' > \zeta$ such that (z, ζ) is equivalent under $SL(2, \vartheta_K)$ to (z', ζ'), for some z', and so, after a finite number of steps, (z, ζ) will lie in F'.

If $d \in \{2, 7, 11\}$, then $SL(2, \vartheta_K)$ is generated by S, T, U and $-I$ and $D = F'$ is a fundamental region for its action on $\mathcal{H}_3^*$.

If $d = 1$ then $SL(2, \vartheta_K)$ is generated by S, T, U and $R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, (i = \sqrt{-1})$ and a fundamental region is given by cutting F' in half to give:

$$D := \{(x + iy, \zeta) : -\frac{1}{2} < x < \frac{1}{2}, 0 < y < \frac{1}{2}, x^2 + y^2 + \zeta^2 > 1\}.$$

If $d = 3$ then $SL(2, \vartheta_K)$ is generated by S, T, U and $R = \begin{pmatrix} 0 & \rho \\ \rho^2 & 0 \end{pmatrix}, (\rho = \frac{1+\sqrt{-3}}{2})$ and a fundamental region is obtained by cutting F' in three to give:

$$D := \{(x + \rho^2 y, \zeta) : 0 < x, y < \frac{1}{2}, |x + \rho^2 y|^2 + \zeta^2 > 1\}.$$

We will also be interested in fundamental regions for certain subgroups of $SL(2, \vartheta_K)$. Let G be a subgroup of finite index in $SL(2, \vartheta_K)$ and let $\mathcal{G} = \{\gamma_1, \ldots, \gamma_n\}$ be a complete set of right coset representatives for G in $SL(\vartheta_K)$. Then, if D is a fundamental region for the action of $SL(2, \vartheta_K)$, a fundamental region for the action of G on $\mathcal{H}_3^*$ is given by $\bigcup_{i=1}^{n} \gamma_i D$. This is a hyperbolic polyhedron with certain be a complete set of right coset representatives for G in $SL(\vartheta_K)$. Then, if D is a fundamental region for the action of $SL(2,\vartheta_K)$, a fundamental region for the action of G on $\mathcal{H}_3^*$ is given by $\bigcup_{i=1}^{n} \gamma_i D$. This is a hyperbolic polyhedron with certain faces identified by elements of G and will form the basis for a tessellation of $\mathcal{H}_3^*$ by hyperbolic polyhedra in the next section.

§1.2 Tessellation of $\mathcal{H}_3^*$ by Hyperbolic Polyhedra

The 2-dimensional rational case is the simplest to illustrate so I will begin with a description of this and then show how to extend the method to the 5 Euclidean fields.

Let $\Gamma := SL(2,\mathbf{Z})$, with the usual generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, be acting on the extended hyperbolic upper-half plane, $\mathcal{H}_2^* = \mathcal{H}_2 \cup \{\text{cusps}\}$, in the usual way. A fundamental region for the action of Γ on $\mathcal{H}_2^*$ is the hyperbolic triangle, F, with vertices at $0, \rho = \frac{1+\sqrt{-3}}{2}$ and $i\infty$ as shown in Figure (i).

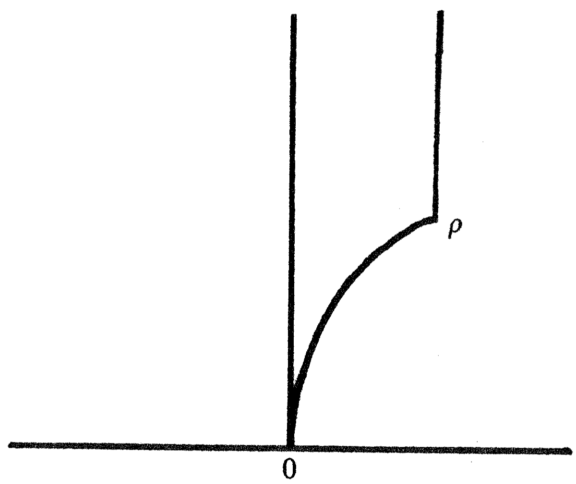


Figure (i)

Notice that we can form the larger hyperbolic triangle, F^+, with vertices at $0, 1$ and $i\infty$, simply by "gluing together" the triangles $F, (TS)F$ and $(TS)^2F$ along common edges as in Figure (ii).

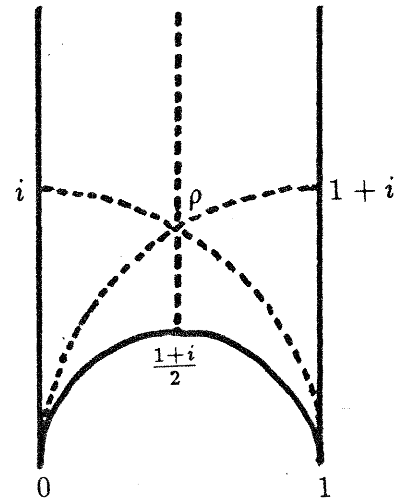


Figure (ii)

Effectively we are fixing the point ρ and gluing together translates of F by elements of the stabilizer of ρ, $G_\rho = \{I, TS, (TS)^2\}$ around ρ.

Notice that the edges of the triangle F^+ are all transforms of the basic edge from 0 to $i\infty$ by elements of G_ρ.

If we take as our basis element the larger triangle F^+, and let γ run through all elements of Γ then $\bigcup_{\gamma \in \Gamma} \gamma F^+$ gives a tessellation of $\mathcal{H}_2^*$ by hyperbolic triangles as shown in Figure (iii).

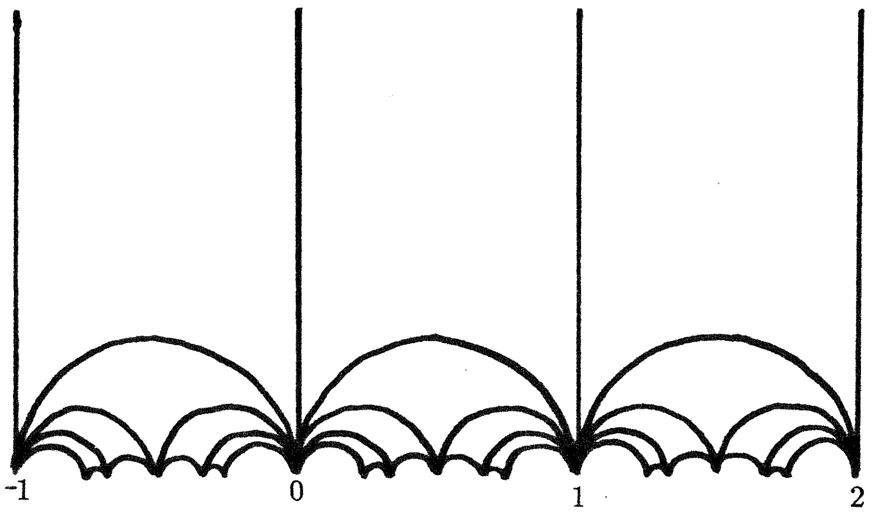


Figure (iii)

The 1-skeleton of this tessellation consists entirely of edges which are transforms of the basic edge from 0 to $i\infty$ by $\gamma \in \Gamma$.

Now consider G, a subgroup of finite index in Γ, and let $\mathcal{G}$ be a complete set of right coset representatives for G in Γ. Thus $\Gamma = \bigcup_{\gamma \in \mathcal{G}} G\gamma$. Then a fundamental region for the action of G in $\mathcal{H}_2^*$ is given by $\bigcup_{\gamma \in \mathcal{G}} \gamma F$.

Passing to the quotient space $G \backslash \mathcal{H}_2^*$, the tessellation of $\mathcal{H}_2^*$ by the transforms of F^+ gives a triangulation of $G \backslash \mathcal{H}_2^*$ by a finite number of hyperbolic triangles. The triangulation is finite because $[\Gamma : G] < \infty$. Every edge of this triangulation is the image in $G \backslash \mathcal{H}_2^*$ of a transform of the basic path in $\mathcal{H}_2^*$ from 0 to ∞ by some element of G. So we can generate the homology of $G \backslash \mathcal{H}_2^*$ by the paths $\{\gamma(0), \gamma(\infty)\}, \gamma \in \mathcal{G}$. Moreover, there are certain relations between the paths of the triangulation. These relations arise in two different ways:

(i) The edges of the basic triangle adding to give zero:

$$\{\gamma(0), \gamma(i\infty)\} + \{\gamma TS(0), \gamma TS(i\infty)\} + \{\gamma (TS)^2(0), \gamma (TS)^2(i\infty)\} = 0.$$

(ii) The gluing together of 2 triangles along a common edge:

$$\{\gamma(0), \gamma(i\infty)\} + \{\gamma S(0), \gamma S(i\infty)\} = 0.$$

We now turn our attention to the case where K is one of the five Euclidean fields. As in the rational case, $\Gamma = SL(2, \vartheta_{\mathrm{K}})$ acts on $\mathcal{H}_3^*$ and G is a subgroup of finite index in Γ. We begin with the fundamental region, D, for the action of Γ on $\mathcal{H}_3^*$, defined in §1.1, chosen to contain the edge from 0 to ∞. We then proceed as follows:

Fix some vertex p, which is not a cusp of D, and find G_p, the stabilizer of p. In each case, G_p is generated by S, TS and one other element of Γ of finite order. We obtain the larger basic polyhedron $D^+ = \bigcup_{\gamma \in G_p} D$ by gluing together translates of D by elements of G_p around the vertex p. The exact nature of D^+ depends on the field in question (see [4]). The results are summarised in the following table:

d	D^+
1	Octahedron
2	Cuboctahedron
3	Tetrahedron
7	Triangular Prism
11	Truncated Tetrahedron

As in the rational case, all the edges of these polyhedra are translates of the basic edge from 0 to ∞ by some $\gamma \in \Gamma$. Again $\bigcup_{\gamma\in\Gamma} \gamma D^+$ gives a tessellation of $\mathcal{H}_3^\star$, this time by hyperbolic polyhedra of the appropriate type. If $\mathcal{G}$ is a complete set of right coset representatives for G in Γ then $\Gamma = \bigcup_{\gamma\in\mathcal{G}} G\gamma$, and a fundamental region for G in $\mathcal{H}_3^\star$ is given by $\bigcup_{\gamma\in\mathcal{G}} \gamma D$. Passing to the quotient gives a tessellation of the space $G\backslash\mathcal{H}_3^\star$ by a finite number of hyperbolic polyhedra; we are particularly interested in its 1-skeleton. Every edge of this skeleton is a transform of the basic edge from $\{0, \infty\}$ by some $\gamma \in \Gamma$ so we can generate the homology $H_1(G\backslash\mathcal{H}_3^\star,\mathbf{Q})$ by the edges $\{\gamma(0), \gamma(\infty)\}, \gamma \in \Gamma$.

Relations between the edges arise from the gluing together of corresponding edges and from the edges of a face of a polyhedron adding together to give zero. We shall consider these further in the following section.

§1.3 Calculation of Homology

Recall G is a subgroup of finite index in Γ. We wish to calculate $H_1(G\backslash\mathcal{H}_3^*,\mathbf{Q})$. In the previous section we saw that we can generate the homology by paths $\{\gamma(0),\gamma(\infty)\},\gamma \in \Gamma$. Denote the path $\{\gamma(0),\gamma(\infty)\}$ by (γ). (By abuse of notation, the same symbol, (γ), will also be used to denote both the image of this path in the quotient space $G\backslash\mathcal{H}_3^*$, and its homology class.) The symbol (γ) refers implicitly to the group G, as it is an element of $H_1(G\backslash\mathcal{H}_3^*,\mathbf{Q})$; so

$$(\gamma) = (\gamma'\gamma), \forall\gamma' \in G. \tag{1.3.1}$$

Therefore, it suffices to take as generators for $H_1(G\backslash\mathcal{H}_3^*,\mathbf{Q})$ the set $\{(\gamma) : \gamma \in \mathcal{G}\}$. This is a **finite** set as the index $[\Gamma : G] < \infty$ and so the homology is finitely generated. Denote the $\mathbf{Q}$-vector space spanned by these symbols by $C(G)$.

We now return to the question of determining the relations between these paths, mentioned in §1.2. Recall that these come from two sources:

(i) the gluing together of polyhedra along a common face,

(ii) the adding together of the edges of a face to zero.

The exact nature of these relations depends on the geometry of the field in question. The explicit calculations for the 5 Euclidean fields are found in [4]. We give a summary here.

In each case the edge $\{0,\infty\}$ is self identified by S with reversed orientation, ie $S(0) = \infty$ and $S(\infty) = 0$. We represent this by the relation:

$$(\gamma) + (\gamma S) = 0. \tag{1.3.2}$$

Also, in each case, the relation

$$(\gamma) + (\gamma TS) + (\gamma (TS)^2) = 0, \tag{1.3.3}$$

is satisfied.

Let ε be a generator of $\vartheta_{\mathrm{K}}^{\star}$.

It transpires that it is more natural to consider the action of the larger group

$$\Gamma^{\pm} := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \vartheta_{\mathrm{K}}, ad - bc \in \vartheta_{\mathrm{K}}^{\star} \}.$$

$\Gamma^{\pm}$ contains Γ with index $|\vartheta_{\mathrm{K}}^{\star}|$. The corresponding projective groups $\overline{\Gamma}$ and $\overline{\Gamma^{\pm}}$, obtained by factoring out the scalar matrices, satisfy $[\overline{\Gamma^{\pm}} : \overline{\Gamma}] = 2$ and $\overline{\Gamma^{\pm}} = \overline{\Gamma} \cup J\overline{\Gamma}$ where $J = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \varepsilon \in \vartheta_{\mathrm{K}}^{\star}$. The definition of the symbol $(\gamma) = \{\gamma(0), \gamma(\infty)\}_G$ extends to γ in $\Gamma^{\pm}$ in the obvious way. So we have a further relation

$$(\gamma J) - (\gamma) = 0. \tag{1.3.4}$$

In each field there is one additional relation. We summarise these in the following:

Table (1.3.5)

d	Relation	Matrix X
1	$(\gamma) + (\gamma X) + (\gamma X^2) = 0$	$\begin{pmatrix} i & 1 \\ 1 & 0 \end{pmatrix}$
2	$(\gamma) + (\gamma X) + (\gamma X^2) + (\gamma X^3) = 0$	$\begin{pmatrix} \omega & 1 \\ 1 & 0 \end{pmatrix}$
3	$(\gamma) + (\gamma X) + (\gamma X^2) = 0$	$\begin{pmatrix} 1 & \omega - 1 \\ \omega & 0 \end{pmatrix}$
7	$(\gamma) - (\gamma U) + (\gamma X) - (\gamma XU) = 0$	$\begin{pmatrix} -1 & \omega \\ \omega - 1 & 1 \end{pmatrix}$
11	$(\gamma) - (\gamma U) + (\gamma X) - (\gamma XU) + (\gamma X^2) - (\gamma X^2 U) = 0$	$\begin{pmatrix} -1 & \omega \\ \omega - 1 & 2 \end{pmatrix}$

By means of relations (1.3.1) and (1.3.4), any symbol (γ), for $\gamma \in \Gamma^{\pm}$, can be identified with a unique basis element of $C(G)$. Thus, we can define an action of $\mathbf{Z}\Gamma^{\pm}$ on $C(G)$ via:

$$\gamma^{\pm} : (\gamma) \mapsto (\gamma\gamma^{\pm}),$$

where $\gamma^{\pm} \in \Gamma^{\pm}$, extended to $\mathbf{Z}\Gamma^{\pm}$ by linearity.

The relations (1.3.2), (1.3.3), (1.3.4) and (1.3.5) generate a left ideal of $\mathbf{Z}\Gamma^{\pm}$, which we call the "relation ideal" of $\mathbf{Z}\Gamma^{\pm}$, and denote by $\mathcal{R}$.

Set $B(G) := C(G)\mathcal{R}$.

Now consider the **boundary map** from the space of 1-cycles to the space of 0-cycles:

$$\delta : C(G) \to H_0(G),$$

defined by

$$\delta : (\gamma) \mapsto [\gamma(\infty)] - [\gamma(0)], \qquad (1.3.6)$$

(where $[\alpha]$ denotes the class of cusps equivalent to α, as defined in Lemma 0.3.1).

Denote the kernel of this mapping, $\mathrm{Ker}(\delta)$, by $Z(G)$. From the definition of $\mathcal{R}$ it can be seen that $B(G) \subset Z(G)$.

Finally set $H(G) := Z(G)/B(G)$.

Then the following is proved by Cremona in [4]:

Theorem 1.3.1:

There is an isomorphism from $H(G)$ to $H_1(G\backslash\mathcal{H}_3^*, \mathbf{Q})$ induced by the natural map

$$\xi : (\gamma) \mapsto \{\gamma(0), \gamma(\infty)\}_G.$$

Note: $\mathcal{R}$ is independent of the choice of G so we need only calculate it once for each field.

So, for any non-zero ideal $\mathbf{a}$ of ϑ_K, we now have an algorithm for computing $H_1(\Gamma_0(\mathbf{a})\backslash\mathcal{H}_3^*,\mathbf{Q})$ explicitly in terms of right coset representatives for $\Gamma_0(\mathbf{a})$.

We are generating the homology entirely by paths of the form $\{\gamma(0),\gamma(\infty)\}$, so we need to be able to express any path between cusps α and $\beta \in$ K as a sum of such paths. It will suffice to do so for paths of the form $\{0,\alpha\}$ since

$$\begin{aligned}\{\alpha,\beta\} &= \{\alpha,0\} + \{0,\beta\} \\ &= -\{0,\alpha\} + \{0,\beta\}.\end{aligned}$$

Let $\{0,\alpha\} = \{0,\frac{a}{b}\}$ be such a path where $a,b \in \vartheta_K$ and $(a,b) = \vartheta_K$ ie α is written in lowest terms. Write down the continued fraction convergents of $\frac{a}{b}$. We can do this because K is Euclidean and therefore ϑ_K has a Euclidean algorithm in the usual sense. These convergents are:

$$\frac{a}{b} = \frac{a_n}{b_n}, \frac{a_{n-1}}{b_{n-1}}, \ldots, \frac{a_0}{b_0} = \frac{a_0}{1}, \frac{a_{-1}}{b_{-1}} = \frac{1}{0}, \frac{a_{-2}}{b_{-2}} = \frac{0}{1}. \tag{1.3.7}$$

It is well known that $a_k b_{k-1} - a_{k-1} b_k = (-1)^{k-1}$ so

$$\gamma_k := \begin{pmatrix} a_k & a_{k-1} \\ b_k & b_{k-1} \end{pmatrix} \in \Gamma^{\pm}.$$

Now we can write:

$$\begin{aligned}\{0,\frac{a}{b}\} = \{0,\alpha\} &= \sum_{k=-1}^{n} \{\frac{a_{k-1}}{b_{k-1}}, \frac{a_k}{b_k}\} \\ &= \sum_{k=-1}^{n} \{\gamma_k(0),\gamma_k(\infty)\} \qquad (1.3.8) \\ &= \sum_{k=-1}^{n} (\gamma_k)\end{aligned}$$

as required.

Finally notice, that when the subgroup G is normalised by J, the space $H(G)$ decomposes according to the eigenvalues of the J involution as:

$$H(G) = H^+(G) \oplus H^-(G),$$

where J acts as $+1$ on $H^+(G)$ and -1 on $H^-(G)$. In particular, for any non-zero ideal $\mathbf{a}$ in ϑ_K, J normalises $\Gamma_0(\mathbf{a})$. We write $V(\mathbf{a})$ for $H(\Gamma_0(\mathbf{a}))$ and decompose it, as above, into

$$V(\mathbf{a}) = V^+(\mathbf{a}) \oplus V^-(\mathbf{a})$$

where J acts as $+1$ on $V^+(\mathbf{a})$ and -1 on $V^-(\mathbf{a})$.

§1.4 M-symbols

In the previous section I described a method for calculating $H_1(\Gamma_0(\mathbf{a})\backslash\mathcal{H}_3^*, \mathbf{Q})$ in terms of right coset representatives for $\Gamma_0(\mathbf{a})$ in Γ, some non-zero ideal $\mathbf{a}$ of ϑ_K. We therefore need a convenient way of writing down these coset representatives which will enable us to perform calculations upon them. In this section we introduce **M-symbols**, named by Cremona in [4], and originally due to Manin (in the rational case) [12].

Consider the set of all ordered pairs (c, d) where $c, d \in \vartheta_K$ and $\gcd(c, d, \mathbf{a}) = 1$.

Definition 1.4.1:

Two such pairs (c_1, d_1) *and* (c_2, d_2) *are equivalent* $\Leftrightarrow c_1 d_2 \equiv c_2 d_1 \pmod{\mathbf{a}}$.

We denote the equivalence class of (c, d) by $(c : d)$.

The set of all such equivalence classes is the projective line, $P^1(\vartheta_K/\mathbf{a})$.

There is a one-to-one correspondence between right coset representatives of $\Gamma_0(\mathbf{a})$ in Γ, and $P^1(\vartheta_K/\mathbf{a})$, given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow (c:d). \tag{1.4.1}$$

Manin [12] proves this in the rational case, using a method which generalises immediately to the imaginary quadratic case (or indeed to any number field).

The action of Γ on $P^1(\vartheta_K/\mathbf{a})$ is given by:

$$(c:d)\begin{pmatrix} p & q \\ r & s \end{pmatrix} = (cp+dr : cq+ds), \tag{1.4.2}$$

since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}.$$

In order to simplify the homology relations we will work with the larger group $\Gamma^{\pm}$ and so we extend the definition as follows:

If $\{\gamma\}$ is a complete set of right coset representatives for $\Gamma_0(\mathbf{a})$ in Γ, then $\{\gamma\} \cup \{J\gamma\}$ is a complete set of right coset representatives for $\Gamma_0(\mathbf{a})$ in $\Gamma^{\pm}$. We extend the set of M-symbols accordingly; We will represent the symbol (γ), where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad-bc=1$, by $(c:d)^+$. Then the symbol $(c:d)^-$ will denote (γ'), where $\gamma' = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$ with $a'd-b'c=\varepsilon$. The action of the matrix $g = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma^+$ on symbol $(c:d)^-$ is again defined by (1.4.2), except that if $\det g \neq 1$ then g also changes the sign on the superscript. However, under the map ξ (defined in Theorem 1.3.1), $(c:d)^-$ and $(\varepsilon c:d)^+$ have the same image. Thus, we can identify $(c:d)^-$ and $(\varepsilon c:d)^+$ with no loss. Therefore we can, and will, drop the superscripts from now on.

In terms of M-symbols, the boundary map δ defined in (1.3.6) is:

$$\delta : (c : d) \mapsto [\frac{a}{c}] - [\frac{b}{d}]. \tag{1.4.3}$$

Also, in terms of M-symbols, the main involution J is:

$$(c : d) \mapsto (\varepsilon c : d). \tag{1.4.4}$$

Finally it will be useful for us to be able to convert modular symbols to M-symbols and vice-versa.

To convert the modular symbol $\{\alpha, 0\} = \{\frac{a}{b}, 0\}$ to M-symbols we use continued fractions as described in §1.3. Take the continued fraction convergents to $\frac{a}{b}$ given by (1.3.7). Then we can write

$$\{0, \frac{a}{b}\} = \sum_{k=-1}^{n} (b_k : (-1)^{k-1} b_{k-1}). \tag{1.4.5}$$

To convert M-symbols to modular symbols we use the isomorphism:

$$\xi : (\gamma) \mapsto \{\gamma(0), \gamma(\infty)\}_{\Gamma_0(\mathfrak{a})},$$

defined in Theorem 1.3.1. Then the M-symbol $(c : d)$ corresponds to (γ) where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, ie

$$(c : d) \mapsto \{\gamma(0), \gamma(\infty)\} = \{\frac{b}{d}, \frac{a}{c}\}. \tag{1.4.6}$$

Note: The first of these conversions, (1.4.5), uses the fact that K is Euclidean. In the non-Euclidean case there is no Euclidean algorithm and so we must use a different method to replace the use of the continued fraction convergents to $\frac{a}{b}$. We will discuss this problem further in §2.7.

§1.5 Hecke Algebra

Just as in §0.5 we considered the Hecke algebra acting on $S(\mathbf{a})$, the space of cusp forms for $\Gamma_0(\mathbf{a})$, so we also consider the Hecke algebra acting on the corresponding space $V(\mathbf{a})$, defined in §1.3.

For prime ideals $\wp$ of ϑ_K, which do not divide $\mathbf{a}$, we define the Hecke operator T_π, where π is a generator of $\wp$. We cannot define the Hecke operator for $\wp$ itself since, in general, $T_\pi \neq T_{\varepsilon\pi}, \varepsilon \in \vartheta_K^*$. So the Hecke operator depends on the choice of π. In fact $T_{\varepsilon\pi} = JT_\pi$.

We define the action of T_π on cusps as follows:

$$T_\pi : [\beta] \mapsto \sum_{x \bmod \pi} [\frac{\beta + x}{\pi}] + [\pi\beta]. \tag{1.5.1}$$

We cannot compute the action of T_π on M-symbols directly. Instead, we first convert to modular symbols using (1.4.6) and then use the definition (0.6.4) of the action on modular symbols. Finally we convert back to M-symbols, using (1.4.5).

The W-involution, W_π, for prime ideals $\wp = (\pi)$ which divide $\mathbf{a}$, is defined on cusps as follows:

$$W_\pi : [\beta] \mapsto [\frac{\pi^r x\beta + y}{\alpha z\beta + \pi^r w}]. \tag{1.5.2}$$

where $\mathbf{a} = (\alpha), r$ is the highest power of π dividing α and x, y, z, w are chosen so that $\pi^{2r}xw - \alpha zy = \pi^r$.

Again, to calculate the action on M-symbols we convert to modular symbols, use the action defined in (0.6.5) and then convert back to M-symbols.

Both the actions of T_π and W_π on cusps and paths are induced from the action of matrices on $\mathrm{K} \cup \{\infty\}$. The only complication is that, for T_π, the image of a point is a set of $N(\pi) + 1$ points.

Chapter 2: A Fundamental Region for $\mathbf{SL(2,\vartheta_K)}$

In Chapter 1 we saw that the fundamental region for the action of $SL(2,\vartheta_K)$ on $\mathcal{H}_3^*$ plays an important part in generating a tessellation of $\mathcal{H}_3^*$ and, therefore, in the calculation of homology. For the five Euclidean fields considered in Chapter 1, the fundamental region is part (or possibly all) of the rectangular or hexagonal chimney with a curved base arising from the unit hemisphere centred on the origin. In the non-Euclidean, class number 1 cases, $K = \mathbf{Q}(\sqrt{-d}), d \in \{19, 43, 67, 163\}$, which we will be concerned with here and in the remainder of Part I, the fundamental region, D, is again a hexagonal chimney (as $d \equiv 3 \pmod 4$) but the curved base does not arise from just one hemisphere.

Because, in Chapter 1, K is Euclidean, the distance of every $k \in K$ from an "integer" is no more than 1 and so the unit hemispheres centred on the integers are sufficient to define D. This is not the case if K is not Euclidean. However, in the case where $h(K) = 1$, the base of D can be described as the intersection of a particular set of hemispheres, centred on the "floor" - $\{(z, \zeta) : \zeta = 0\}$. It is the problem of determining this curved base of D which will concern us in this chapter.

In §2.1 I will introduce the hemispheres of interest and define some notation and properties of them which will be useful in later sections. §2.2 will contain a brief description of part of an algorithm due to Swan for finding a presentation of the group $SL(2,\vartheta_K)$ for imaginary quadratic field K. I will describe only those parts of the algorithm which are used in the determination of D; for full details see [17]. I will then give details of the calculation of D for each of the 4 fields of interest in §2.3 – §2.6.

Finally, because the fields are no longer Euclidean, there is no longer a Euclidean algorithm associated with them. We not only use the Euclidean algorithm for basic arithmetic but also in the continued fraction convergents calculation for converting between modular symbols and M-symbols (§1.4). However because, in our 4 fields, ϑ_{K} is a principal ideal domain we can generate a "pseudo-Euclidean algorithm" so that, given $\alpha, \beta \in \vartheta_{\mathrm{K}}$ we can find $\delta \in \vartheta_{\mathrm{K}}$ such that $(\alpha, \beta) = (\delta)$. This pseudo-Euclidean algorithm uses the fundamental regions defined in this chapter and it will be the subject of §2.7.

§2.1 The Theory of Hemispheres

Recall that the geometry of $\mathcal{H}_3^*$ is hyperbolic. So geodesic lines are vertical half-lines and semi-circles, centred on the floor, while geodesic surfaces are vertical half-planes and hemispheres, centred on the floor. Let $\mathrm{K} = \mathbf{Q}(\sqrt{-d})$ be an imaginary quadratic number field with ring of integers ϑ_{K}.

Definition 2.1.1:

The quotient $\frac{\lambda}{\mu} \in \mathrm{K}$ *is said to be* **reduced** *if:*

(i) $(\lambda, \mu) = \vartheta_{\mathrm{K}}$*;*

(ii) (a) If $d \neq 1$ *or* 3 *then* $\Re(\mu) \geq 0$ *and if* $\Re(\mu) = 0$ *then* $\Im(\mu) > 0$*,*

(b) If $d = 1$ *or* 3 *then* $\mu = x + y\omega$*, where* $x \geq 0$ *and* $y > 0$*.*

Every $\alpha \in \mathrm{K}$ can be written uniquely as a reduced fraction $\frac{\lambda}{\mu}$. From now on we will assume that all fractions are reduced unless otherwise stated.

Definition 2.1.2:

For each $\alpha = \frac{\lambda}{\mu} \in \mathrm{K}$ *we define the hemisphere,* $S_\alpha = S_{\mu,\lambda}$, *with centre at* $(\alpha, 0)$ *and radius* $\frac{1}{|\mu|}$, *by:*

$$S_\alpha := S_{\mu,\lambda} := \{(z,\zeta) \in \mathcal{H}_3^\star : |\mu z - \lambda|^2 + |\mu|^2\zeta^2 = 1\}. \tag{2.1.1}$$

Denote the set of all such S_α by $\mathbf{S}$.

Definition 2.1.3:

We shall say that the hemisphere $S_\alpha \in \mathbf{S}$ **covers** *the point* $(z,\zeta) \in \mathcal{H}_3^\star$ *if* $\exists$ *a point* $(z,\zeta') \in S_\alpha$ *with* $\zeta' > \zeta$.

Equivalently, S_α covers (z,ζ) if $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 < 1$.

Definition 2.1.4:

We say that the hemisphere $S_{\mu,\lambda}$ **covers** *the hemisphere* $S_{\tau,\sigma}$ *at the point* $z \in \mathbf{C}$ *if the inequality*

$$|z - \frac{\lambda}{\mu}|^2 - \frac{1}{|\mu|^2} < |z - \frac{\sigma}{\tau}|^2 - \frac{1}{|\tau|^2}$$

is satisfied.

In fact, this is an abuse of language as there may not be any points on $S_{\mu,\lambda}$ or $S_{\tau,\sigma}$ with co-ordinates (z,ζ). However, it is easy to see that if $(z,\zeta) \in S_{\mu,\lambda}$ then the inequality becomes

$$-\zeta^2 < |z - \frac{\sigma}{\tau}|^2 - \frac{1}{|\tau|^2} = -(\zeta')^2,$$

where $(z, \zeta') \in S_{\tau,\sigma}$ and $\zeta' < \zeta$. So Definition 2.1.4 agrees with the obvious one.

Definition 2.1.5:

A subset $\mathcal{S}$ of hemispheres in $\mathbf{S}$ *is said to be a* **covering** *of the floor if,* $\forall z \in \mathbf{C}$, $\exists$ $S_{\mu,\lambda} \in \mathcal{S}$ *such that* $S_{\mu,\lambda}$ *covers* $(z, 0)$.

Set

$$\mathcal{C} := \{(z,\zeta) \in \mathcal{H}_3^\star : \ |\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geq 1, \forall \lambda, \mu \text{ s.t. } S_{\mu,\lambda} \in \mathcal{S}\}. \tag{2.1.2}$$

So $\mathcal{C}$ is that part of $\mathcal{H}_3^\star$ which lies on, or above, all the hemispheres in $\mathcal{S}$.

In [17] Swan proves the following

Theorem 2.1.6: (Thoerem 2.1, [17])

Let K *be an imaginary quadratic number field with ring of integers* ϑ_{K}. *Then* $\exists$ *a constant* C, *which depends on* K, *with the following property.*

If z is any complex number not in K, *there are an infinite number of solutions* $\lambda, \mu \in \vartheta_{\mathrm{K}}$ *to the inequality*

$$|z - \frac{\lambda}{\mu}| \leq \frac{C}{|\mu|^2}$$

with $(\lambda, \mu) = \vartheta_{\mathrm{K}}$.

ie $\exists$ infinitely many $S_{\mu,\lambda} \in \mathbf{S}$ such that $S_{\mu,\lambda}$ covers z.

Furthermore, because our fields will always have class number 1, ie ϑ_{K} is a principal ideal domain, any $\frac{\lambda}{\mu} \in \mathrm{K}$, written in lowest terms will be such that $(\lambda, \mu) = \vartheta_{\mathrm{K}}$.

ie $S_{\mu,\lambda}$ covers $\frac{\lambda}{\mu}$.

Thus, in our 4 fields, $\forall z \in \mathbf{C}$, $\exists$ at least one $S_{\mu,\lambda} \in \mathbf{S}$ which covers $(z,0)$ and so it will always be possible to find a subset $\mathcal{S}$ of $\mathbf{S}$ which is a covering of the floor.

Note: Swan considers the general case and introduces the notion of singular points. These are points, all lying in K, for which there is no pair (λ,μ) with $(\lambda,\mu) = \vartheta_{\mathrm{K}}$ and $|\mu z - \lambda| < 1$, ie for which there are no $S_{\mu,\lambda}$ which cover them. There are no singular points in K if $h(\mathrm{K}) = 1$ and so these will not concern us here.

Definition 2.1.7:

Let $\mathcal{S}$ be some covering of the floor, with $\mathcal{C}$ as defined in Definition 2.1.5.

Then the **boundary** *of $\mathcal{C}$ is denoted by*

$$\delta\mathcal{C} := \{(z,\zeta) \in \mathcal{H}_3 : (z,\zeta) \in \mathcal{C} \text{ and } (z,\zeta) \in S_{\mu,\lambda} \text{ for at least one } S_{\mu,\lambda} \in \mathcal{S}\},$$

while the **interior** *of $\mathcal{C}$ is given by*

$$int\ \mathcal{C} := \{(z,\zeta) \in \mathcal{H}_3 : \ |\mu z - \lambda|^2 + |\mu|^2\zeta^2 > 1, \forall \lambda,\mu \text{ s.t. } S_{\mu,\lambda} \in \mathcal{S}\}.$$

Definition 2.1.8:

Two hemispheres $S_{\mu,\lambda}$ and $S_{\tau,\sigma}$ intersect in the vertical half plane $P_{\mu,\lambda;\tau,\sigma}$ given by

$$|z - \frac{\lambda}{\mu}|^2 - \frac{1}{|\mu|^2} = |z - \frac{\sigma}{\tau}|^2 - \frac{1}{|\tau|^2}. \tag{2.1.3}$$

On one side of $P_{\mu,\lambda;\tau,\sigma}$, $S_{\mu,\lambda}$ covers $S_{\tau,\sigma}$ while on the other side, $S_{\tau,\sigma}$ covers $S_{\mu,\lambda}$. On the plane itself $S_{\mu,\lambda}$ and $S_{\tau,\sigma}$ meet in a geodesic semicircle, if they meet at all.

Definition 2.1.9:

Given a covering $\mathcal{S}$, with $\mathcal{S}$ as defined in Definition 2.1.5, *define*

$$e_{\mu,\lambda} := \mathcal{C} \cap S_{\mu,\lambda},$$

$\forall\ S_{\mu,\lambda} \in \mathcal{S}$.

Then $e_{\mu,\lambda}$ is the intersection of $S_{\mu,\lambda}$ with a finite number of vertical half spaces. So **either** $e_{\mu,\lambda}$ is empty, **or** a point, **or** an arc of a geodesic **or** a 2-cell bounded by a finite number of geodesic arcs. The $e_{\mu,\lambda}$ which are 2-cells determine a cell division of $\delta\mathcal{C}$. Moreover, the projection of $e_{\mu,\lambda}$ onto $\mathbf{C}$ is a convex polygon set, so the projection of the cell division of $\delta\mathcal{C}$ onto $\mathbf{C}$ is a convex polygonal subdivision of $\mathbf{C}$.

Problem:

Given a point $p = (z, \zeta) \in \mathcal{H}_3$, find all hemispheres $S_{\mu,\lambda} \in \mathbf{S}$ which cover p. ie find all $S_{\mu,\lambda}$ such that $(z, \zeta') \in S_{\mu,\lambda}$ with $\zeta' > \zeta$.

Solution:

We wish to find all $\lambda, \mu \in \vartheta_{\mathrm{K}}$, $(\lambda, \mu) = \vartheta_{\mathrm{K}}$ which satisfy

$$|\mu z + \lambda|^2 + |\mu|^2 \zeta^2 < 1. \qquad (*)$$

Now

$$|\mu z + \lambda|^2 \geq 0 \quad \text{so} \quad |\mu|^2 \zeta^2 < 1,$$

ie.

$$|\mu|^2 < \zeta^{-2}.$$

This gives an upper bound for $|\mu|$.

Now, for each such (fixed) μ, we seek λ, satisfying

$$|\mu z + \lambda|^2 < 1 - \zeta^2 |\mu|^2.$$

Set $\delta = \mu z + \lambda$. We wish to find all δ with $|\delta|^2 < 1 - \zeta^2|\mu|^2$, then $\lambda = \delta - \mu z$. Thus we have upper bounds for λ, μ. Those which satisfy $(\lambda, \mu) = \vartheta_{\mathrm{K}}$ will define a hemisphere in **S** which covers p.

Notice that there are only a finite number of solutions to the inequality $(*)$ and so there are only finitely many hemispheres which cover p.

In Chapter 3 we will also be interested in those hemispheres $S_{\mu,\lambda} \in$ **S** which **contain** a given point p. ie. all $S_{\mu,\lambda}$ such that $p = (z, \zeta) \in S_{\mu,\lambda}$.

In this case we consider all those $\lambda, \mu \in \vartheta_{\mathrm{K}}$ which satisfy

$$|\mu z + \lambda|^2 + |\mu|^2 \zeta^2 = 1, \qquad (\dagger)$$

and proceed as above to find upper bounds for λ and μ. Those which satisfy $(\lambda, \mu) = \vartheta_{\mathrm{K}}$ and $(\dagger)$ are the ones which define a hemisphere of the appropriate type.

§2.2 Swan's algorithm

Let K be one of our 4 fields with class number 1. We wish to find a fundamental region for the action of $SL(2, \vartheta_{\mathrm{K}})$ on $\mathcal{H}_3^\star$. We begin by considering the covering of $\{\zeta = 0\}$ by **S** itself.

Definition 2.2.1:

Let

$$B := \{(z,\zeta) \in \mathcal{H}_3^\star : |\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geq 1, \forall \lambda, \mu \text{ s.t. } S_{\mu,\lambda} \in \mathbf{S}\}.$$

So B is that part of $\mathcal{H}_3^\star$ which lies on, or above, the union of all the hemispheres in $\mathbf{S}$.

Define δB and $intB$, the boundary and interior of B, as in Definition 2.1.7.

If $z \in \mathrm{K}$ then it is clear that S_z covers z; if $z \notin \mathrm{K}$ then $(z, 0)$ is covered, by Theorem 2.1.6. Thus every point $(z,\zeta) \in B$ has $\zeta > 0$.

Given $p = (z,\zeta) \in \mathcal{H}_3^\star$ and $\sigma \in SL(2,\vartheta_{\mathrm{K}})$, define $\sigma p = (z', \zeta')$ according to (0.2.2). Then we have the following:

Lemma 2.2.2: (Lemma 3.5, [17])

For every $p \in \mathcal{H}_3^\star$, $\exists \sigma \in SL(2,\vartheta_{\mathrm{K}})$ *such that* $\sigma p \in B$.

Lemma 2.2.3: (Lemma 3.6, [17])

A point $p \in \mathcal{H}_3^\star$ *is in* $B \Leftrightarrow \forall \sigma \in SL(2,\vartheta_{\mathrm{K}})$ *,*$\sigma p = (z', \zeta')$ *is such that* $\zeta' \leq \zeta$.

So, for any point $p \in \mathcal{H}_3^\star$, the transform of p (by an element of $SL(2,\vartheta_{\mathrm{K}})$) which has the largest ζ co-ordinate is in B.

We now define a fundamental region, F, for the action of T and $U \in SL(2,\vartheta_{\mathrm{K}})$ on $\mathcal{H}_3^\star$.

Definition 2.2.4:

$$F := \{z \in \mathbf{C} : |z| \leq |z - z_0|, \forall z_0 \in \vartheta_{\mathrm{K}}\}. \tag{2.2.1}$$

F is the set of points whose distance from the origin is less than their distance from any other integer. If K=$\mathbf{Q}(\sqrt{-d})$, $d \equiv 3 \pmod 4$, this is a hexagon.

Definition 2.2.5:

Set $D := \{(z, \zeta) \in B : z \in F\}$.

Then we have:

Lemma 2.2.6:

$SL(2, \vartheta_{\mathrm{K}})D = \mathcal{H}_3^\star$.

Proof:

Given some point $p \in \mathcal{H}_3^\star$ we can transform p into B using Lemma 2.2.2 and then translate into F, using suitable powers of T and U. ∎

Theorem 2.2.7: (Theorem 3.13, [17])

There are only finitely many $\alpha \in \mathrm{K}$ *such that* $S_\alpha \in \mathbf{S}$ *and* $D \cap S_\alpha \neq \phi$.

So D is bounded below by finitely many S_α.

Definition 2.2.8:

Set

$$B(\alpha_1, \ldots, \alpha_n) := \{(z, \zeta) \in \mathcal{H}_3 : (z, \zeta) \textit{ lies on, or above, all } S_{\alpha_i + \delta}, i = 1, \ldots, n; \delta \in \vartheta_{\mathrm{K}}\}.$$

It follows from Theorem 2.2.7 that $B = B(\alpha_1, \ldots, \alpha_n)$, for some finite set $\{\alpha_1, \ldots, \alpha_n\}$. Given any finite set $\{\alpha_1, \ldots, \alpha_n\}$, Swan's algorithm gives a method for deciding when $B(\alpha_1, \ldots, \alpha_n) = B$ and for finding the cell decomposition of δB.

If $B(\alpha_1, \ldots, \alpha_n) \neq B$ it gives a method for determining which other α_j should be included.

Note: Because B is obtained from D by translating by T and U it is sufficient to find $\{\alpha_1, \ldots, \alpha_n\}$ such that

$$\begin{aligned} D(\alpha_1, \ldots, \alpha_n) &:= \{(z, \zeta) \in B(\alpha_1, \ldots, \alpha_n) : z \in F\} \\ &= D \end{aligned}$$

Then $B(\alpha_1, \ldots, \alpha_n) = B$

Swan's algorithm is as follows:

[**1**] Begin with a set $\{\alpha_1, \ldots, \alpha_n\}$ such that $D(\alpha_1, \ldots, \alpha_n)$ covers F. Then

$$\mathcal{S} := \{S_{\alpha_i + \delta} : i = 1, \ldots, n, \delta \in \vartheta_{\mathrm{K}}\}$$

is a covering of $\{\zeta = 0\}$.

The most efficient way of finding such an initial covering set is to start with $\alpha_1 = \frac{0}{1}$, ie the hemisphere with the largest possible radius, and then add $\alpha_i = \frac{\lambda_i}{\mu_i}$ with $\alpha_i \in F$ and $N(\mu_i)$ strictly increasing. Thus, we begin with $\alpha_1 = \frac{0}{1}$, then add $\alpha_2 = \frac{\omega}{2}$, then $\alpha_3, \alpha_4 = \frac{\omega}{3}, \frac{\omega+1}{3}$ and so on. Because K has class number 1 this process will terminate (ie all (z, ζ) will be covered) after a finite number of steps.

[**2**] Find $V := \{p \in \mathcal{H}_3 : p \in S_{\alpha_i}$ for 3 or more $i \in \{1, 2, \ldots, n\}\}$.

[**3**] For each $v \in V$ find $H_v :=$ { hemispheres in **S** which cover v}, using the method described in §2.1.

[**4**] If $H_v = \phi, \forall v \in V$ then $B(\alpha_1, \ldots, \alpha_n) = B$ and we are done.

Otherwise:

[5] $\forall v = (z,\zeta) \in V$ with $H_v \neq \phi$ find $S_\beta \in H_v$ such that $(z,\zeta') \in S_\beta$ has the largest ζ co-ordinate.

[6] Set $B(\alpha_1,\ldots,\alpha_n,\alpha_{n+1}) = B(\alpha_1,\ldots,\alpha_n,\beta)$.

[7] Goto [2].

The termination of the algorithm is guaranteed by Theorem 2.2.7.

The claim in step [4] follows from:

Proposition 2.2.9: (Proposition 8.4, [17])

$B(\alpha_1,\ldots,\alpha_n) = B \Leftrightarrow$ there is no $v \in V$ for which $H_v \neq \phi$.

For convenience we make the following:

Definition 2.2.10:

If $B(\alpha_1,\ldots,\alpha_n) = B$ then let

$$A := \{\alpha_1,\ldots,\alpha_n\}.$$

For each $\alpha = \frac{\lambda}{\mu} \in A$ define the matrix

$$M_\alpha := \begin{pmatrix} a & b \\ \mu & -\lambda \end{pmatrix}$$

to be such that $M_\alpha \in SL(2,\vartheta_{\mathrm{K}})$, $M_\alpha(\alpha) = \infty$ and $M_\alpha(\infty) \in A$.

Set

$$M := \{M_{\alpha_i} : \alpha_i \in A\}.$$

Then $SL(2,\vartheta_{\mathrm{K}})$ is generated by T, U, M and $R = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$, and $D = D(\alpha_1,\ldots,\alpha_n)$ is a fundamental region for $SL(2,\vartheta_{\mathrm{K}})$.

Note: If, in addition to translations by ϑ_K, we also consider two more symmetries of the configuration; namely negation, $(z, \zeta) \mapsto (-z, \zeta)$, and conjugation, $(z, \zeta) \mapsto (\bar{z}, \zeta)$; we can further reduce the region we need to cover in order to define B. This smaller region is

$$F_0 := \{z = \xi + i\eta : 0 \leq \xi \leq \frac{1}{2}, 0 \leq \eta \leq \frac{\sqrt{d}}{4}\}. \tag{2.2.2}$$

In the next 4 sections I will give details of the calculations for each of the 4 non-euclidean, class number 1 fields. For each field I will give A, M and V_0, the points of intersection of the S_α with $\alpha \in A$ in the reduced region F_0, along with pictures of the cell decomposition of F_0 and the cell structure of δB above the region $F_0^* := \{z = \xi + i\eta : 0 \leq \xi \leq 1, 0 \leq \eta \leq \frac{\sqrt{d}}{4}\}$.

We shall use A and M in the pseudo-euclidean algorithm in §2.7 and A and V in Chapter 3, where we will find a tessellation of $\mathcal{H}_3^*$ by hyperbolic polyhedra.

Note: The centre of S_α doesn't always lie directly under $S_\alpha \cap B$ - see §2.6.

§2.3 The Calculation for $\mathbf{Q}(\sqrt{-19})$

This case is already very well documented; in particular we reference [17]. The initial choice of covering of F_0 by hemispheres of radii 1 and $\frac{1}{2}$ is sufficient to determine B. ie $B(\frac{0}{1}, \frac{\omega-1}{2}, \frac{\omega}{2}) = B$.

δB over F_0^* is shown in Figure (2.3.1) while the cell decomposition of F_0 is illustrated in Figure (2.3.2). Only the solid lines represent edges of the decomposition; the dotted lines indicate the boundaries of F_0. The $\alpha \in A$ and $M_\alpha \in M$ are given in Table (i) while the $v \in V_0$ are given in Table (ii).

Table (i):

α	M_α	$M_\alpha(\infty)$
0	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	0
$\frac{\omega-1}{2}$	$\begin{pmatrix} \omega & 2 \\ 2 & 1-\omega \end{pmatrix}$	$\frac{\omega}{2}$
$\frac{\omega}{2}$	$\begin{pmatrix} \omega-1 & 2 \\ 2 & -\omega \end{pmatrix}$	$\frac{\omega-1}{2}$

Table (ii):

v_1	$(0, \frac{4}{\sqrt{19}}; \sqrt{\frac{3}{19}})$
v_2	$(\frac{1}{2}, \frac{7}{2\sqrt{19}}; \sqrt{\frac{2}{19}})$

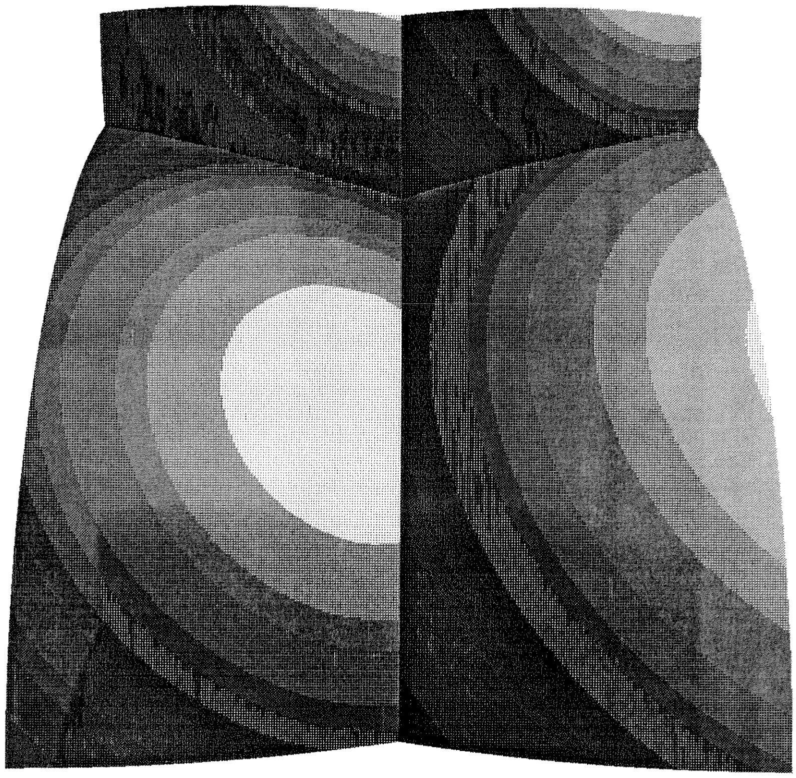


Figure (2.3.1)

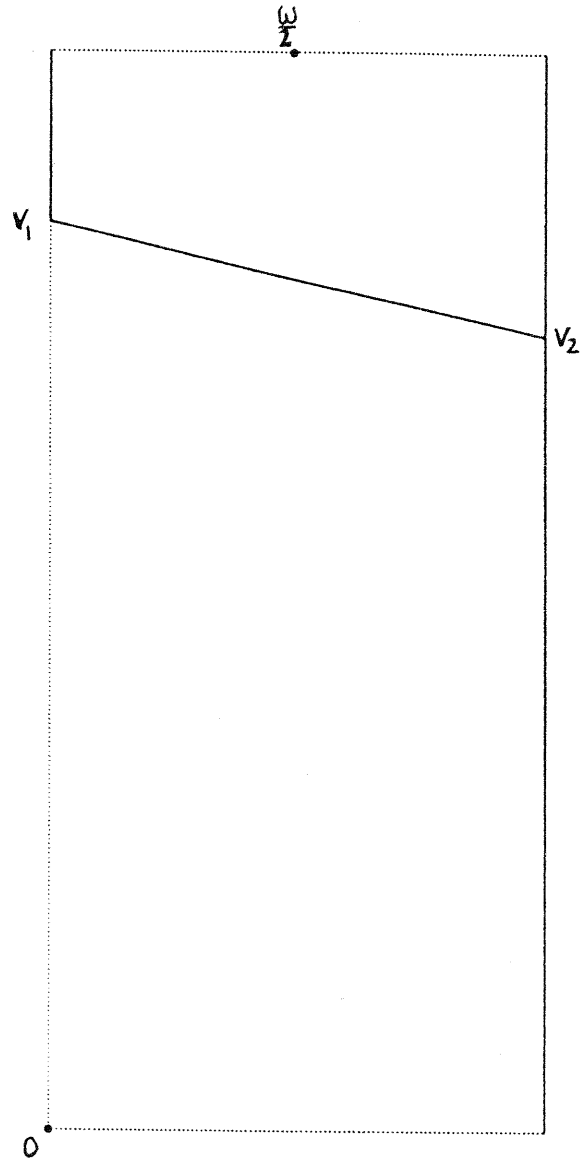


Figure (2.3.2)

§2.4 The Calculation for $\mathbf{Q}(\sqrt{-43})$

This case is dealt with in [8]. Again, the initial covering of F_0 is sufficient to define B. ie $B(0, \frac{\omega-1}{2}, \frac{\omega}{2}, \frac{\omega-1}{3}, \frac{\omega}{3}, \frac{\omega+1}{3}, \frac{2-\omega}{3}, \frac{-\omega}{3}, \frac{1-\omega}{3}) = B$.

δB over F_0^* is shown in Figure (2.4.1); the cell decomposition of F_0 is shown in Figure (2.4.2). The elements of A and M are listed in Table (i); the $v \in V_0$ are listed in Table (ii).

Table (i):

α	M_α	$M_\alpha(\infty)$
0	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	0
$\frac{\omega-1}{2}$	$\begin{pmatrix} \omega & 5 \\ 2 & 1-\omega \end{pmatrix}$	$\frac{\omega}{2}$
$\frac{\omega}{2}$	$\begin{pmatrix} \omega-1 & 5 \\ 2 & -\omega \end{pmatrix}$	$\frac{\omega-1}{2}$
$\frac{\omega-1}{3}$	$\begin{pmatrix} -\omega & -4 \\ 3 & 1-\omega \end{pmatrix}$	$\frac{-\omega}{3}$
$\frac{\omega}{3}$	$\begin{pmatrix} 1-\omega & -4 \\ 3 & -\omega \end{pmatrix}$	$\frac{1-\omega}{3}$
$\frac{\omega+1}{3}$	$\begin{pmatrix} 1+\omega & 3-\omega \\ 3 & -1-\omega \end{pmatrix}$	$\frac{\omega+1}{3}$
$\frac{2-\omega}{3}$	$\begin{pmatrix} 2-\omega & 2+\omega \\ 3 & 2-\omega \end{pmatrix}$	$\frac{2-\omega}{3}$
$\frac{-\omega}{3}$	$\begin{pmatrix} \omega-1 & -4 \\ 3 & \omega \end{pmatrix}$	$\frac{\omega-1}{3}$
$\frac{1-\omega}{3}$	$\begin{pmatrix} \omega & -4 \\ 3 & \omega-1 \end{pmatrix}$	$\frac{\omega}{3}$

Table (ii):

v_1	$(0, \frac{25}{3\sqrt{43}}; \frac{2}{3}\sqrt{\frac{5}{43}})$
v_2	$(\frac{1}{3}, \frac{8}{\sqrt{43}}; \frac{1}{3}\sqrt{\frac{26}{43}})$
v_3	$(\frac{1}{2}, \frac{17}{2\sqrt{43}}; \sqrt{\frac{3}{43}})$
v_4	$(0, \frac{19}{3\sqrt{43}}; \frac{1}{3}\sqrt{\frac{26}{43}})$
v_5	$(\frac{1}{3}, \frac{6}{\sqrt{43}}; \frac{2}{3}\sqrt{\frac{5}{43}})$
v_6	$(\frac{1}{2}, \frac{11}{2\sqrt{43}}; \sqrt{\frac{2}{43}})$

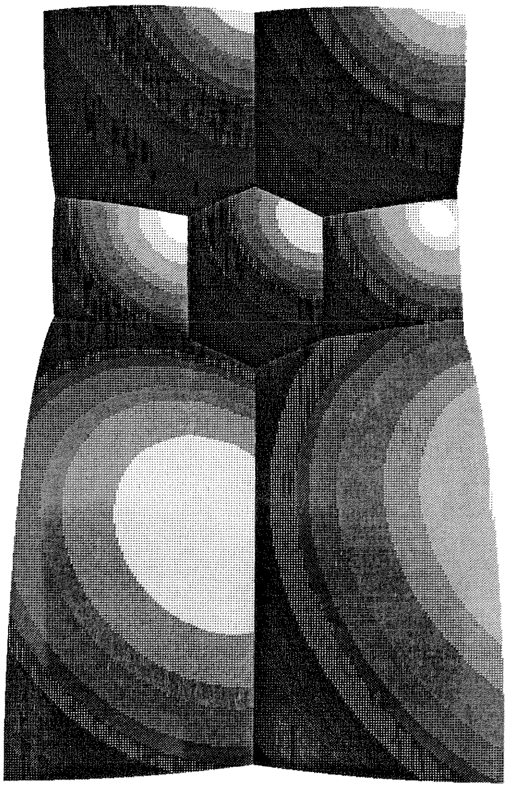


Figure (2.4.1)

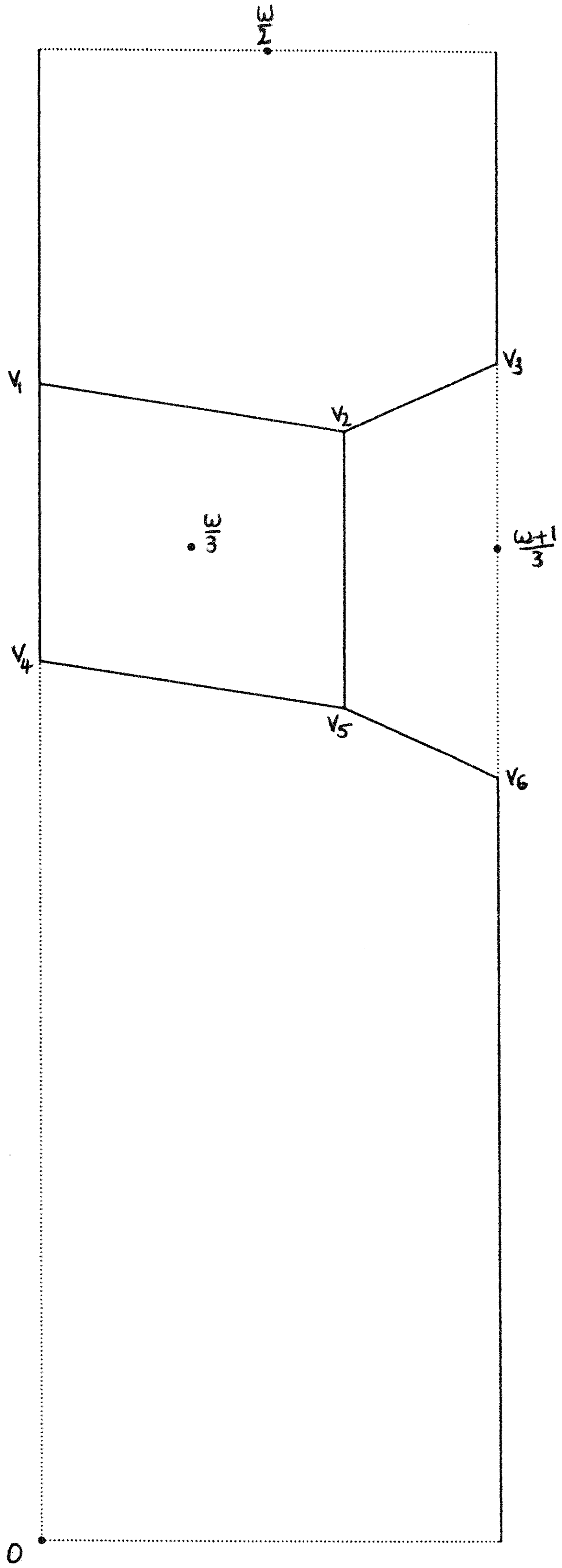


Figure (2.4.2)

§2.5 The Calculation for $\mathbf{Q}(\sqrt{-67})$

As before, we begin with a covering of F_0 by hemispheres in $\mathbf{S}$ with radii as large as possible. We now wish to acertain whether

$$B(0, \tfrac{\omega-1}{2}, \tfrac{\omega}{2}, \tfrac{\omega-1}{3}, \tfrac{\omega}{3}, \tfrac{\omega+1}{3}, \tfrac{-\omega}{3}, \tfrac{1-\omega}{3}, \tfrac{2-\omega}{3},$$

$$\tfrac{\omega-1}{4}, \tfrac{\omega}{4}, \tfrac{\omega+1}{4}, \tfrac{1-\omega}{4}, \tfrac{2-\omega}{4}, \tfrac{-1-\omega}{4}, \tfrac{\omega-2}{4}, \tfrac{-\omega}{4}) = B.$$

The associated cell decomposition of F_0 is shown in Figure (2.5.1a) where the $v_i \in V$ are listed in Table (i):

Table (i):

v_1	$(0, \frac{40}{3\sqrt{67}}; \frac{2}{3}\sqrt{\frac{2}{67}})$
v_2	$(\frac{1}{3}, \frac{13}{\sqrt{67}}; \frac{2}{3}\sqrt{\frac{5}{67}})$
v_3	$(\frac{1}{2}, \frac{27}{2\sqrt{67}}; \sqrt{\frac{2}{67}})$
v_4	$(0, \frac{28}{3\sqrt{67}}; \frac{2}{3}\sqrt{\frac{5}{67}})$
v_5	$(\frac{1}{4}, \frac{109}{12\sqrt{67}}; \frac{1}{6}\sqrt{\frac{95}{67}})$
v_6	$(\frac{1}{3}, \frac{19}{2\sqrt{67}}; \frac{1}{6}\sqrt{\frac{101}{67}})$
v_7	$(\frac{1}{2}, \frac{9}{\sqrt{67}}; \frac{1}{2}\sqrt{\frac{11}{67}})$
v_8	$(0, \frac{8}{\sqrt{67}}; \sqrt{\frac{3}{67}})$
v_9	$(\frac{1}{4}, \frac{31}{4\sqrt{67}}; \frac{1}{2}\sqrt{\frac{11}{67}})$
v_{10}	$(\frac{1}{2}, \frac{7}{\sqrt{67}}; \frac{1}{2}\sqrt{\frac{5}{67}})$

For each v_i in turn we find H_{v_i}. For $2 \leq i \leq 9, H_{v_i} = \phi$. However this is not the case when $i = 1$ and $i = 10$. We consider both of these vertices in turn.

Case 1: $\mathbf{v_1} = (\mathbf{0}, \frac{40}{3\sqrt{67}}; \frac{2}{3}\sqrt{\frac{2}{67}})$

$$H_{v_1} = \{S_\alpha : \alpha \in \{\tfrac{\omega-7}{\omega+2}, \tfrac{2\omega-1}{5}\}\},$$

and the corresponding ζ co-ordinates of $(0, \frac{40}{3\sqrt{67}}; \zeta)$ are:

$\{\sqrt{\frac{586}{13869}}, \sqrt{\frac{602}{15075}}\}$.

So the "best" covering of v_1, ie the hemisphere which has the largest ζ co-ordinate, is the hemisphere with centre $\frac{\omega-7}{\omega+2}$.

We add this to our set of α_i.

The new cell decompostion of F_0 is shown in Figure (2.5.1b) and the new vertices are listed in Table (ii).

Table (ii):

v_{11}	$(0, \frac{68}{5\sqrt{67}}; \frac{1}{5}\sqrt{\frac{66}{67}})$
v_{12}	$(\frac{1}{12}, \frac{53}{4\sqrt{67}}; \frac{1}{6}\sqrt{\frac{95}{67}})$
v_{13}	$(0, \frac{77}{6\sqrt{67}}; \frac{1}{6}\sqrt{\frac{101}{67}})$

Repeating the calculation on these new vertices, we find that $H_{v_{11}} = H_{v_{12}} = H_{v_{13}} = \phi$. So none of the new vertices can be covered by any hemisphere in **S**.

Case 2: $\mathbf{v_{10}} = (\frac{1}{2}, \frac{7}{\sqrt{67}}; \frac{1}{2}\sqrt{\frac{5}{67}})$

$H_{v_{10}} = \{S_\alpha : \alpha \in \{\frac{\omega-3}{\omega+2}, \frac{\omega+2}{5}\}\}$,

with corresponding ζ co-ordinates:

$\{\sqrt{\frac{249}{6164}}, \sqrt{\frac{259}{6700}}\}$.

So the best covering of v_{10} is by the hemisphere with centre $\frac{\omega-3}{\omega+2}$. We add this centre to the list of α_i.

The new cell decomposition of F_0 is shown in Figure (2.5.1c) where the new vertices are listed in Table (iii).

Table (iii):

v_{14}	$(\frac{5}{12}, \frac{29}{4\sqrt{67}}; \frac{1}{6}\sqrt{\frac{101}{67}})$
v_{15}	$(\frac{1}{2}, \frac{23}{3\sqrt{67}}; \frac{1}{6}\sqrt{\frac{95}{67}})$
v_{16}	$(\frac{1}{2}, \frac{69}{10\sqrt{67}}; \frac{1}{5}\sqrt{\frac{66}{67}})$

Again, $H_{v_{14}} = H_{v_{15}} = H_{v_{16}} = \phi$.

Now no vertex of V can be covered by any $S_{\mu,\lambda} \in \mathbf{S}$, so we have found A. The final set, V_0, of vertices is given in Table (iv). δB over F_0^* is shown in Figure (2.5.2) and A and M are listed in Table (v).

Table (iv):

v_2	$(\frac{1}{3}, \frac{13}{\sqrt{67}}; \frac{2}{3}\sqrt{\frac{5}{67}})$
v_3	$(\frac{1}{2}, \frac{27}{2\sqrt{67}}; \sqrt{\frac{2}{67}})$
v_4	$(0, \frac{28}{3\sqrt{67}}; \frac{2}{3}\sqrt{\frac{5}{67}})$
v_5	$(\frac{1}{4}, \frac{109}{12\sqrt{67}}; \frac{1}{6}\sqrt{\frac{95}{67}})$
v_6	$(\frac{1}{3}, \frac{19}{2\sqrt{67}}; \frac{1}{6}\sqrt{\frac{101}{67}})$
v_7	$(\frac{1}{2}, \frac{9}{\sqrt{67}}; \frac{1}{2}\sqrt{\frac{11}{67}})$
v_8	$(0, \frac{8}{\sqrt{67}}; \sqrt{\frac{3}{67}})$
v_9	$(\frac{1}{4}, \frac{31}{4\sqrt{67}}; \frac{1}{2}\sqrt{\frac{11}{67}})$
v_{11}	$(0, \frac{68}{5\sqrt{67}}; \frac{1}{5}\sqrt{\frac{66}{67}})$
v_{12}	$(\frac{1}{12}, \frac{53}{4\sqrt{67}}; \frac{1}{6}\sqrt{\frac{95}{67}})$
v_{13}	$(0, \frac{77}{6\sqrt{67}}; \frac{1}{6}\sqrt{\frac{101}{67}})$
v_{14}	$(\frac{5}{12}, \frac{29}{4\sqrt{67}}; \frac{1}{6}\sqrt{\frac{101}{67}})$
v_{15}	$(\frac{1}{2}, \frac{23}{3\sqrt{67}}; \frac{1}{6}\sqrt{\frac{95}{67}})$
v_{16}	$(\frac{1}{2}, \frac{69}{10\sqrt{67}}; \frac{1}{5}\sqrt{\frac{66}{67}})$

Table (v):

α	M_α	$M_\alpha(\infty)$
0	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	0
$\frac{\omega-1}{2}$	$\begin{pmatrix} \omega & 8 \\ 2 & 1-\omega \end{pmatrix}$	$\frac{\omega}{2}$
$\frac{\omega}{2}$	$\begin{pmatrix} \omega-1 & 8 \\ 2 & -\omega \end{pmatrix}$	$\frac{\omega-1}{2}$
$\frac{\omega+6}{3-\omega}$	$\begin{pmatrix} 2+\omega & 7-\omega \\ 3-\omega & -6-\omega \end{pmatrix}$	$\frac{2+\omega}{3-\omega}$
$\frac{\omega+2}{3-\omega}$	$\begin{pmatrix} 6+\omega & 7-\omega \\ 3-\omega & -2-\omega \end{pmatrix}$	$\frac{6+\omega}{3-\omega}$
$\frac{-2-\omega}{3-\omega}$	$\begin{pmatrix} -6-\omega & 7-\omega \\ 3-\omega & 2+\omega \end{pmatrix}$	$\frac{-6-\omega}{3-\omega}$
$\frac{-6-\omega}{3-\omega}$	$\begin{pmatrix} -2-\omega & 7-\omega \\ 3-\omega & 6+\omega \end{pmatrix}$	$\frac{-2-\omega}{3-\omega}$
$\frac{\omega-7}{\omega+2}$	$\begin{pmatrix} \omega-3 & \omega+6 \\ \omega+2 & 7-\omega \end{pmatrix}$	$\frac{\omega-3}{\omega+2}$
$\frac{\omega-3}{\omega+2}$	$\begin{pmatrix} \omega-7 & \omega+6 \\ \omega+2 & 3-\omega \end{pmatrix}$	$\frac{\omega-7}{\omega+2}$
$\frac{7-\omega}{\omega+2}$	$\begin{pmatrix} 3-\omega & \omega+6 \\ \omega+2 & \omega-7 \end{pmatrix}$	$\frac{3-\omega}{\omega+2}$
$\frac{3-\omega}{\omega+2}$	$\begin{pmatrix} 7-\omega & \omega+6 \\ \omega+2 & \omega-3 \end{pmatrix}$	$\frac{7-\omega}{\omega+2}$
$\frac{\omega-1}{3}$	$\begin{pmatrix} -\omega & -6 \\ 3 & 1-\omega \end{pmatrix}$	$\frac{-\omega}{3}$
$\frac{\omega}{3}$	$\begin{pmatrix} 1-\omega & -6 \\ 3 & -\omega \end{pmatrix}$	$\frac{1-\omega}{3}$
$\frac{\omega+1}{3}$	$\begin{pmatrix} \omega+1 & 5-\omega \\ 3 & -1-\omega \end{pmatrix}$	$\frac{1+\omega}{3}$
$\frac{-\omega}{3}$	$\begin{pmatrix} \omega-1 & -6 \\ 3 & \omega \end{pmatrix}$	$\frac{\omega-1}{3}$
$\frac{1-\omega}{3}$	$\begin{pmatrix} \omega & -6 \\ 3 & \omega-1 \end{pmatrix}$	$\frac{\omega}{3}$
$\frac{2-\omega}{3}$	$\begin{pmatrix} 2-\omega & \omega+4 \\ 3 & \omega-2 \end{pmatrix}$	$\frac{2-\omega}{3}$

Table (v) continued:

α	M_α	$M_\alpha(\infty)$
$\frac{\omega-1}{4}$	$\begin{pmatrix} \omega & 4 \\ 4 & 1-\omega \end{pmatrix}$	$\frac{\omega}{4}$
$\frac{\omega}{4}$	$\begin{pmatrix} \omega-1 & 4 \\ 4 & -\omega \end{pmatrix}$	$\frac{\omega-1}{4}$
$\frac{\omega+1}{4}$	$\begin{pmatrix} 2-\omega & -5 \\ 4 & -1-\omega \end{pmatrix}$	$\frac{2-\omega}{4}$
$\frac{1-\omega}{4}$	$\begin{pmatrix} -\omega & 4 \\ 4 & \omega-1 \end{pmatrix}$	$\frac{-\omega}{4}$
$\frac{2-\omega}{4}$	$\begin{pmatrix} 1+\omega & -5 \\ 4 & \omega-2 \end{pmatrix}$	$\frac{1+\omega}{4}$
$\frac{-1-\omega}{4}$	$\begin{pmatrix} \omega-2 & -5 \\ 4 & \omega+1 \end{pmatrix}$	$\frac{\omega-2}{4}$
$\frac{\omega-2}{4}$	$\begin{pmatrix} -1-\omega & -5 \\ 4 & 2-\omega \end{pmatrix}$	$\frac{-1-\omega}{4}$
$\frac{-\omega}{4}$	$\begin{pmatrix} 1-\omega & 4 \\ 4 & \omega \end{pmatrix}$	$\frac{1-\omega}{4}$

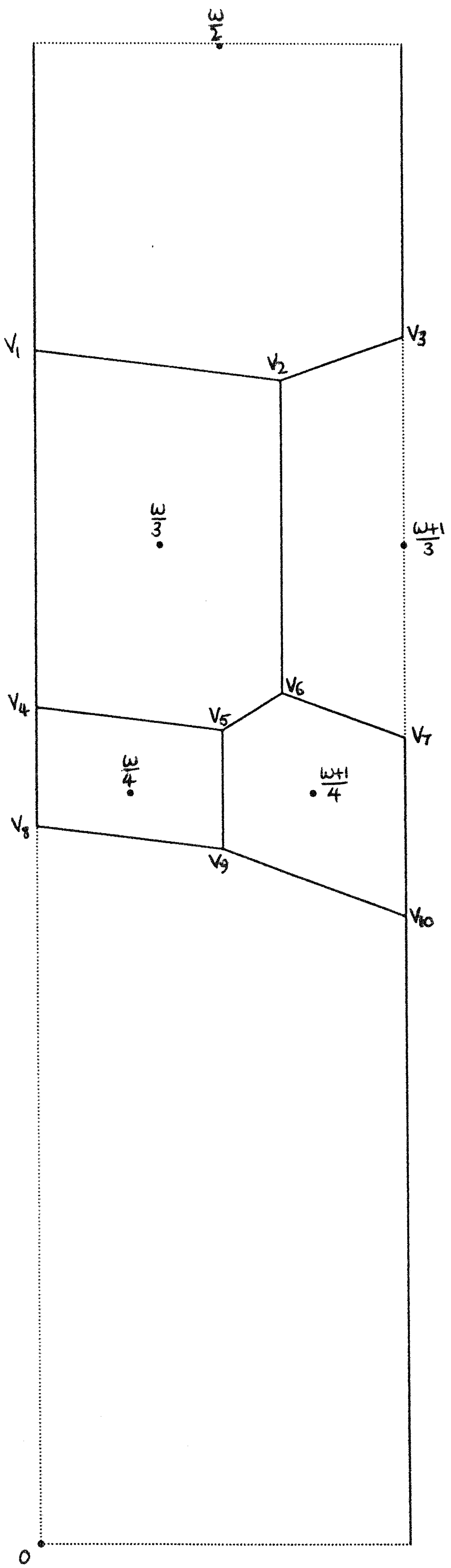


Figure (2.5.1a)

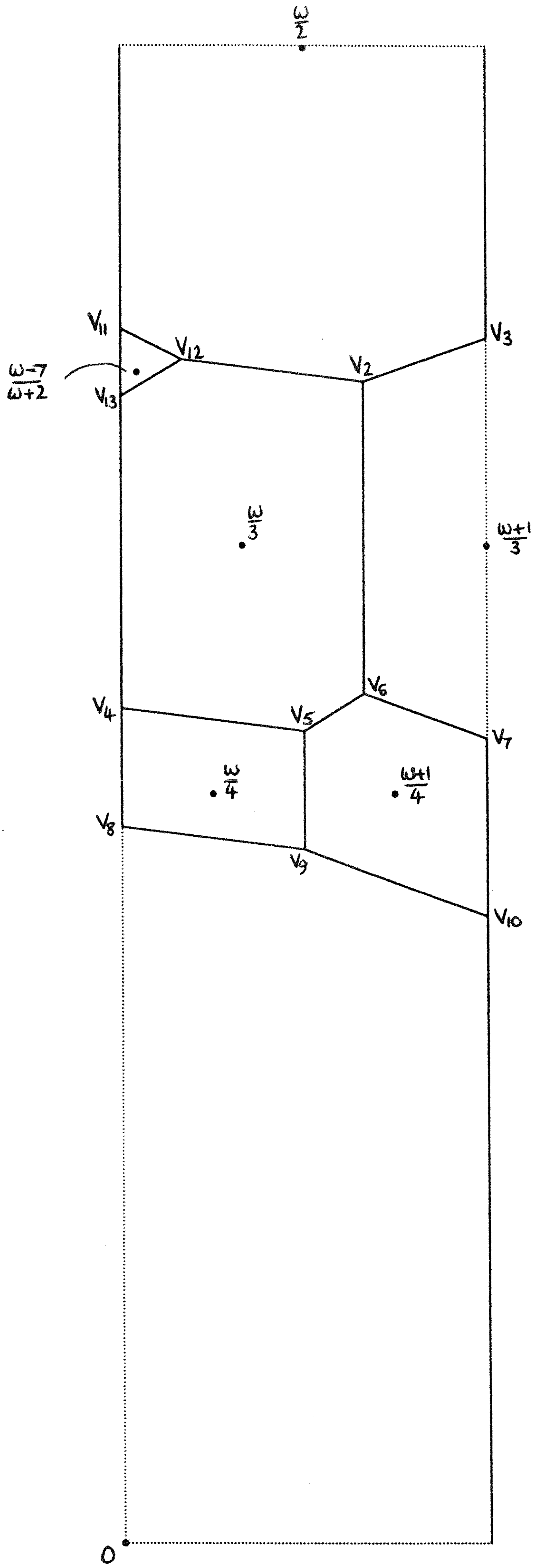


Figure (2.5.1b)

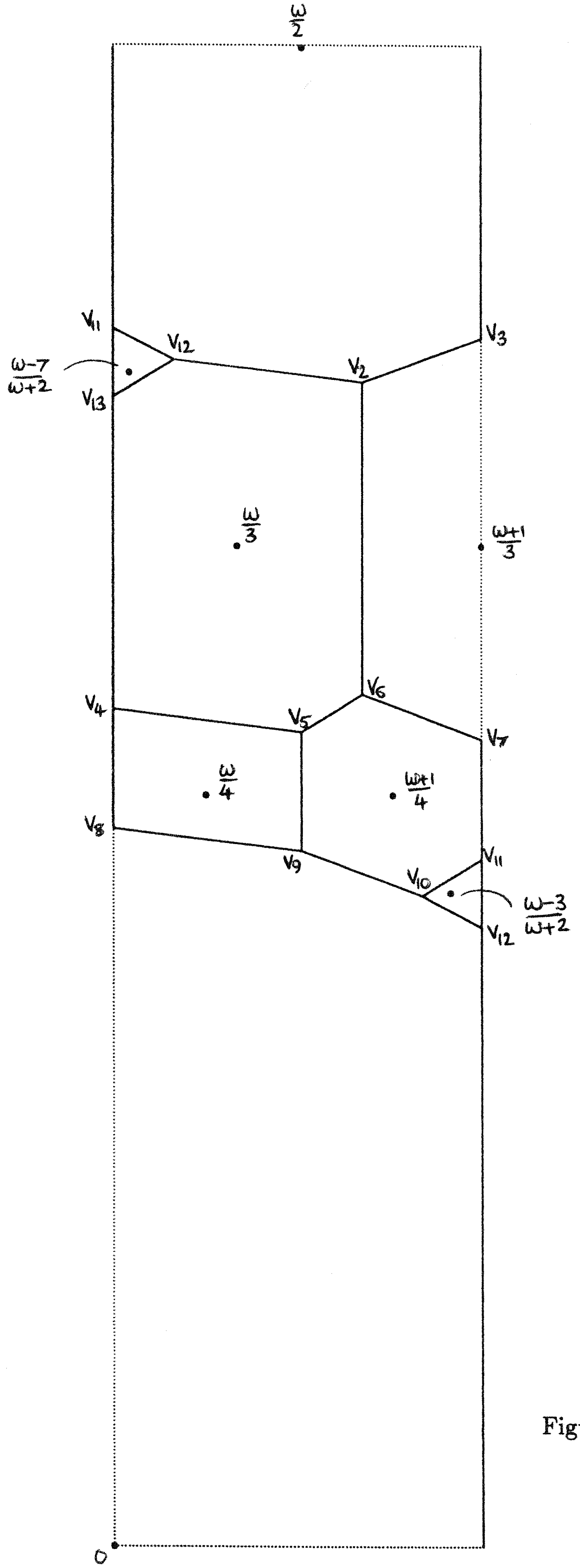

ω/2
V11
V12
(ω−7)/(ω+2)
V13
V2
V3
ω/3
(ω+1)/3
V4
V6
V5
V7
ω/4
(ω+1)/4
V8
V9
V10
V11
(ω−3)/(ω+2)
V12
0

Figure (2.5.1c)

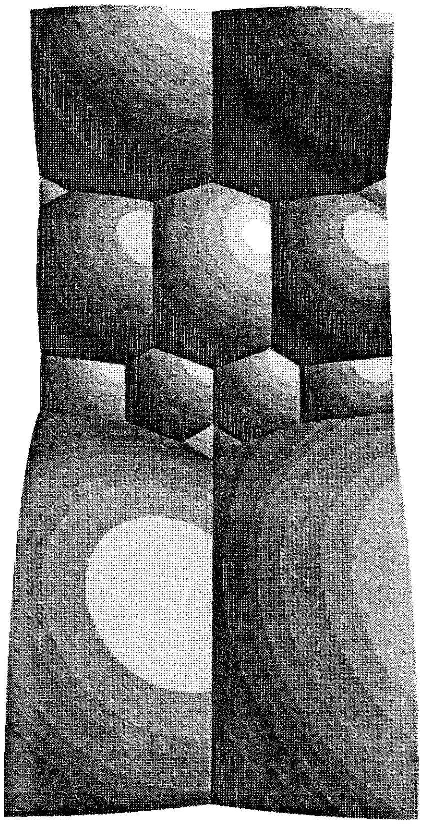


Figure (2.5.2)

§2.6 The Calculation for $\mathbf{Q}(\sqrt{-163})$

We proceed as in the other 3 cases. I will not give details of the calculation in this field as it is straightforward but long. The progressive stages of the calculation can be seen in the cell decompositions of F_0, shown in Figures (2.6.1a) to (2.6.1e), where the vertices of the final resulting set, V_0, are listed in Table (i). The elements of A are listed in Table (ii) along with $M_\alpha(\infty), \forall\alpha \in A$. δB over F_0^* is shown in Figure (2.6.2).

Note: In this field it is not always the case that $\alpha \in A$ lies directly beneath $S_\alpha \cap B$.

Table (i):

v_1	$\left(0, \frac{248}{7\sqrt{163}}; \sqrt{\frac{110}{7987}}\right)$	v_{29}	$\left(\frac{1}{14}, \frac{323}{14\sqrt{163}}; \sqrt{\frac{120}{7987}}\right)$
v_2	$\left(\frac{1}{7}, \frac{243}{7\sqrt{163}}; \sqrt{\frac{120}{7987}}\right)$	v_{30}	$\left(\frac{3}{14}, \frac{321}{14\sqrt{163}}; \sqrt{\frac{116}{7987}}\right)$
v_3	$\left(\frac{2}{7}, \frac{242}{7\sqrt{163}}; \sqrt{\frac{116}{7987}}\right)$	v_{31}	$\left(\frac{5}{21}, \frac{159}{7\sqrt{163}}; \sqrt{\frac{1160}{71883}}\right)$
v_4	$\left(\frac{3}{7}, \frac{35}{\sqrt{163}}; \sqrt{\frac{122}{7987}}\right)$	v_{32}	$\left(\frac{73}{210}, \frac{977}{42\sqrt{163}}; \sqrt{\frac{27982}{1797075}}\right)$
v_5	$\left(\frac{1}{2}, \frac{71}{2\sqrt{163}}; \sqrt{3/163}\right)$	v_{33}	$\left(\frac{7}{20}, \frac{461}{20\sqrt{163}}; \sqrt{\frac{293}{16300}}\right)$
v_6	$\left(0, \frac{518}{15\sqrt{163}}; \sqrt{\frac{626}{36675}}\right)$	v_{34}	$\left(\frac{1}{2}, \frac{113}{5\sqrt{163}}; \sqrt{\frac{269}{16300}}\right)$
v_7	$\left(\frac{1}{10}, \frac{1021}{30\sqrt{163}}; \sqrt{\frac{638}{36675}}\right)$	v_{35}	$\left(\frac{1}{4}, \frac{451}{20\sqrt{163}}; \sqrt{\frac{291}{16300}}\right)$
v_8	$\left(\frac{9}{70}, \frac{7211}{210\sqrt{163}}; \sqrt{\frac{29408}{1797075}}\right)$	v_{36}	$\left(0, \frac{18}{\sqrt{163}}; \sqrt{\frac{2}{163}}\right)$
v_9	$\left(\frac{61}{70}, \frac{7177}{210\sqrt{163}}; \sqrt{\frac{29782}{1797075}}\right)$	v_{37}	$\left(\frac{1}{5}, \frac{89}{5\sqrt{163}}; \sqrt{\frac{66}{4075}}\right)$
v_{10}	$\left(\frac{3}{10}, \frac{34}{\sqrt{163}}; \sqrt{\frac{293}{16300}}\right)$	v_{38}	$\left(\frac{1}{4}, \frac{363}{20\sqrt{163}}; \sqrt{\frac{269}{16300}}\right)$
v_{11}	$\left(\frac{9}{20}, \frac{689}{20\sqrt{163}}; \sqrt{\frac{269}{16300}}\right)$	v_{39}	$\left(\frac{2}{5}, \frac{177}{10\sqrt{163}}; \sqrt{\frac{293}{16300}}\right)$
v_{12}	$\left(\frac{1}{2}, \frac{341}{10\sqrt{163}}; \sqrt{\frac{66}{4075}}\right)$	v_{40}	$\left(\frac{1}{2}, \frac{91}{5\sqrt{163}}; \sqrt{\frac{291}{16300}}\right)$
v_{13}	$\left(0, \frac{307}{10\sqrt{163}}; \sqrt{\frac{291}{16300}}\right)$	v_{41}	$\left(0, \frac{44}{3\sqrt{163}}; \sqrt{\frac{20}{1467}}\right)$
v_{14}	$\left(\frac{1}{10}, \frac{156}{5\sqrt{163}}; \sqrt{\frac{293}{16300}}\right)$	v_{42}	$\left(\frac{1}{6}, \frac{29}{2\sqrt{163}}; \sqrt{\frac{23}{1467}}\right)$
v_{15}	$\left(\frac{23}{210}, \frac{1303}{42\sqrt{163}}; \sqrt{\frac{29782}{1797075}}\right)$	v_{43}	$\left(\frac{1}{5}, \frac{74}{5\sqrt{163}}; \sqrt{\frac{66}{4075}}\right)$
v_{16}	$\left(\frac{19}{70}, \frac{6481}{210\sqrt{163}}; \sqrt{\frac{29408}{1797075}}\right)$	v_{44}	$\left(\frac{1}{3}, \frac{72}{5\sqrt{163}}; \sqrt{\frac{614}{36675}}\right)$
v_{17}	$\left(\frac{3}{10}, \frac{187}{6\sqrt{163}}; \sqrt{\frac{638}{36675}}\right)$	v_{45}	$\left(\frac{2}{5}, \frac{223}{15\sqrt{163}}; \sqrt{\frac{638}{36675}}\right)$
v_{18}	$\left(\frac{11}{30}, \frac{307}{10\sqrt{163}}; \sqrt{\frac{614}{36675}}\right)$	v_{46}	$\left(\frac{1}{2}, \frac{431}{30\sqrt{163}}; \sqrt{\frac{626}{36675}}\right)$
v_{19}	$\left(\frac{1}{2}, \frac{311}{10\sqrt{163}}; \sqrt{\frac{66}{4075}}\right)$	v_{47}	$\left(0, \frac{38}{3\sqrt{163}}; \sqrt{\frac{23}{1367}}\right)$
v_{20}	$\left(0, \frac{640}{21\sqrt{163}}, \sqrt{\frac{1160}{71883}}\right)$	v_{48}	$\left(\frac{1}{6}, \frac{25}{2\sqrt{163}}; \sqrt{\frac{20}{1467}}\right)$
v_{21}	$\left(\frac{1}{3}, \frac{213}{7\sqrt{163}}; \sqrt{\frac{1298}{71883}}\right)$	v_{49}	$\left(\frac{2}{7}, \frac{85}{7\sqrt{163}}; \sqrt{\frac{110}{7987}}\right)$
v_{22}	$\left(0, \frac{502}{21\sqrt{163}}; \sqrt{\frac{1298}{71883}}\right)$	v_{50}	$\left(\frac{1}{3}, \frac{64}{5\sqrt{163}}; \sqrt{\frac{626}{36675}}\right)$
v_{23}	$\left(\frac{13}{210}, \frac{1643}{70\sqrt{163}}; \sqrt{\frac{29408}{1797075}}\right)$	v_{51}	$\left(\frac{13}{30}, \frac{123}{10\sqrt{163}}; \sqrt{\frac{638}{36675}}\right)$
v_{24}	$\left(\frac{47}{210}, \frac{979}{42\sqrt{163}}; \sqrt{\frac{29782}{1797075}}\right)$	v_{52}	$\left(\frac{1}{2}, \frac{383}{30\sqrt{163}}; \sqrt{\frac{614}{36675}}\right)$
v_{25}	$\left(\frac{1}{3}, \frac{167}{7\sqrt{163}}; \frac{1160}{71883}\right)$	v_{53}	$\left(\frac{46}{105}, \frac{414}{35\sqrt{163}}; \sqrt{\frac{29408}{1797075}}\right)$
v_{26}	$\left(\frac{5}{14}, \frac{331}{14\sqrt{163}}; \sqrt{\frac{116}{7987}}\right)$	v_{54}	$\left(\frac{1}{2}, \frac{515}{42\sqrt{163}}; \sqrt{\frac{1298}{71883}}\right)$
v_{27}	$\left(\frac{1}{2}, \frac{325}{14\sqrt{163}}; \sqrt{\frac{122}{7987}}\right)$	v_{55}	$\left(\frac{3}{7}, \frac{80}{7\sqrt{163}}; \sqrt{\frac{120}{7987}}\right)$
v_{28}	$\left(0, \frac{158}{7\sqrt{163}}; \sqrt{\frac{138}{7987}}\right)$	v_{56}	$\left(\frac{1}{2}, \frac{153}{14\sqrt{163}}; \sqrt{\frac{138}{7987}}\right)$

Table (ii):

α	$M_\alpha(\infty)$
0	0
$\frac{\omega-1}{2}$	$\frac{\omega}{2}$
$\frac{\omega}{2}$	$\frac{\omega-1}{2}$
$\frac{\omega-1}{3}$	$\frac{-\omega}{3}$
$\frac{\omega}{3}$	$\frac{1-\omega}{3}$
$\frac{\omega+1}{3}$	$\frac{\omega+1}{3}$
$\frac{-\omega}{3}$	$\frac{\omega-1}{3}$
$\frac{1-\omega}{3}$	$\frac{\omega}{3}$
$\frac{2-\omega}{3}$	$\frac{2-\omega}{3}$
$\frac{\omega}{4}$	$\frac{\omega-1}{4}$
$\frac{\omega+1}{4}$	$\frac{2-\omega}{4}$
$\frac{\omega-1}{4}$	$\frac{\omega}{4}$
$\frac{\omega-2}{4}$	$\frac{-\omega-1}{4}$
$\frac{-\omega}{4}$	$\frac{1-\omega}{4}$
$\frac{1-\omega}{4}$	$\frac{-\omega}{4}$
$\frac{2-\omega}{4}$	$\frac{1+\omega}{4}$
$\frac{-\omega-1}{4}$	$\frac{\omega-2}{4}$
$\frac{2\omega-1}{5}$	$\frac{3-\omega}{5}$
$\frac{2\omega}{5}$	$\frac{2-2\omega}{5}$
$\frac{2\omega+1}{5}$	$\frac{\omega+1}{5}$
$\frac{2\omega-2}{5}$	$\frac{-2\omega}{5}$
$\frac{2\omega-3}{5}$	$\frac{\omega-2}{5}$
$\frac{1-2\omega}{5}$	$\frac{\omega+2}{5}$
$\frac{2-2\omega}{5}$	$\frac{2\omega}{5}$
$\frac{3-2\omega}{5}$	$\frac{2-\omega}{5}$

α	$M_\alpha(\infty)$
$\frac{-2\omega}{5}$	$\frac{2\omega-2}{5}$
$\frac{-2\omega-1}{5}$	$\frac{-\omega-1}{5}$
$\frac{\omega-1}{5}$	$\frac{\omega}{5}$
$\frac{\omega}{5}$	$\frac{\omega-1}{5}$
$\frac{\omega+1}{5}$	$\frac{2\omega+1}{5}$
$\frac{\omega+2}{5}$	$\frac{1-2\omega}{5}$
$\frac{\omega-2}{5}$	$\frac{2\omega-3}{5}$
$\frac{-\omega}{5}$	$\frac{1-\omega}{5}$
$\frac{1-\omega}{5}$	$\frac{-\omega}{5}$
$\frac{2-\omega}{5}$	$\frac{3-2\omega}{5}$
$\frac{3-\omega}{5}$	$\frac{2\omega-1}{5}$
$\frac{-1-\omega}{5}$	$\frac{-1-2\omega}{5}$
$\frac{\omega-1}{6}$	$\frac{-\omega}{6}$
$\frac{\omega}{6}$	$\frac{1-\omega}{6}$
$\frac{\omega+1}{6}$	$\frac{\omega-2}{6}$
$\frac{\omega+2}{6}$	$\frac{3-\omega}{6}$
$\frac{\omega-3}{6}$	$\frac{-\omega-2}{6}$
$\frac{\omega-2}{6}$	$\frac{\omega+1}{6}$
$\frac{-\omega}{6}$	$\frac{\omega-1}{6}$
$\frac{1-\omega}{6}$	$\frac{\omega}{6}$
$\frac{2-\omega}{6}$	$\frac{-\omega-1}{6}$
$\frac{3-\omega}{6}$	$\frac{\omega+2}{6}$
$\frac{-\omega-2}{6}$	$\frac{\omega-3}{6}$
$\frac{-\omega-1}{6}$	$\frac{2-\omega}{6}$
$\frac{12}{1-\omega}$	$\frac{17}{1-\omega}$

α	$M_\alpha(\infty)$
$\frac{17}{1-\omega}$	$\frac{12}{1-\omega}$
$\frac{-17}{1-\omega}$	$\frac{-12}{1-\omega}$
$\frac{-12}{1-\omega}$	$\frac{-17}{1-\omega}$
$\frac{-17}{\omega}$	$\frac{-12}{\omega}$
$\frac{-12}{\omega}$	$\frac{-17}{\omega}$
$\frac{12}{\omega}$	$\frac{17}{\omega}$
$\frac{17}{\omega}$	$\frac{12}{\omega}$
$\frac{12}{2-\omega}$	$\frac{-\omega-16}{2-\omega}$
$\frac{16+\omega}{2-\omega}$	$\frac{-12}{2-\omega}$
$\frac{-\omega-16}{2-\omega}$	$\frac{12}{2-\omega}$
$\frac{-12}{2-\omega}$	$\frac{\omega+16}{2-\omega}$
$\frac{\omega-17}{1+\omega}$	$\frac{12}{1+\omega}$
$\frac{-12}{1+\omega}$	$\frac{17-\omega}{1+\omega}$
$\frac{12}{1+\omega}$	$\frac{\omega-17}{1+\omega}$
$\frac{17-\omega}{1+\omega}$	$\frac{-12}{1+\omega}$
$\frac{17+\omega}{3-\omega}$	$\frac{7}{3-\omega}$
$\frac{7}{3-\omega}$	$\frac{\omega+17}{3-\omega}$
$\frac{\omega+15}{3-\omega}$	$\frac{\omega+10}{3-\omega}$
$\frac{10+\omega}{3-\omega}$	$\frac{\omega+15}{3-\omega}$
$\frac{-\omega-15}{3-\omega}$	$\frac{-\omega-10}{3-\omega}$
$\frac{-\omega-10}{3-\omega}$	$\frac{-\omega-15}{3-\omega}$
$\frac{-\omega-17}{3-\omega}$	$\frac{-7}{3-\omega}$
$\frac{-7}{3-\omega}$	$\frac{-\omega-17}{3-\omega}$
$\frac{\omega-16}{\omega+2}$	$\frac{\omega-11}{2+\omega}$
$\frac{\omega-11}{\omega+2}$	$\frac{\omega-16}{2+\omega}$

α	$M_\alpha(\infty)$
$\frac{\omega-18}{\omega+2}$	$\frac{-7}{\omega+2}$
$\frac{-7}{\omega+2}$	$\frac{\omega-18}{\omega+2}$
$\frac{18-\omega}{\omega+2}$	$\frac{7}{\omega+2}$
$\frac{7}{\omega+2}$	$\frac{18-\omega}{\omega+2}$
$\frac{16-\omega}{\omega+2}$	$\frac{11-\omega}{\omega+2}$
$\frac{11-\omega}{\omega+2}$	$\frac{16-\omega}{\omega+2}$
$\frac{2\omega-1}{7}$	$\frac{\omega+3}{7}$
$\frac{2\omega+1}{7}$	$\frac{3-2\omega}{7}$
$\frac{3\omega+2}{7}$	$\frac{3\omega+2}{7}$
$\frac{\omega+3}{7}$	$\frac{2\omega-1}{7}$
$\frac{2\omega-3}{7}$	$\frac{-2\omega-1}{7}$
$\frac{1-2\omega}{7}$	$\frac{4-\omega}{7}$
$\frac{3-2\omega}{7}$	$\frac{2\omega+1}{7}$
$\frac{5-3\omega}{7}$	$\frac{5-3\omega}{7}$
$\frac{4-\omega}{7}$	$\frac{1-2\omega}{7}$
$\frac{-2\omega-1}{7}$	$\frac{2\omega-3}{7}$
$\frac{16+\omega}{4-\omega}$	$\frac{-\omega-4}{4-\omega}$
$\frac{\omega+4}{4-\omega}$	$\frac{-\omega-16}{4-\omega}$
$\frac{-\omega-4}{4-\omega}$	$\frac{\omega+16}{4-\omega}$
$\frac{-\omega-16}{4-\omega}$	$\frac{\omega+4}{4-\omega}$
$\frac{\omega-5}{3+\omega}$	$\frac{17-\omega}{3+\omega}$
$\frac{\omega-17}{\omega+3}$	$\frac{5-\omega}{3+\omega}$
$\frac{17-\omega}{3+\omega}$	$\frac{\omega-5}{3+\omega}$
$\frac{5-\omega}{3+\omega}$	$\frac{\omega-17}{3+\omega}$

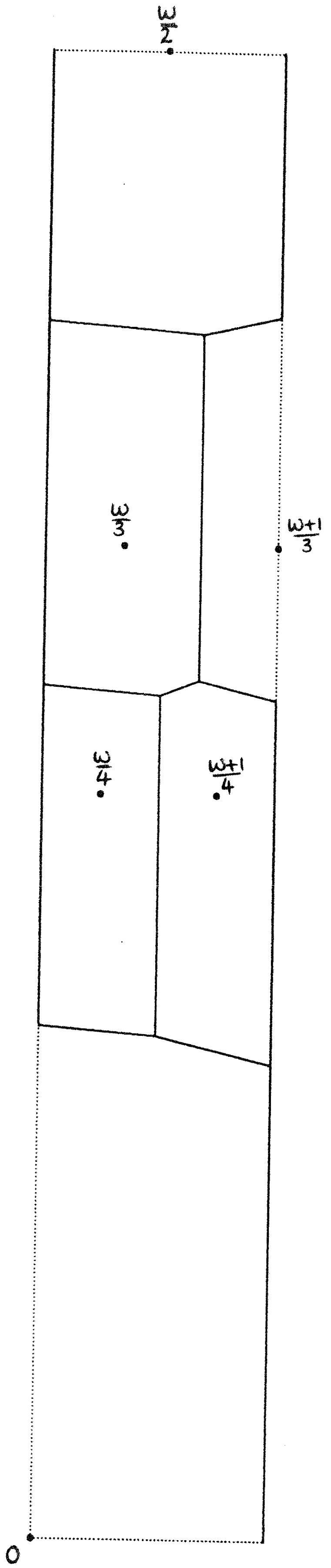


Figure (2.6.1a)

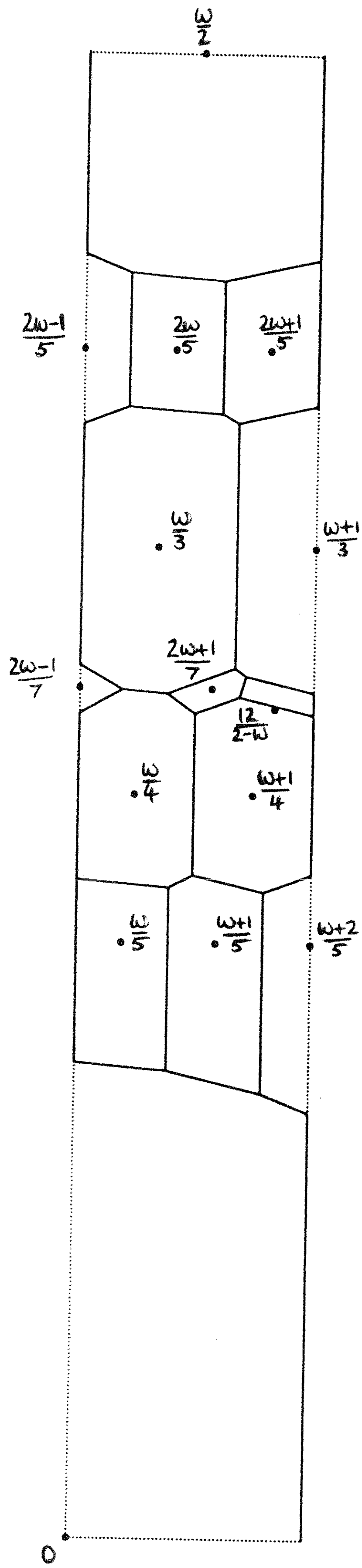


Figure (2.6.1b)

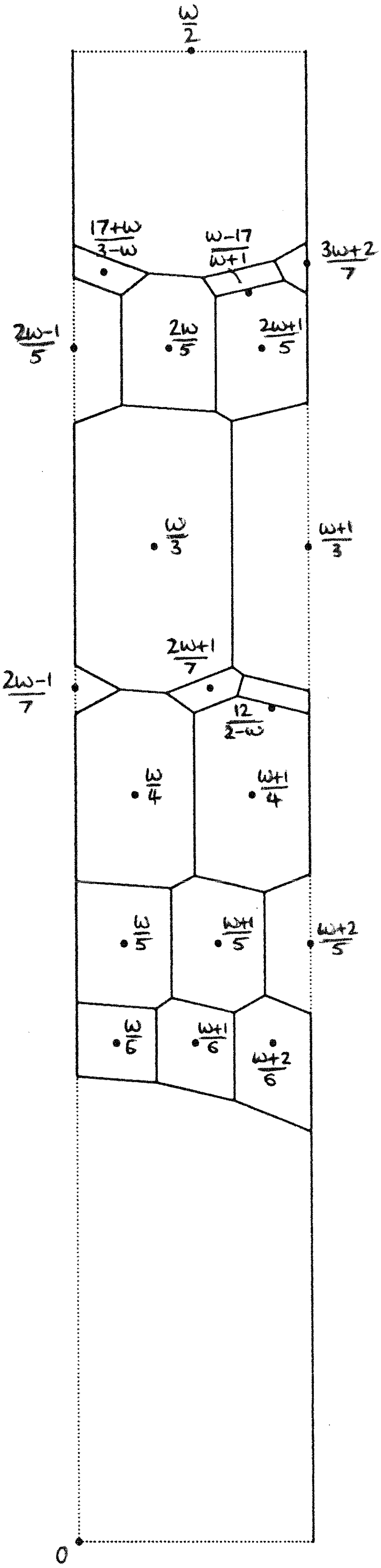


Figure (2.6.1c)

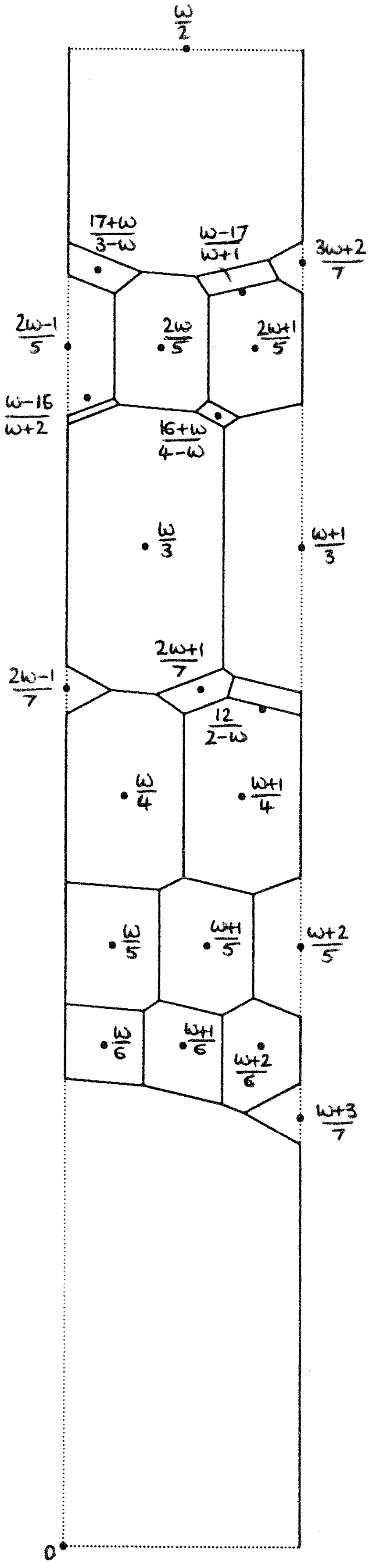
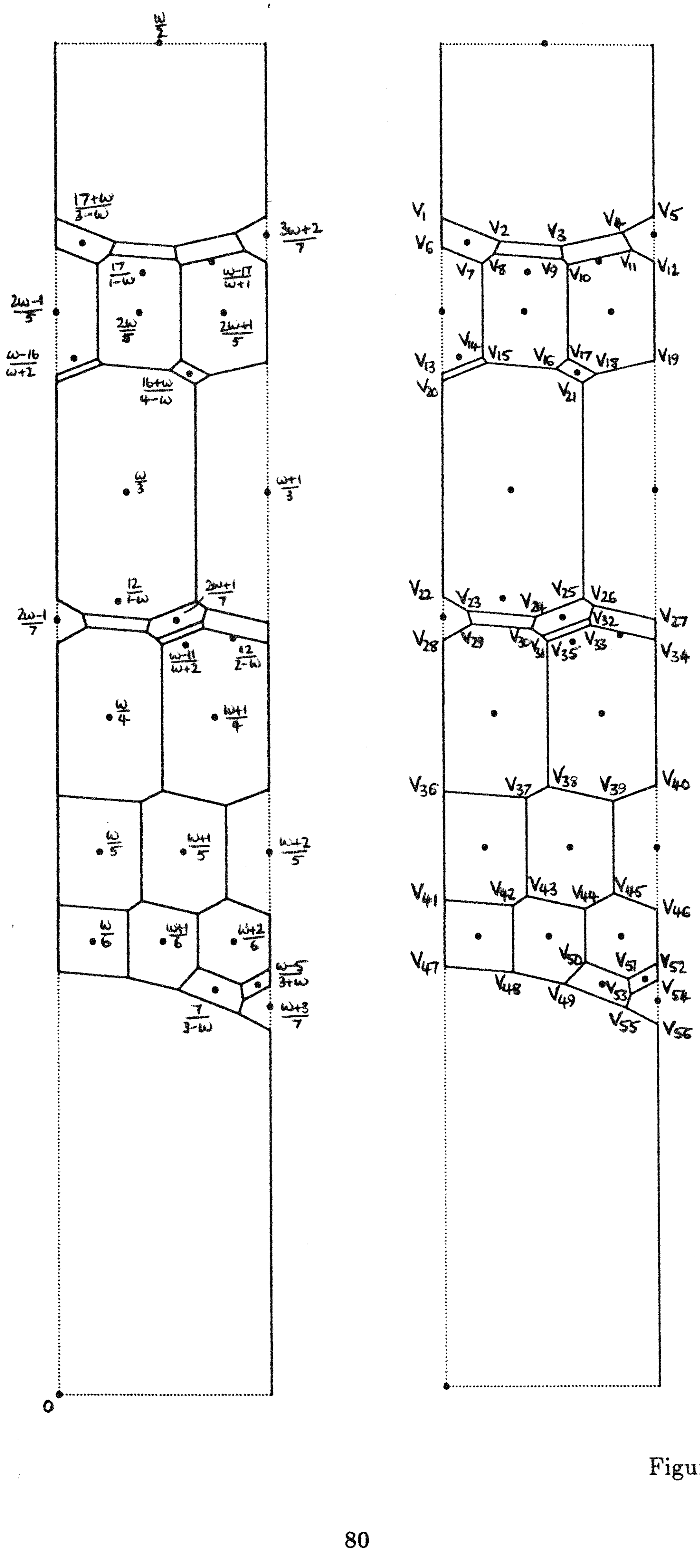


Figure (2.6.1d)



ω/2
17+ω / 3−ω
3ω+2 / 7
17 / 1−ω
ω−17 / ω+1
2ω−1 / 5
2ω / 8
2ω+1 / 5
ω−16 / ω+2
16+ω / 4−ω
ω/3
ω+1 / 3
12 / 1−ω
2ω+1 / 7
2ω−1 / 7
ω−11 / ω+2
12 / 2−ω
ω/4
ω+1 / 4
ω/5
ω+1 / 5
ω+2 / 5
ω/6
ω+1 / 6
ω+2 / 6
ω−5 / 3+ω
ω+3 / 7
7 / 3−ω
0
V1 V2 V3 V4 V5 V6 V7 V8 V9 V10 V11 V12
V13 V14 V15 V16 V17 V18 V19 V20 V21
V22 V23 V24 V25 V26 V27 V28 V29 V30 V31 V32 V33 V34 V35
V36 V37 V38 V39 V40
V41 V42 V43 V44 V45 V46
V47 V48 V49 V50 V51 V52 V53 V54 V55 V56

Figure (2.6.1e)

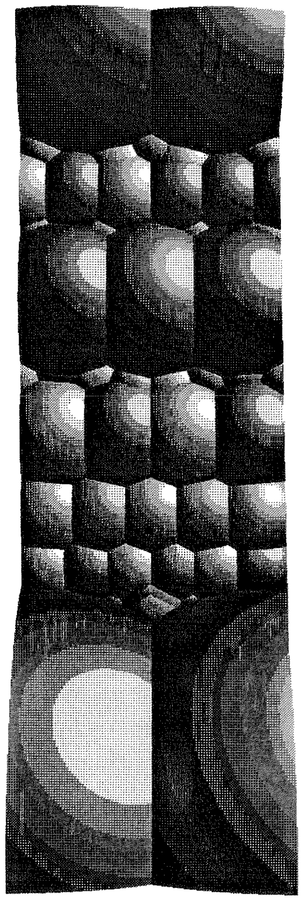


Figure (2.6.2)

§2.7 A Pseudo-Euclidean Algorithm

We begin by recalling the Euclidean algorithm defined over $\mathbf{Q}$. Given $a, b \in \mathbf{Z}$ we wish to find $g = (a, b)$, and write it in the form $g = ax + by$, with $x, y \in \mathbf{Z}$. We do this as follows:

Repeat

$$a = q_1 b + r_1$$

$$b = q_2 r_1 + r_2$$

$$\vdots$$

$$r_{i-1} = q_{i+1} r_i + r_{i+1}$$

$$\vdots$$

until $\quad r_n = 0.$

Then $g = r_{n-1}$.

To express g as a sum of a and b, we define sequences $\{x_i\}, \{y_i\}$ recursively by:

$$\begin{cases} x_0 = 1, x_1 = 0, x_{i+1} = q_i x_i + x_{i-1}, \ 1 \le i \le n \\ y_0 = 0, y_1 = 1, y_{i+1} = q_i y_i + y_{i-1}, \ 1 \le i \le n. \end{cases} \tag{2.7.1}$$

It is easy to prove that $x_i y_{i+1} - x_{i+1} y_i = (-1)^i$ and $x_{n+1} = b_0 = \frac{b}{g}$; $y_{n+1} = a_0 = \frac{a}{g}$. So $x_n a - y_n b = (-1)^n g$.

The quotients $\frac{y_i}{x_i}$ are the **continued fraction convergents** to $\frac{a}{b}$, where two consecutive convergents give a "determinant" of ± 1 with alternating sign.

We now view the algorithm geometrically, as a process applied to the fraction $\alpha = \frac{a}{b}$, as a point of $\mathcal{H}_2^\star$.

At each stage we:

[1] "Translate" α mod $\mathbf{Z}$ until $0 \leq \alpha < 1$,

[2] "Invert" : $\alpha \mapsto \frac{1}{\alpha}$.

These operations can be described in terms of matrices and linear functional transformations. Recall, the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ maps $\alpha \mapsto \frac{p\alpha+q}{r\alpha+s}$.

So step [1] is acheived via

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : \alpha \mapsto \alpha + n,$$

(where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, n \in \mathbf{Z}$)

while step [2] is achieved via

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \alpha \mapsto \frac{1}{\alpha}.$$

Then the **total** transformation, which sends α to ∞, is

$$RT^{-q_n}RT^{-q_{n-1}} \ldots RT^{-q_2}RT^{-q_1}.$$

So

$$\alpha = T^{q_1}RT^{q_2}R \ldots T^{q_n}R(\infty). \tag{2.7.2}$$

Now define:

$$M_k = T^{q_1}R \ldots T^{q_k}R,$$

so that

$$M_0 = I,$$

$$M_{k+1} = M_k T^{q_{k+1}} R,$$

and

$M_n(\infty) = \alpha.$

Thus, if $M_n = \begin{pmatrix} x_{n+1} & x_n \\ y_{n+1} & y_n \end{pmatrix}$, then $\alpha = \frac{x_{n+1}}{y_{n+1}}$ in lowest terms, and $x = y_n, y = x_n$.

Finally, write $M_k = \begin{pmatrix} x_{k+1} & x_k \\ y_{k+1} & y_k \end{pmatrix}$. Then $\det(M_k) = (-1)^k$, since $\det(T) = +1$ and $\det(R) = -1$, and $M_k(\infty) = \frac{x_k}{y_k}$ is the kth continued fraction convergent to $\frac{a}{b}$.

Note: It makes things easier in the geometric Euclidean algorithm if one uses the transformation

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \alpha \mapsto -\frac{1}{\alpha}$$

instead of $R : \alpha \mapsto \frac{1}{\alpha}$ in the inversion step.

This is because S has determinant 1 and so all the M_k will have determinant 1 also. So, as before, for any $\alpha \in \mathbf{Q}$ there is a sequence of integers $q_1, q_2, \ldots, q_n$ such that

$$\alpha = T^{q_1} S^{-1} T^{q_2} S^{-1} \ldots T^{q_n} S^{-1}(\infty).$$

We deduce from this that $SL(2,\mathbf{Z})$ is generated by S, T and $-I$.

We now consider the imaginary quadratic **Euclidean** case.

We replace

$\mathbf{Q}$ by $\mathrm{K} = \mathbf{Q}(\sqrt{-d})$;

$\mathbf{Z}$ by ϑ_{K};

$SL(2,\mathbf{Z})$ by $\Gamma = SL(2,\vartheta_{\mathrm{K}})$;

$\mathcal{H}_2^*$ by $\mathcal{H}_3^*$.

Then, given $\lambda, \mu \in \vartheta_K$, we wish to find $\delta = (\lambda, \mu) \in \vartheta_K$ and $x, y \in \vartheta_K$ such that $\lambda x + \mu y = \delta$.

The algebraic Euclidean algorithm extends immediately. We now consider the geometric Euclidean algorithm.

Recall that Γ is generated by $\{T, U, S, R\}$, where $R = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$.
A fundamental region D for the action of Γ on $\mathcal{H}_3^*$ is given in §1.1.

If we now consider the quotient $\alpha = \frac{\lambda}{\mu} \in \mathrm{K}$ as a point of $\mathcal{H}_3^*$ then the geometrical Euclidean algorithm gives a method for transposing α into D.

Equivalently, it takes α to ∞ by repeatedly

[1] Translating α mod ϑ_K until $|\alpha| < 1$,

[2] Inverting in the unit sphere : $\alpha \mapsto -\frac{1}{\alpha}$ until $\alpha \mapsto \infty$.

This process will terminate after a finite number of steps. We can see this by considering the quaternion notation for the action of $SL(2, \vartheta_K)$ on $\mathcal{H}_3^*$ given by $(0.2.2')$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : p = z + j\zeta \mapsto Mp = (ap + b)(cp + d)^{-1}$$

So $\zeta(Mp) \mapsto \frac{\zeta(p)}{|cp+d|^2}$.

For a fixed point p there are only finitely many many pairs (c, d) with $|cp+d|^2 < 1$. But each inversion (step **[2]**) **raises** p. So only finitely many inversions can occur before the process terminates.

Because ϑ_K is Euclidean, any point $z \in \mathbf{C}$ is no further than 1 away from an integer, so we need only use inversion in the unit hemisphere, centred on the origin.

We now consider the continued fraction convergents to α. Let $q_1, \ldots, q_n$ be the translations used in step [1] of the Euclidean algorithm as before. Then we have:

$$\alpha = T^{q_1} S^{-1} T^{q_2} S^{-1} \ldots T^{q_n} S^{-1}(\infty).$$

Notice that $q_i \in \vartheta_{\mathrm{K}}$ and not just in $\mathbf{Z}$. So, strictly, $T^{q_i} = T^n U^m$ where $n, m \in \mathbf{Z}$ and $q_i = n + m\omega$.

Define $M_0 = I$ and $M_k = M_{k-1} T^{q_k} S^{-1}$ as before. Notice that M_k has determinant 1 as $\det T = \det S = 1$. If $M_n = \begin{pmatrix} x_{n+1} & x_n \\ y_{n+1} & y_n \end{pmatrix}$ then $\alpha = \frac{\lambda}{\mu} = \frac{x_{n+1}}{y_{n+1}}$ in lowest terms. Then $\delta = (\lambda, \mu) = \frac{\lambda}{x_{n+1}} = \frac{\mu}{y_{n+1}}$ and $x_{n+1} y_n - y_{n+1} x_n = 1 \Rightarrow \lambda y_n - \mu x_n = \delta$.

We saw in §1.3, that to get from one cusp to another in $\mathcal{H}_3^*$ we use a succession of paths of the form $\{\gamma(0), \gamma(\infty)\}$, some $\gamma \in \Gamma$. Setting $\gamma = M_k = \begin{pmatrix} x_{k+1} & x_k \\ y_{k+1} & y_k \end{pmatrix}$, we get $M_k(0) = \frac{x_k}{y_k}$ and $M_k(\infty) = \frac{x_{k+1}}{y_{k+1}}$. So the M_k are sufficient to define a path from 0 to $\frac{a}{c}$ and

$$\frac{x_0}{y_0}, \frac{x_1}{y_1}, \ldots, \frac{x_{n+1}}{y_{n+1}} = \frac{a}{c}$$

are the continued fraction convergents to $\frac{a}{c}$.

We now consider the **non-Euclidean**, class number 1 case.

Recall that the imaginary quadratic number field, K, is Euclidean $\Leftrightarrow$ every $k \in \mathrm{K}$ has distance less than one from the nearest element of ϑ_{K} $\Leftrightarrow$ the unit hemispheres $\{S_\alpha : \alpha \in \vartheta_{\mathrm{K}}\}$ cover the floor

If K is non-Euclidean but still has class number one, then a larger set of S_α is required to cover the floor. This gives a geometric algorithm which we can translate back into algebraic terms to give a new algebraic algorithm. The "inversion" steps in this new algorithm are rather more complicated than in the ordinary Euclidean case.

If K has class number > 1 then there is no covering of the floor by hemispheres (see [17]) - this case will not concern us.

Let K be one of the four non-Euclidean class number 1 fields and let A and M be as defined in Definition 2.2.10. Then $SL(2,\vartheta_{\mathrm{K}})$ is generated by $\{S,T,U\}\cup M$ and D is a fundamental region for its action on $\mathcal{H}_3^*$. As before we wish to construct an algorithm for transposing any point $p\in\mathcal{H}_3^*$ into D. Equivalently we want a method which will take any $p\in\mathrm{K}$ to ∞ in a finite number of steps. Set $p=\frac{\lambda}{\mu}$.

Repeat

[1] Translate p mod ϑ_{K} (using T and $U=T^{\omega}$) until $p\in F$

[2] Invert in S_α, using M_α, where S_α is the hemisphere which "best" covers p, ie $p=(z,\zeta)\in S_\alpha$ has the largest ζ co-ordinate,

until $p\mapsto\infty$.

Consider the effect of inverting p in the hemisphere S_α. We can only invert p using M_α when $S_\alpha=S_{\tau,\sigma}$ covers p, ie when $|\tau p-\sigma|^2<1$. Therefore, each inversion **raises** p. But there are only finitely many pairs $(\tau,\sigma)\in\vartheta_{\mathrm{K}}$ such that $|\tau p-\sigma|^2<1$, so the process will terminate after a finite number of steps.

Let q_i be the power of T, in the generalised sense, which we used at the ith step, and let N_j be the matrix used in at the jth inversion step. Then we have:

$$p=T^{q_1}N_1^{-1}T^{q_2}N_2^{-1}\ldots T^{q_n}N_n^{-1}(\infty).$$

Now set $M_0 = I, M_k = M_{k-1}T^{q_k}N_k^{-1}$ and

$$M_n = T^{q_1}N_1^{-1}\dots T^{q_n}N_n^{-1},$$

so that $p = M_n(\infty)$.

It is no longer the case that the last column of M_k agrees with the first of M_{k+1}, and so we use a different notation:

Set $M_k = \begin{pmatrix} x_k & x'_k \\ y_k & y'_k \end{pmatrix}$.

Then, if $M_n = \begin{pmatrix} x_n & x'_n \\ y_n & y'_n \end{pmatrix}$, $p = \frac{x_n}{y_n}$ in lowest terms while $\delta = (\lambda, \mu) = \lambda y'_n - \mu x'_n$. and the generalised continued fraction convergents to $\frac{\lambda}{\mu}$ are:

$$\frac{0}{1}, \frac{1}{0} = \frac{x_0}{y_0}, \frac{x_1}{y_1}, \dots, \frac{x_n}{y_n}.$$

These convergents will be vital in later chapters in the conversion between M-symbols and Modular symbols.

We can no longer express any path between cusps in terms of those of the form $\{M_k(0), M_k(\infty)\}$. However, we **can** express it in terms of the paths

$$\{M_k(\alpha_k), M_k(\infty)\},$$

where α_k is the centre of the hemisphere which we inverted in at the kth step of the algorithm. The set of such α is finite and is determined, in advance, once and for all for each field.

Example:

Let K $= \mathbf{Q}(\sqrt{-19})$.

Find the gcd of 4 *and* $2+3\omega$ *and express the path* $\{\frac{2+3\omega}{4}, \infty\}$ *in terms of paths of the form* $\{M_k(\alpha_k), M_k(\infty)\}$.

Solution:

We use the region above F, shown in Figure (2.7.1), as our fundamental region.

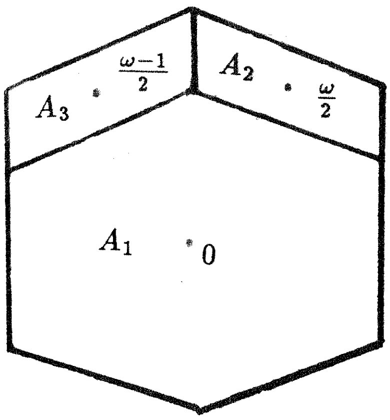


Figure (2.7.1)

So, setting $z = \frac{2+3\omega}{4}$, we repeatedly

[1] Translate z mod ϑ_K until $z \in A_1 \cup A_2 \cup A_3$,

[2] Invert, using S_1, S_2, S_3 respectively,

until $z \mapsto \infty$.

$$S_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} \omega - 1 & 2 \\ 2 & -\omega \end{pmatrix}, S_3 = \begin{pmatrix} \omega & 2 \\ 2 & 1-\omega \end{pmatrix}.$$

$T^{-\omega} : z = \frac{2+3\omega}{4} \mapsto \frac{2-\omega}{4}$ which has z co-ordinate which lies in F.

It lies beneath the hemisphere which defines A_1 and so we must invert using S_1.

$S_1 : \frac{2-\omega}{4} \mapsto \frac{-4\omega-4}{7}$.

Now translate again, using T^{ω}.

$T^{\omega} : \frac{-4\omega-4}{7} \mapsto \frac{3\omega-4}{7}$ and $\frac{3\omega-4}{7}$ lies below A_3.

$S_3 : \frac{3\omega-4}{7} \mapsto 1.$

$T^{-1} : 1 \mapsto 0$, which lies below A_1, and $S_1 : 0 \mapsto \infty$.

So the alogrithm terminates after 3 steps.

Now set $M_0 = I$,

$$M_1 = T^{\omega} S_1^{-1} = \begin{pmatrix} -\omega & 1 \\ -1 & 0 \end{pmatrix},$$

$$M_2 = M_1 T^{-\omega} S_3^{-1} = \begin{pmatrix} 2\omega - 3 & 5+\omega \\ \omega+1 & 3-\omega \end{pmatrix},$$

$$M_3 = M_2 T S_1^{-1} = T^{\omega} S_1^{-1} T^{-\omega} S_3^{-1} T S_1^{-1} = \begin{pmatrix} -2-3\omega & 2\omega-3 \\ -4 & 1+\omega \end{pmatrix},$$

so that $M_3(\infty) = z$ in lowest terms.

So $z = \frac{2+3\omega}{4}$ was in lowest terms to begin with,

ie $\quad \gcd(2+3\omega, 4) = 1 = -(2+3\omega)(1+\omega) - 4(3-2\omega).$

The continued fraction convergents to z are

$$\frac{0}{1}, \frac{1}{0}, \frac{\omega}{1}, \frac{2\omega-3}{1+\omega}, \frac{2+3\omega}{4}.$$

Note: The determinant between consecutive convergents is no longer 1.

Finally, notice that

$$M_1\{0, \infty\} = \{\infty, \omega\},$$

$$M_2\{\tfrac{\omega}{2}, \infty\} = \{\omega, \tfrac{2-3\omega}{1+\omega}\},$$

$$M_3\{0, \infty\} = \{\tfrac{2\omega-3}{1+\omega}, \tfrac{2+3\omega}{4}\}.$$

So we can write the path from $\frac{2+3\omega}{4}$ to ∞ as a sum of paths of the form $\{M_k(\alpha), M_k(\infty)\}$ as follows:

$$\{\frac{2+3\omega}{4}, \infty\} = -\{M_3(0), M_3(\infty)\} - \{M_2(\frac{\omega}{2}), M_2(\infty)\} - \{M_1(0), M_1(\infty)\}.$$

Chapter 3: A Tessellation of $\mathcal{H}_3^\star$

In this chapter I will show how to find a tessellation of the upper half space by hyperbolic polyhedra using the fundamental region D for the action of $\Gamma = SL(2,\vartheta_K)$ on $\mathcal{H}_3^\star$ described in Chapter 2. We saw in Chapter 1 that, given a subgroup G of finite index in Γ, by passing to the quotient space, $G\backslash\mathcal{H}_3^\star$, this tessellation gives a tessellation of that space also. Because the index of G in $SL(2,\vartheta_K)$ is finite, this tessellation is by a finite number of polyhedra. From this tessellation we can then calculate the homology $H_1(G\backslash\mathcal{H}_3^\star,\mathbf{Q})$. This will be the subject of Chapter 4.

In the Euclidean case, the fundamental region for the action of $SL(2,\vartheta_K)$ on $\mathcal{H}_3^\star$ had a curved base which came from the unit hemisphere centred on the origin. To find the tessellation of $\mathcal{H}_3^\star$ by hyperbolic polyhedra we fixed a point, p, of D and looked at how translates of D by elements of G_p, the stabilizer of p, glued together around p. This gave the **basic** polyhedron, transforms of which, by elements of $SL(2,\vartheta_K)$ itself, gave the tessellation of $\mathcal{H}_3^\star$.

We saw in the previous chapter that the situation is not so simple in the non-Euclidean class number 1 case. The curved base of the fundamental region, D, comes from the union of several hemispheres and so it is not sufficient to consider the transforms of D around just one vertex. Instead, we must consider each vertex of the base in turn, find the stabilizer of it and look at how translates of D by that subgroup glue together around it. This will lead to more than one basic polyhedron, one for each vertex of D, forming the basic set needed to generate the tessellation of $\mathcal{H}_3^*$.

In §3.1 I will give details of the general method used in the 4 non-Euclidean, class number 1 fields. Details of the calculations for each field in turn will then be given in §3.2 – §3.5. Throughout the remainder of this chapter, unless otherwise stated, K will be one of the four fields $\mathbf{Q}(\sqrt{-d}), d \in \{19, 43, 67, 163\}$. The cases $\mathbf{Q}(\sqrt{-19})$ and $\mathbf{Q}(\sqrt{-43})$ have already been calculated by Hatcher in an unpublished preprint version of [8], about hyperbolic structure on knot compliments.

§3.1 The General Method

Recall the fundamental region, D, calculated in Chapter 2, and the sets associated with it:

$$A = \{\alpha \in \mathrm{K} : S_\alpha \cap D \neq \phi\},$$

$$M = \{M_\alpha \in SL(2,\vartheta_{\mathrm{K}}) : M_\alpha(\alpha) = \infty, M_\alpha(\infty) \in A\},$$

$$V = \{\text{ vertices of } D\}.$$

Then, for each $v \in V$ we make the following:

Definition 3.1.1:

Set $T_v := \{\frac{\lambda}{\mu} \in \mathrm{K} : v \in S_{\mu,\lambda}\} \cup \{\infty\}$.

So T_v *is the set of centres of* all *hemispheres which contain* v*, along with the point at* ∞.

Let $T_v^* := T_v \backslash \{\infty\}$.

Also, set $A_v := \{\alpha \in A : v \in S_\alpha\} \cup \{\infty\}$.

So A_v *is the subset of centres of hemispheres which contain* v *in* A*, along with the point at* ∞.

Let $A_v^\star := A_v \backslash \{\infty\}$.

Notice that $A_v \subset T_v$ *and, hence, that* $A_v^\star \subset T_v^\star$.

We can determine T_v, using the method described in §2.1.

Now, for each $v \in V$, we wish to find the stabilizer, G_v, of it. Recall from §2.2 that Γ is generated by $\{S, T, U, J\} \cup \{$ inversions in $S_\alpha : \alpha \in A\}$. A simple geometrical argument shows that

$$G_v = \{ \text{ inversions in } S_\alpha : \alpha \in T_v\}.$$

and, in fact, G_v is generated by $\{$ inversions in $S_\alpha : \alpha \in A_v\}$. In practise we will use the former formulation, as the following Lemma gives an algorithm for determining G_v explicitly, given T_v.

Lemma 3.1.2:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_v \Rightarrow \tfrac{a}{c}, -\tfrac{d}{c} \in T_v.$$

Proof:

Suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ fixes $v = (z, \zeta)$.

Then it follows that γ fixes some hemisphere containing v. In fact γ must fix $S_{c,-d}$, so $-\frac{d}{c} \in T_v$, by definition.

Similarly, γ^{-1} fixes $S_{c,a}$ and so $\frac{a}{c} \in T_v$, since $\gamma \in G_v \Rightarrow \gamma^{-1} \in G_v$. ∎

So, given T_v, we find G_v as follows:

[1] Fix $\mu \in \vartheta_K$ such that $\exists \lambda \in \vartheta_K$ with $(\lambda, \mu) = \vartheta_K$ and $\frac{\lambda}{\mu} \in T_v$.

[2] Determine **all** $\lambda \in \vartheta_K$ such that $(\lambda, \mu) = \vartheta_K$ and $\frac{\lambda}{\mu} \in T_v$.

[3] Construct all matrices of the form $\begin{pmatrix} \lambda_i & x \\ \mu & -\lambda_j \end{pmatrix}$, with λ_i and λ_j not necessarily distinct and x chosen so that $-\lambda_i\lambda_j - x\mu = 1$.

[4] Keep those matrices which fix v.

[5] Goto [1].

We will also be interested in when two vertices $v, w \in V$ are equivalent under the action of Γ. So we wish to know which $\gamma \in \Gamma$ are such that $\gamma(v) = w$. Using a similar argument to the one above, we can see that if $\gamma(v) = w$ then, $\forall \frac{\lambda}{\mu} \in T_v$, $\gamma(\frac{\lambda}{\mu}) \in T_w$. Again, these γ are all of a particular type and we can determine them explicitly, given T_v and T_w. Denote the set of $\gamma \in \Gamma$ such that $\gamma(v) = w$ by H_{vw}. Then we have:

Lemma 3.1.3:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_{vw} \Rightarrow \frac{a}{c} \in T_w, -\frac{d}{c} \in T_v.$$

Proof:

Use the same argument as Lemma 3.1.2. ∎

Lemma 3.1.3 gives an algorithm, similar to the previous one, for determining H_{vw}, given T_v and T_w.

So $v, w \in V$ are **equivalent** under the action of $\Gamma \Leftrightarrow H_{vw} \neq \phi$.

If $H_{vw} \neq \phi$ then it is a coset of G_v (or of G_w). We have already computed G_v, G_w and so it will suffice to find just one matrix in H_{vw} in order to determine it completely.

Lemma 3.1.4:

If $v, w \in V$ are equivalent then $\zeta(v) = \zeta(w)$.

Proof:

Let $v, w \in D$ be Γ-equivalent. D is a subset of B and so $\zeta(\gamma(v)) \leq \zeta(v), \forall \gamma \in \Gamma$.

But $w = \gamma(v)$, some $\gamma \in H_{vw}$, so $\zeta(w) \leq \zeta(v)$.

Also, $v = \delta(w)$, some $\delta \in H_{wv}$, so $\zeta(v) \leq \zeta(w)$.

Thus, $\zeta(v) = \zeta(w)$. ∎

The converse of Lemma 3.1.4 is not always true (see §3.5); however, we can significantly reduce the work required to split V into equivalence classes by first, crudely, splitting it according to Lemma 3.1.4.

We now turn our attention to determining the exact nature of the hyperbolic polyhedra associated to the vertices of V. Denote the polyhedron associated with $v \in V$ by P_v.

We begin by partitioning the space D into $n = \#V$ subsets, each associated with some vertex $v \in V$. We do this using the vertical half-planes $Q_{\mu,\lambda;\tau,\sigma}$, with corners at $\frac{\lambda}{\mu}, \frac{\sigma}{\tau}$ and ∞, where $\frac{\lambda}{\mu}, \frac{\sigma}{\tau} \in A$ and $S_{\mu,\lambda} \cap S_{\tau,\sigma} \neq 0$.

Each $v \in V$ is the point of intersection of $\{S_\alpha : \alpha \in A_v^\star\}$. So, it is the intersection of 3 or 4 hemispheres. Thus, to each $v \in V$, there is associated a subspace, D_v, of D which is bounded below by the union of $\{S_\alpha : \alpha \in A_v^\star\}$ and with vertical sides which are the intersections of the half-planes passing through $\alpha \in A_v^\star$. Notice that it is not necessarily the case that $v \in D_v$ (see §3.5).

We now construct the polyhedron P_v by gluing together transforms of D_v by elements of G_v along corresponding edges.

If two vertices $v, w \in V$ are equivalent under the action of Γ, then the polyhedra P_v, P_w are also equivalent under this action. Thus we form P_v by gluing together the translates of D_v by elements of G_v, and translates of D_w by elements of $H_{wv}, \forall w \in V$ which are equivalent to v, along common edges. Then $P_w = \gamma(P_v), \gamma \in H_{vw}$.

Finally, the set $D^+ = \bigcup_{v \in V} P_v$. Then transforms of D^+ by elements of Γ give a tessellation of $\mathcal{H}_3^\star$ by hyperbolic polyhedra.

Given a subgroup G of finite index in Γ, we pass to the quotient $G \backslash \mathcal{H}_3^\star$ to obtain a "triangulation" of the quotient space by a finite number of hyperbolic polyhedra. We will use this tessellation in the following chapter where we calculate homology.

Notes:

(1) Once again, for convenience of calculation, we consider the action of the larger group $\Gamma^{\pm}$ along with that of conjugation, and restrict our attention to the smaller region $\{(z, \zeta) \in D : z \in F_0\}$.

(2) If $v, w \in V$ are equivalent then $\exists \gamma \in H_{vw}$ such that $\gamma(P_v) = P_w$. Therefore we need only calculate P_v for one element v in each equivalence class.

In the following 4 sections I will give details of the calculation in each of the 4 non-Euclidean, class number 1 fields. In each case I will give an illustration of how D is partitioned by presenting a plan of the projection onto F_0. I will then give details of the precise nature of D^+ along with suitable descriptions of the polyhedra involved.

§3.2 The Calculation for $\mathbf{Q}(\sqrt{-19})$

Recall the results of §2.3. The vertices v_1 and v_2 are not equivalent under the action of $\Gamma^{\pm}$ and so we have 2 distinct polyhedra, P_{v_1} and P_{v_2}, which form part of D^+.

The projection of the partitioning of D onto F_0 is shown in Figure (3.2.1).

P_{v_1} is the triangular prism shown in Figure (3.2.2) while P_{v_2} is the cuboctahedron shown in Figure (3.2.3).

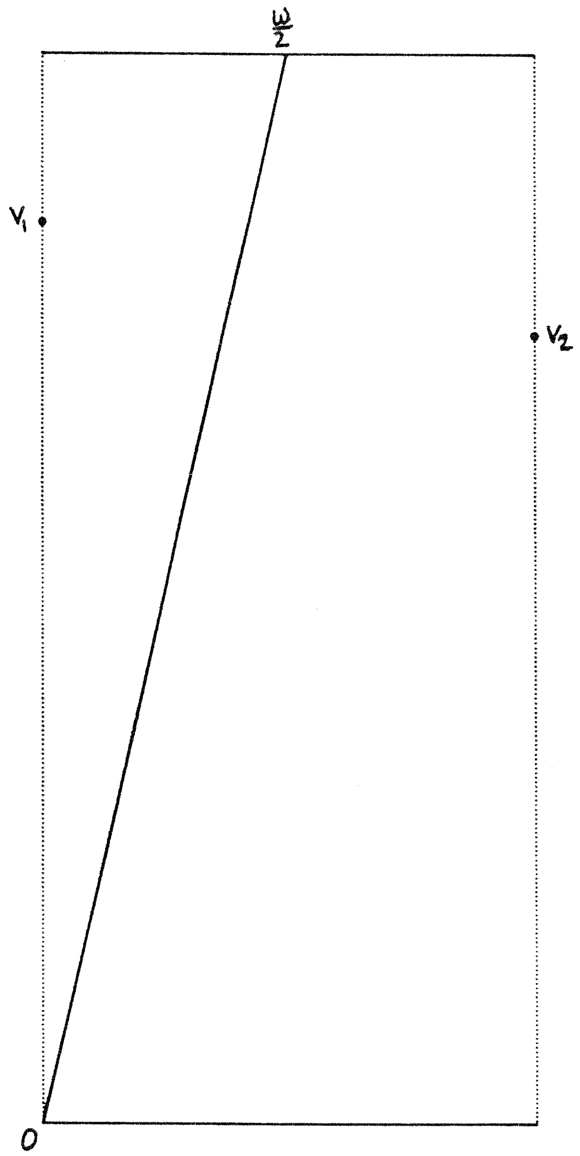


Figure (3.2.1)

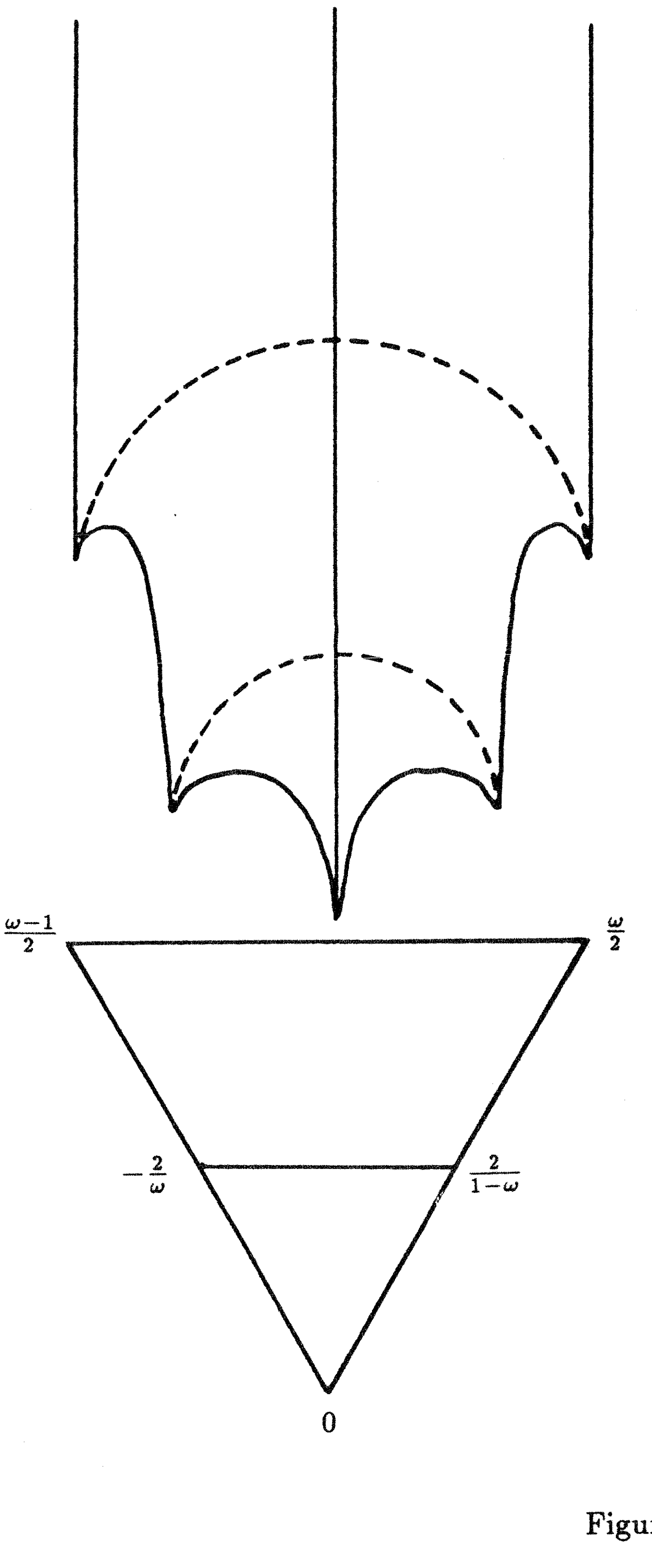


Figure (3.2.2)

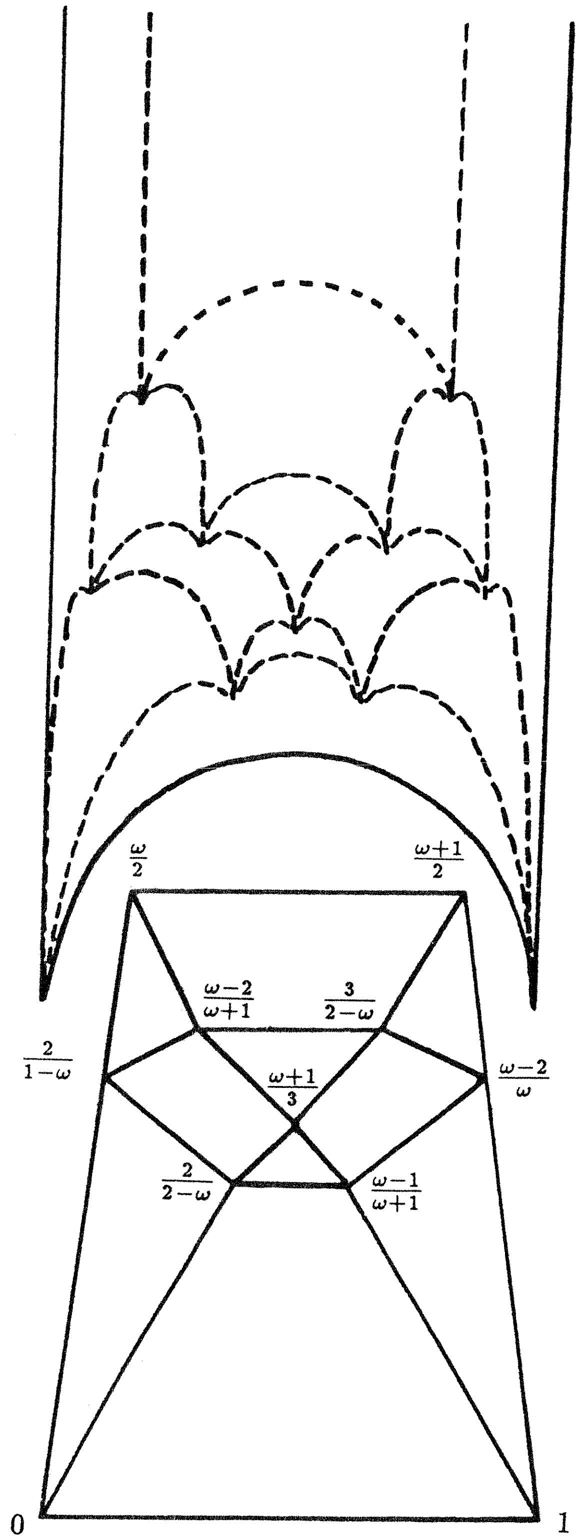

$\frac{\omega}{2}$
$\frac{\omega+1}{2}$
$\frac{\omega-2}{\omega+1}$
$\frac{3}{2-\omega}$
$\frac{2}{1-\omega}$
$\frac{\omega+1}{3}$
$\frac{\omega-2}{\omega}$
$\frac{2}{2-\omega}$
$\frac{\omega-1}{\omega+1}$
0
1

Figure (3.2.3)

§3.3 The Calculation for $\mathbf{Q}(\sqrt{-43})$

Recall the results of §2.4. The projection of the partitioning of D onto F_0 is shown in Figure (3.3.1).

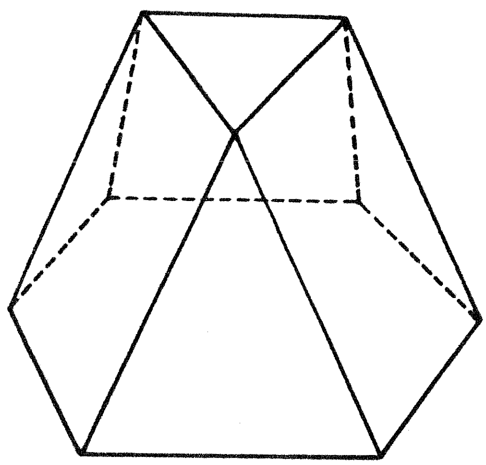
The vertices of V split into 4 distinct equivalence classes as follows:

$$\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}, \{v_6\}.$$

So there are four fundamental polyhedra, one for each equivalence class, which make up D^+. These are as follows:

P_{v_5} is a "hexagonal cap"

ie.



as shown in Figure (3.3.2);

P_{v_4} is the triangular prism shown in Figure (3.3.3);

P_{v_3} is the triangular prism shown in Figure (3.3.4);

P_{v_6} is the truncated tetrahedron shown in Figure (3.3.5).

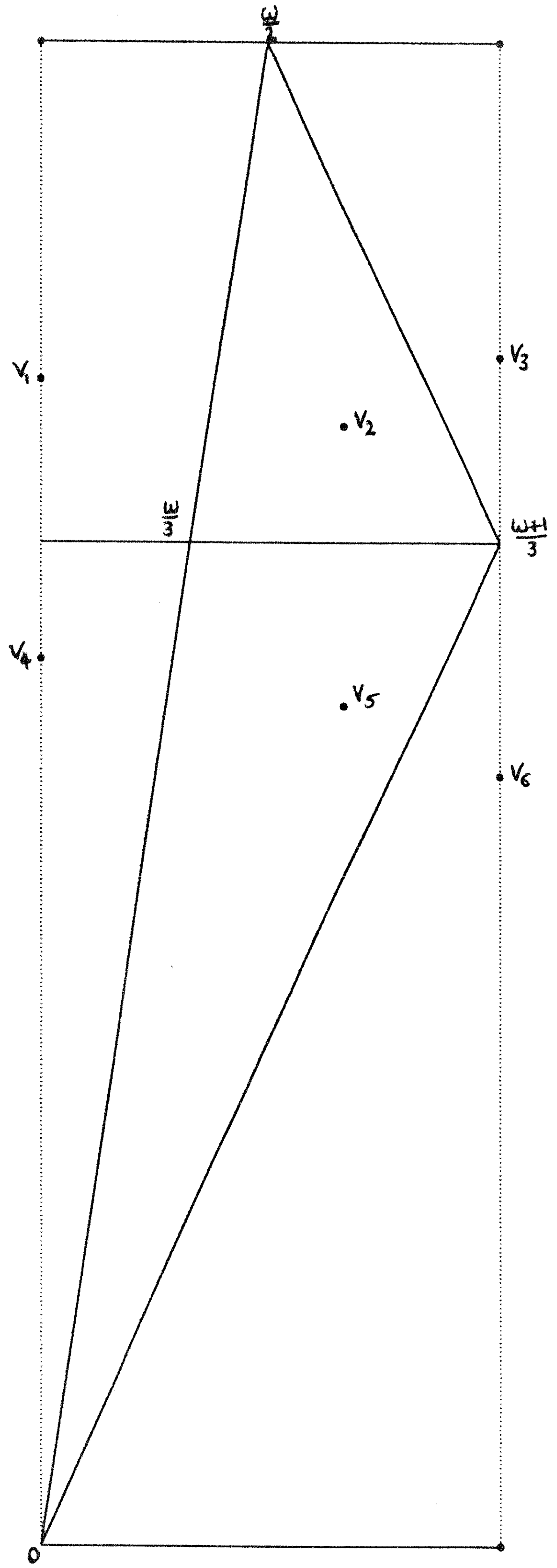


Figure (3.3.1)

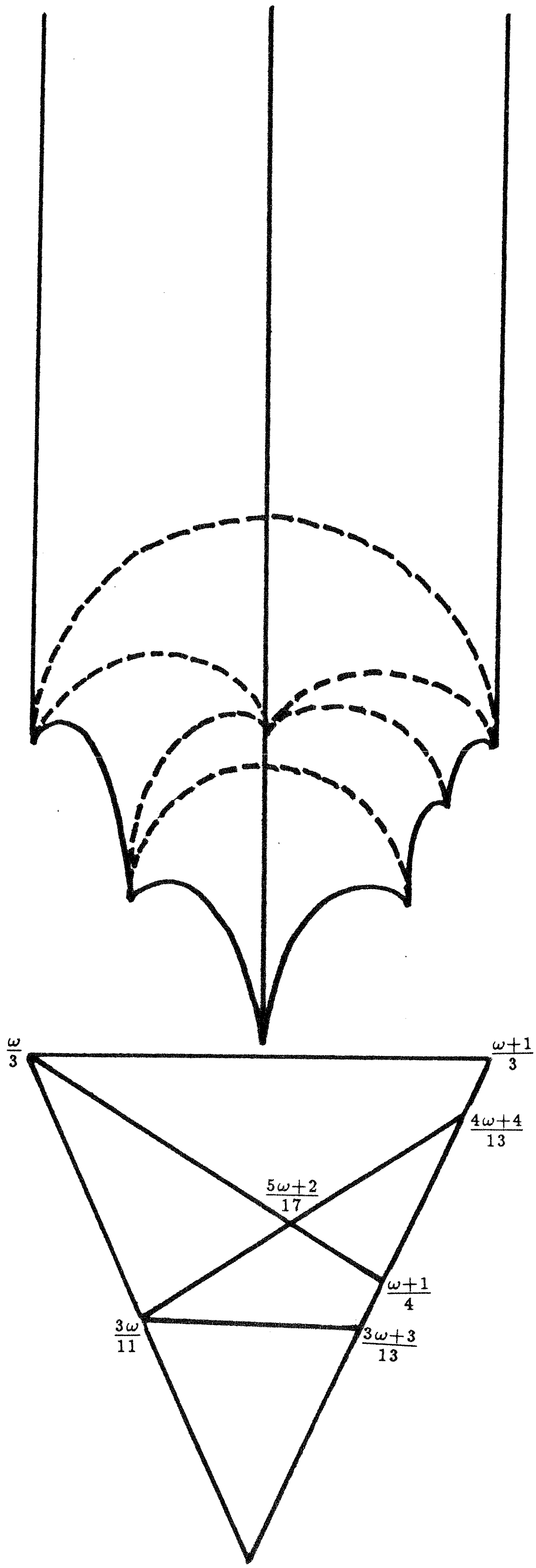

$\frac{\omega}{3}$
$\frac{\omega+1}{3}$
$\frac{4\omega+4}{13}$
$\frac{5\omega+2}{17}$
$\frac{\omega+1}{4}$
$\frac{3\omega}{11}$
$\frac{3\omega+3}{13}$
0

Figure (3.3.2)

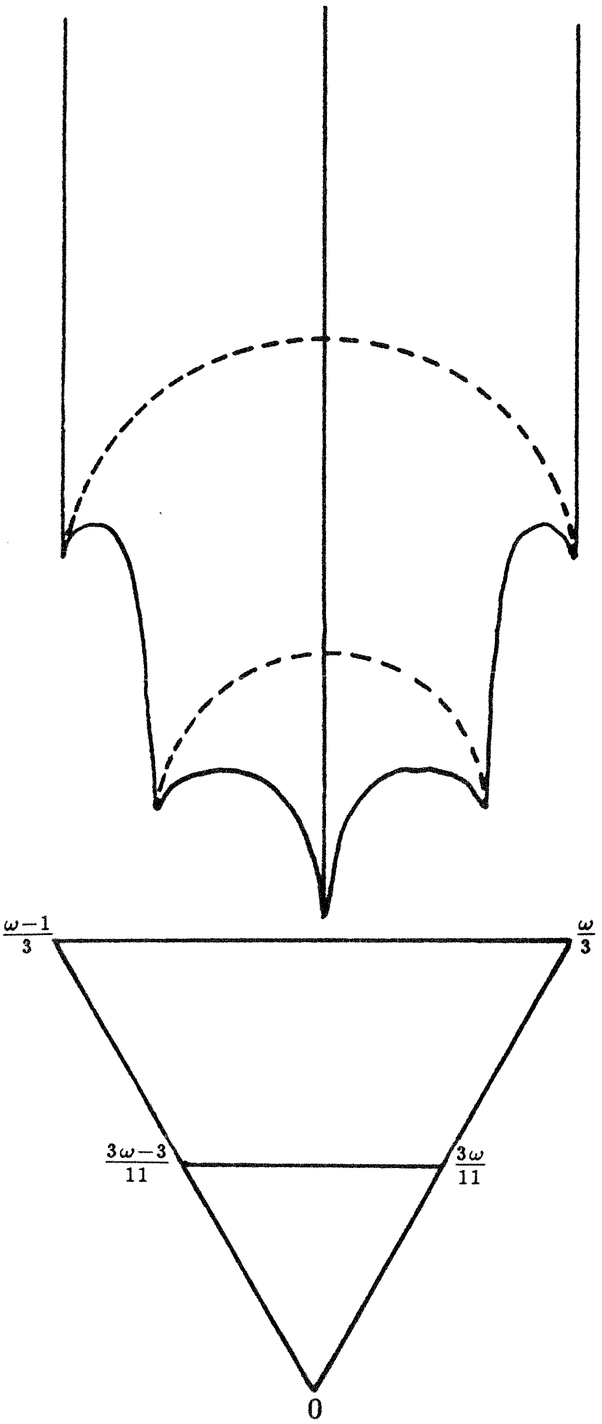


Figure (3.3.3)

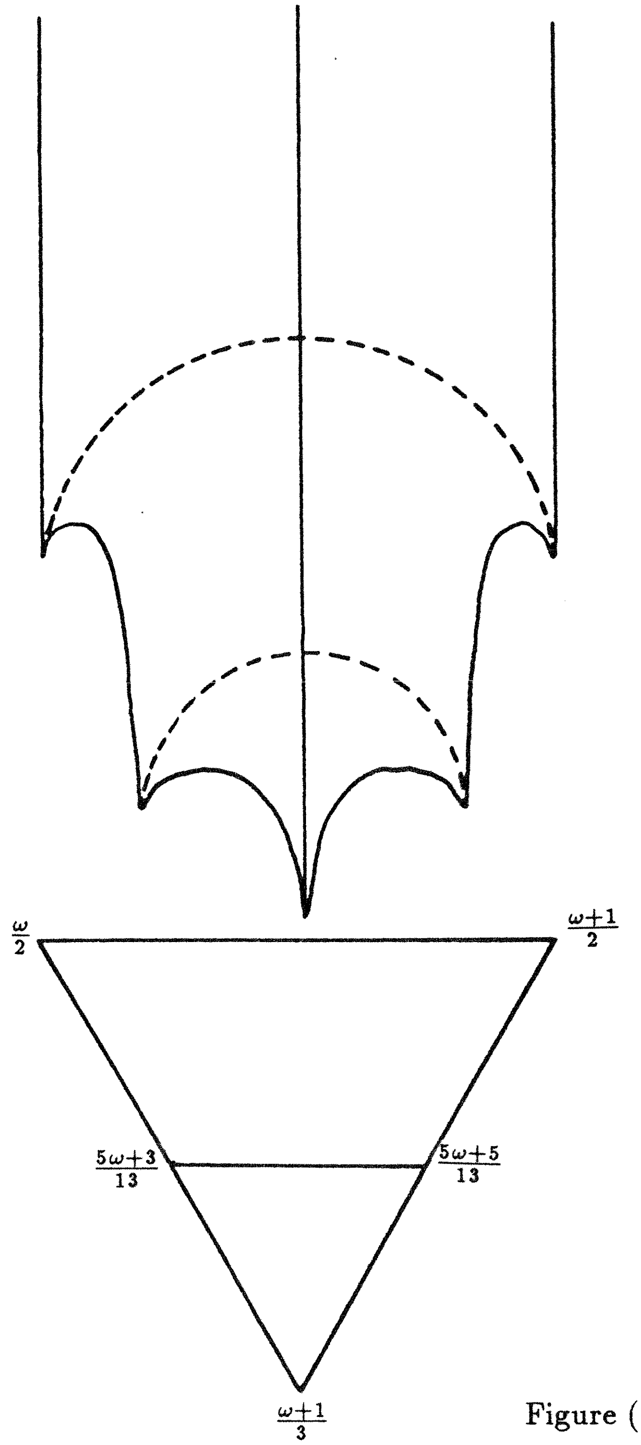
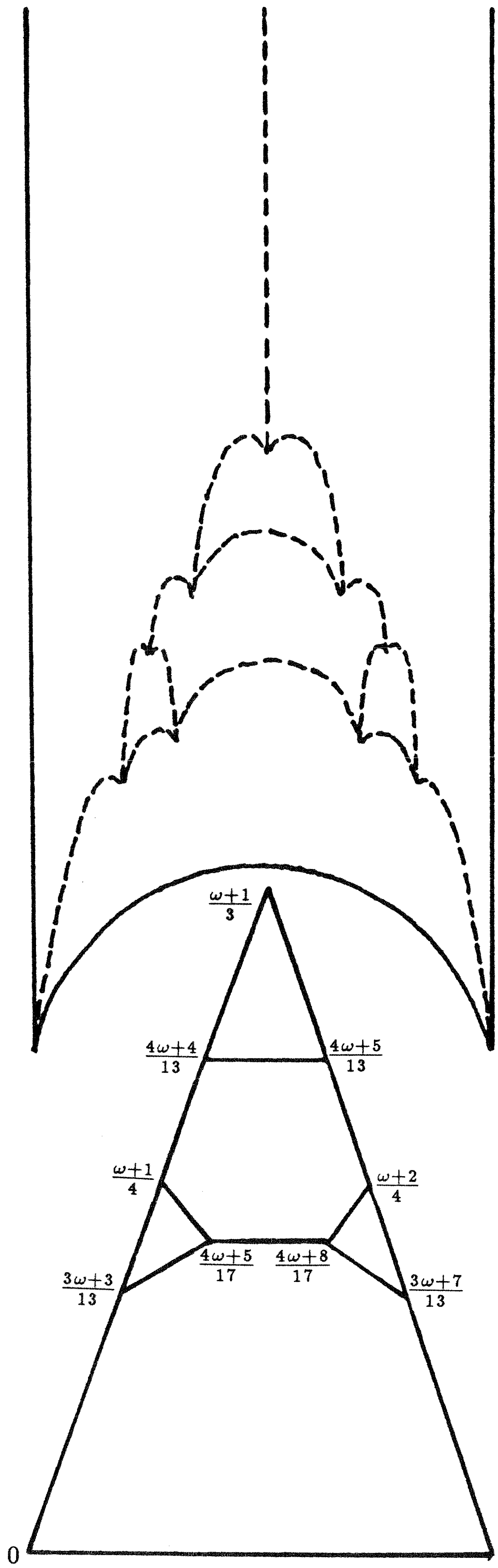

$\frac{\omega}{2}$
$\frac{\omega+1}{2}$
$\frac{5\omega+3}{13}$
$\frac{5\omega+5}{13}$
$\frac{\omega+1}{3}$

Figure (3.3.4)



$\frac{\omega+1}{3}$
$\frac{4\omega+4}{13}$
$\frac{4\omega+5}{13}$
$\frac{\omega+1}{4}$
$\frac{\omega+2}{4}$
$\frac{4\omega+5}{17}$
$\frac{4\omega+8}{17}$
$\frac{3\omega+3}{13}$
$\frac{3\omega+7}{13}$
0
1

Figure (3.3.5)

§3.4 The Calculation for $\mathbf{Q}(\sqrt{-67})$

Recall the results of §2.5. Figure (3.4.1) shows the projection of the partitioning of D onto F_0.

Consider the equivalence of the vertices of V under the action of $\Gamma^{\pm}$. We divide the $v \in V$ into classes using the ζ co-ordinates, according to Lemma 3.1.4. This gives 7 classes as follows:

Class 1: $\{v_3\}$,

Class 2: $\{v_2, v_4\}$,

Class 3: $\{v_5, v_{12}, v_{15}\}$,

Class 4: $\{v_6, v_{13}, v_{14}\}$,

Class 5: $\{v_7, v_9\}$,

Class 6: $\{v_8\}$,

Class 7: $\{v_{11}, v_{16}\}$.

We begin by considering Class 1. Using the method described in §2.1, we find that

$$T_{v_3} = \{\frac{\omega}{2}, \frac{\omega+1}{2}, \frac{\omega+1}{3}, \frac{7}{2-\omega}, \frac{8}{2-\omega}, \frac{\omega-7}{\omega+1}, \frac{\omega-6}{\omega+1}, \frac{2\omega+1}{5}, \frac{2\omega+2}{5}, \frac{\omega+8}{4-\omega},$$
$$\frac{2\omega-6}{\omega+3}, \infty\}.$$

We now wish to find G_{v_3}. We use Lemma 3.1.2 to find those $\gamma \in \Gamma^{\pm}$ such that γ fixes v_3:

Set $\mu = 2$. So $\lambda_1, \lambda_2 \in \{\omega, \omega+1\}$. Then the matrices $\gamma = \begin{pmatrix} \lambda_1 & x \\ \mu & -\lambda_2 \end{pmatrix}$ with determinant ± 1 are:

$$M_1 = \begin{pmatrix} \omega & 8-\omega \\ 2 & -\omega-1 \end{pmatrix}_{+1}$$
$$M_2 = \begin{pmatrix} \omega & 9-\omega \\ 2 & -\omega-1 \end{pmatrix}_{-1}$$
$$M_3 = \begin{pmatrix} \omega+1 & 8-\omega \\ 2 & -\omega \end{pmatrix}_{+1}$$
$$M_4 = \begin{pmatrix} \omega+1 & 9-\omega \\ 2 & -\omega \end{pmatrix}_{-1}$$

There are no solutions in ϑ_{K} to $-\omega^2 - 2x = \pm 1$ or $-(\omega+1)^2 - 2x = \pm 1$. So these four matrices are the only possible ones.

We now consider their action on v_3:

$$M_1 : v_3 \mapsto (\frac{1}{2}, \frac{27}{2\sqrt{67}}; \sqrt{\frac{2}{67}}) = v_3$$
$$M_2 : v_3 \mapsto (0, \frac{20}{\sqrt{67}}; \sqrt{\frac{2}{67}})$$
$$M_3 : v_3 \mapsto (\frac{1}{2}, \frac{27}{2\sqrt{67}}; \sqrt{\frac{2}{67}}) = v_3$$
$$M_4 : v_3 \mapsto (1, \frac{20}{\sqrt{67}}; \sqrt{\frac{2}{67}}).$$

Thus, M_1 and $M_3 \in G_{v_3}$.

Similarly, for all other μ and λ, we find:

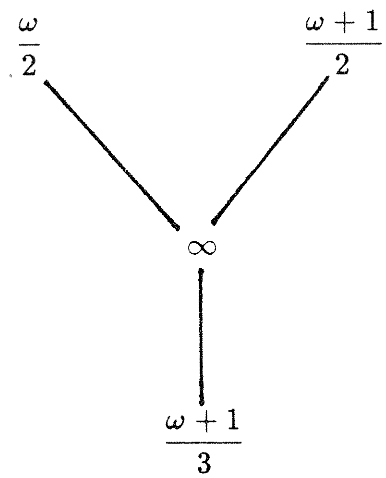
$$G_{v_3} = \{I, \begin{pmatrix} \omega & 8-\omega \\ 2 & -\omega-1 \end{pmatrix}, \begin{pmatrix} \omega+1 & 8-\omega \\ 2 & -\omega \end{pmatrix}, \begin{pmatrix} \omega+1 & 5-\omega \\ 3 & -\omega-1 \end{pmatrix},$$
$$\begin{pmatrix} 7 & -3\omega-3 \\ 2-\omega & -8 \end{pmatrix}, \begin{pmatrix} 8 & -3\omega-3 \\ 2-\omega & -7 \end{pmatrix}, \begin{pmatrix} \omega-7 & 2\omega+8 \\ \omega+1 & 6-\omega \end{pmatrix},$$
$$\begin{pmatrix} \omega-6 & 2\omega+8 \\ \omega+1 & 7-\omega \end{pmatrix}, \begin{pmatrix} 2\omega+1 & 13-2\omega \\ 5 & -2\omega-2 \end{pmatrix}, \begin{pmatrix} 2\omega+2 & 13-2\omega \\ 5 & -2\omega-1 \end{pmatrix},$$
$$\begin{pmatrix} \omega+8 & 5-4\omega \\ 4-\omega & -\omega-8 \end{pmatrix}, \begin{pmatrix} 2\omega-6 & \omega+16 \\ \omega+3 & 6-2\omega \end{pmatrix}\}.$$

G_{v_3} has order 12; eight elements have order 3 and three elements have order 2.

We now wish to know how the translates of D_{v_3} by G_{v_3} glue together. D_{v_3} is the subspace of D which is bounded below by the hemispheres, $S_{2,\omega}, S_{2,\omega+1}, S_{3,\omega+1}$ and whose sides are the intersections of the 3 vertical half-planes, $Q_{2,\omega;2,\omega+1}$, $Q_{2,\omega;3,\omega+1}$ and $Q_{2,\omega+1;3,\omega+1}$.

We begin by considering the translates of D_{v_3} by the inversions in $\{S_\alpha : \alpha \in A^\star_{v_3}\}$ where $A^\star_{v_3} = \{\frac{\omega}{2}, \frac{\omega+1}{2}, \frac{\omega+1}{3}\}$. Suppose α' is the vertex of D_{v_3} lying directly above $\alpha \in A^\star_{v_3}$. Then, because the part of the curved base of D_{v_3} which contains α' is part of the hemisphere S_α, we can see that the translate of D_{v_3} by M_α, the inversion in S_α, glues to D_{v_3} on the curved base. Then, as $M_\alpha(\alpha) = \infty$ and $M_\alpha(\infty) \in A_{v_3}$, there is an edge from β to ∞, $\beta \in A^\star_{v_3}$ in P_{v_3}. Similarly, using inversions in the other hemispheres with centres in $A^\star_{v_3}$, we have an edge from β to ∞ in P_{v_3}, $\forall \beta \in A^\star_{v_3}$.

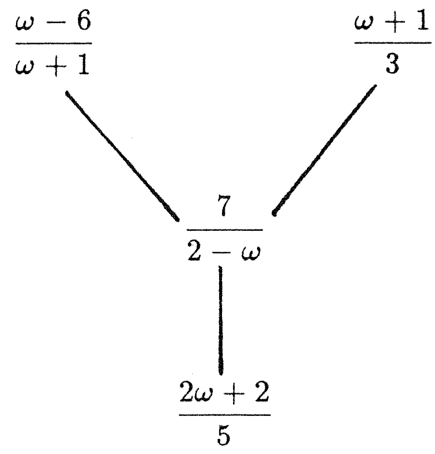
We represent this in the diagram:



where the lines between the points denote an edge in P_{v_3}.

We now consider the effect of $\gamma \in G_{v_3}$ on this diagram, with the adjacency of edges preserved.

$\gamma = \begin{pmatrix} 7 & -3\omega - 3 \\ 2-\omega & -8 \end{pmatrix}$ gives the diagram:



So there is an edge in P_{v_3} from $\frac{\omega+1}{3}$ to $\frac{8}{2-\omega}$, as well as one from $\frac{\omega+1}{3}$ to ∞, and so on.

Continuing in this way we find that P_{v_3} is the polyhedron with vertices $v \in T_{v_3}$ and edges defined by diagrams, similar to the one above, for each $\gamma \in G_{v_3}$. In fact, P_{v_3} is the truncated tetrahedron shown in Figure (3.4.2).

We now consider Class 2: $\{v_2, v_4\}$.

We wish to find P_{v_2}, the polyhedron associated with v_2. Using Lemma 3.1.3, we find that

$$T_{v_2} = \{\frac{\omega}{2}, \frac{\omega}{3}, \frac{\omega+1}{3}, \frac{7}{1-\omega}, \frac{\omega-7}{\omega+1}, \frac{\omega-6}{\omega+1}, \frac{2\omega+1}{5}, \frac{\omega+7}{4-\omega}, \infty\}$$

and

$$T_{v_4} = \{\frac{\omega-1}{3}, \frac{\omega}{3}, \frac{\omega-1}{4}, \frac{\omega}{4}, \frac{5}{1-\omega}, -\frac{5}{\omega}, \frac{\omega+4}{4-\omega}, \frac{\omega-5}{\omega+3}, \infty\}.$$

We use Lemmas 3.1.2 and 3.1.3, as before, to find G_{v_2} and $H_{v_4 v_2}$.

Thus

$$G_{v_2} = \{I, \begin{pmatrix} \omega - 7 & 2\omega + 8 \\ \omega + 1 & 6 - \omega \end{pmatrix}, \begin{pmatrix} \omega - 6 & 2\omega + 8 \\ \omega + 1 & 7 - \omega \end{pmatrix}\}$$

and

$$H_{v_4 v_2} = \{\begin{pmatrix} \omega & 6 \\ 3 & 1 - \omega \end{pmatrix}, \begin{pmatrix} 7 & -2\omega \\ 1 - \omega & -5 \end{pmatrix}, \begin{pmatrix} \omega + 7 & 6 - 2\omega \\ 4 - \omega & -\omega - 4 \end{pmatrix}\}.$$

In particular, $H_{v_4 v_2} \neq \phi$ and so v_2 and v_4 are equivalent under the action of $\Gamma^{\pm}$.

Now proceed as before, gluing together translates of D_{v_2} by G_{v_2} and translates of D_{v_4} by $H_{v_4 v_2}$ along common edges.

We find that P_{v_2} is the hexagonal cap shown in Figure (3.4.3) with vertices in T_{v_2}.

Similarly, in the other 5 cases we find that

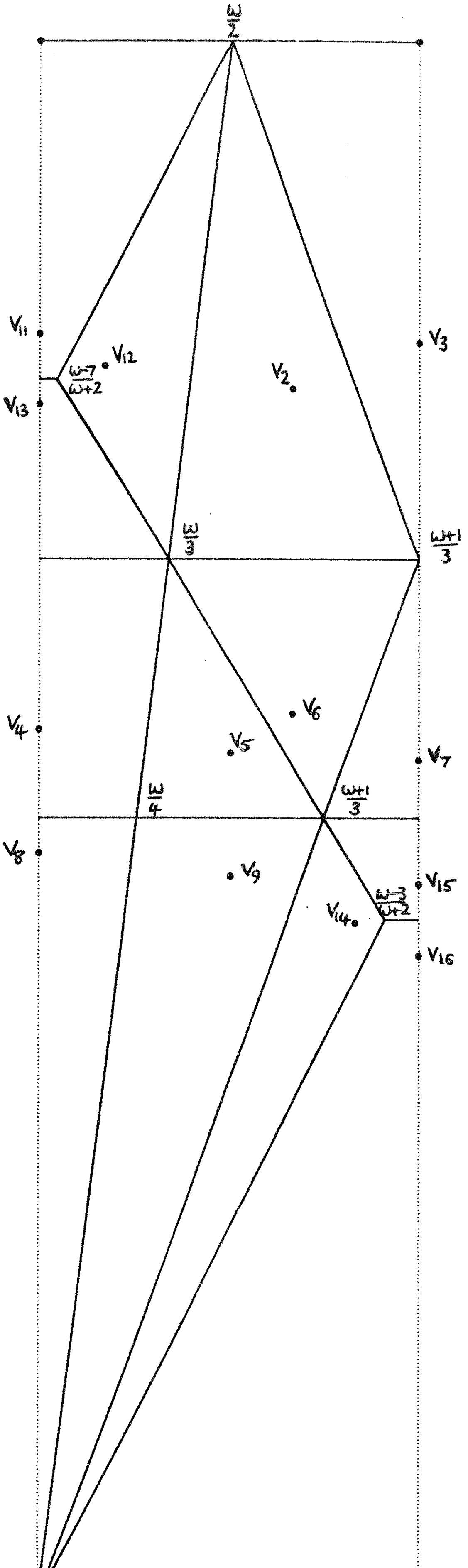
P_{v_5} is the tetrahedron shown in Figure (3.4.4);

P_{v_6} is the tetrahedron shown in Figure (3.4.5);

P_{v_9} is the triangular prism shown in Figure (3.4.6);

P_{v_8} is the triangular prism shown in Figure (3.4.7);

$P_{v_{11}}$ is the octahedron shown in Figure (3.4.8).



ω/2
V11
V12
(ω-7)/(ω+2)
V13
V2
V3
ω/3
(ω+1)/3
V4
V6
V5
V7
ω/4
(ω+1)/3
V8
V9
V15
(ω-3)/(ω+2)
V14
V16
0

Figure (3.4.1)

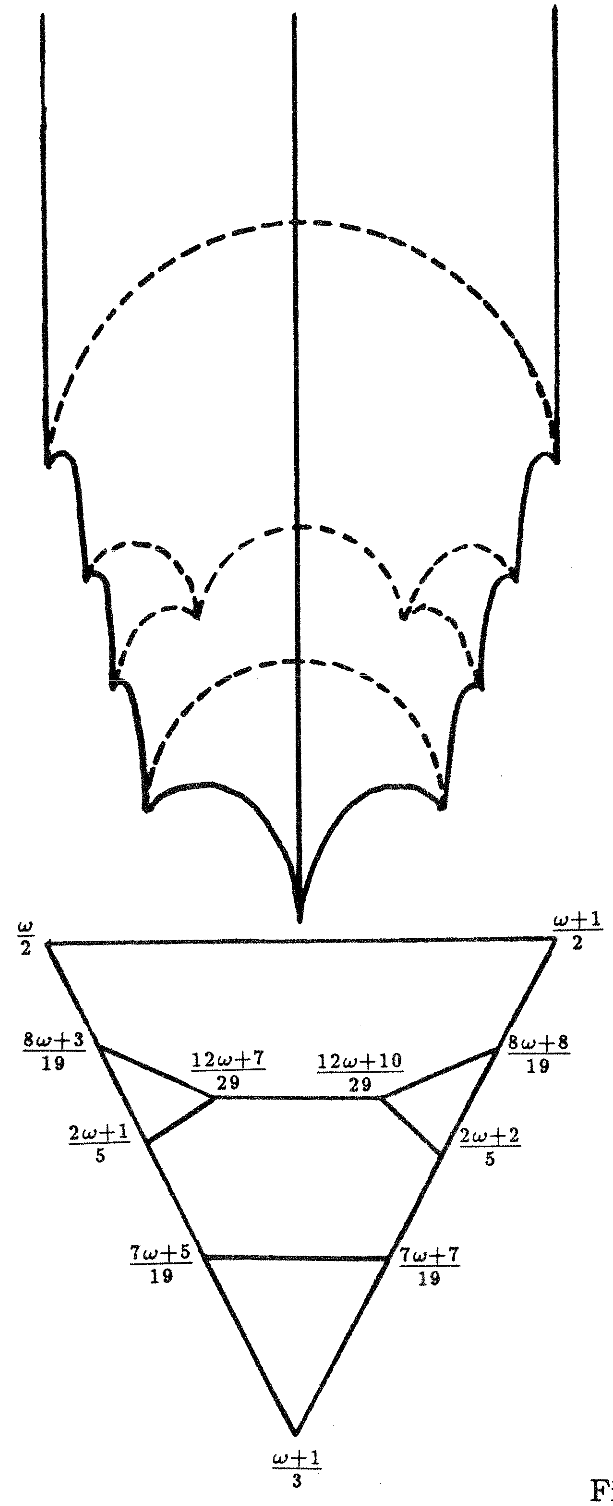

$\frac{\omega}{2}$
$\frac{\omega+1}{2}$
$\frac{8\omega+3}{19}$
$\frac{12\omega+7}{29}$
$\frac{12\omega+10}{29}$
$\frac{8\omega+8}{19}$
$\frac{2\omega+1}{5}$
$\frac{2\omega+2}{5}$
$\frac{7\omega+5}{19}$
$\frac{7\omega+7}{19}$
$\frac{\omega+1}{3}$

Figure (3.4.2)

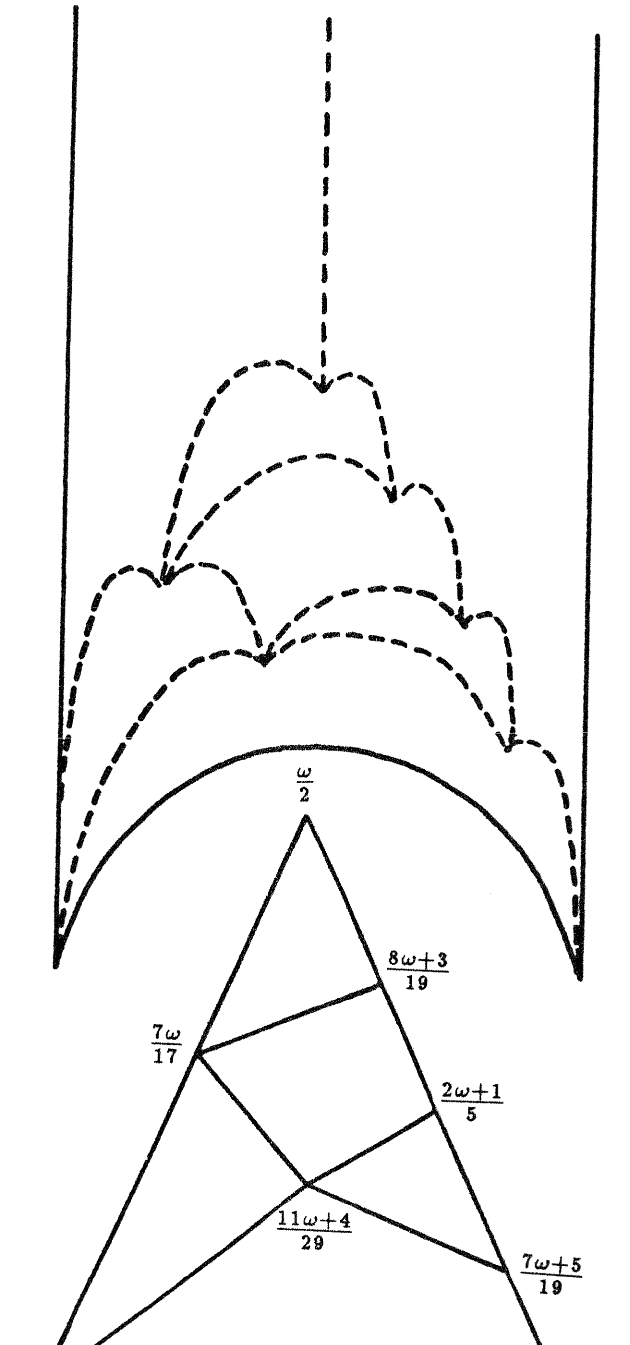


Figure (3.4.3)

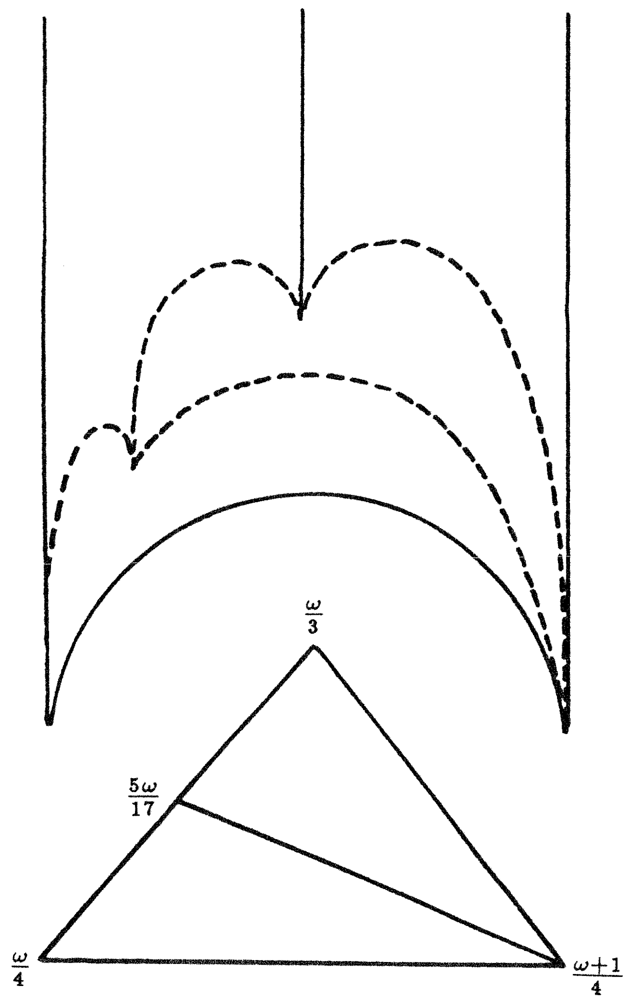


Figure (3.4.4)

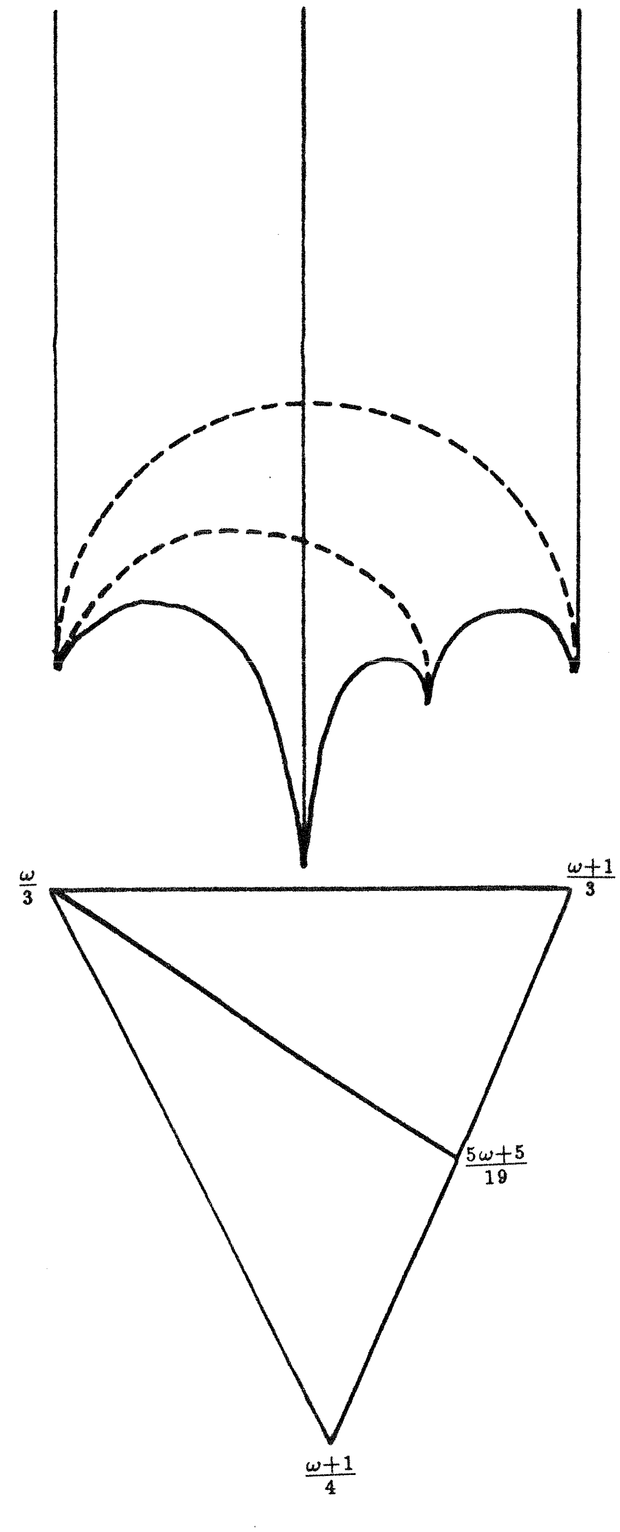


Figure (3.4.5)

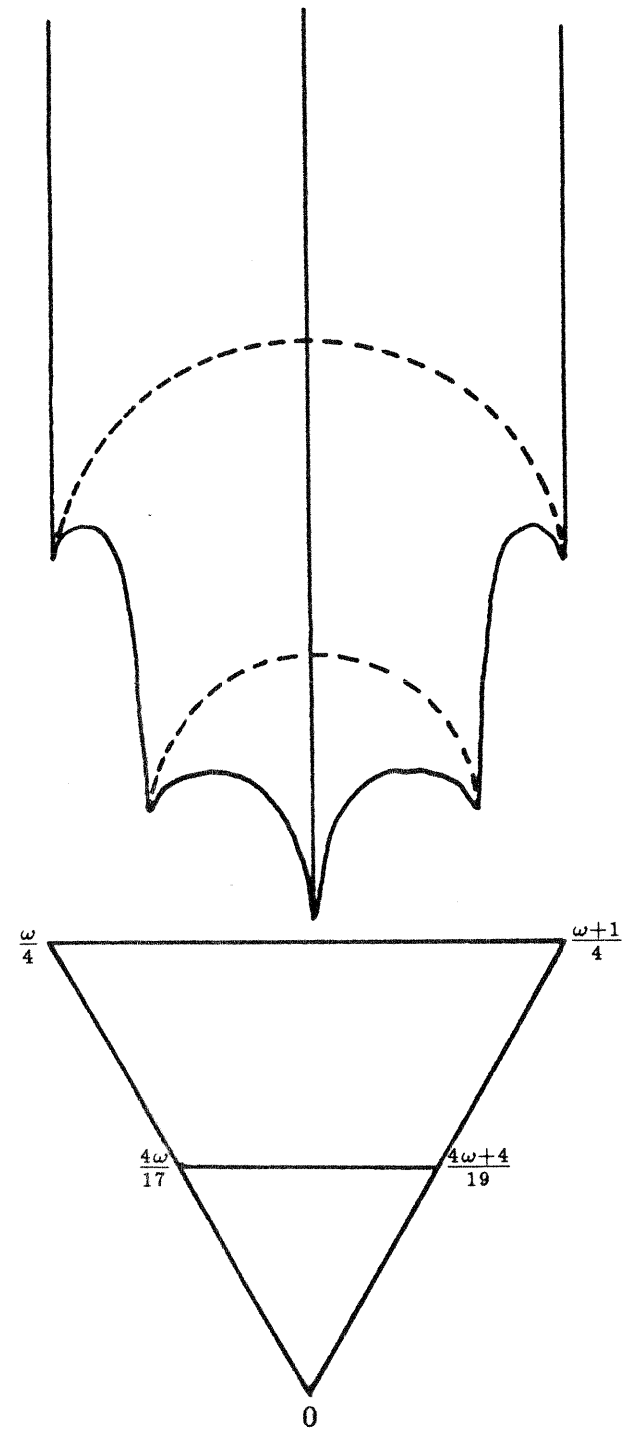


Figure (3.4.6)

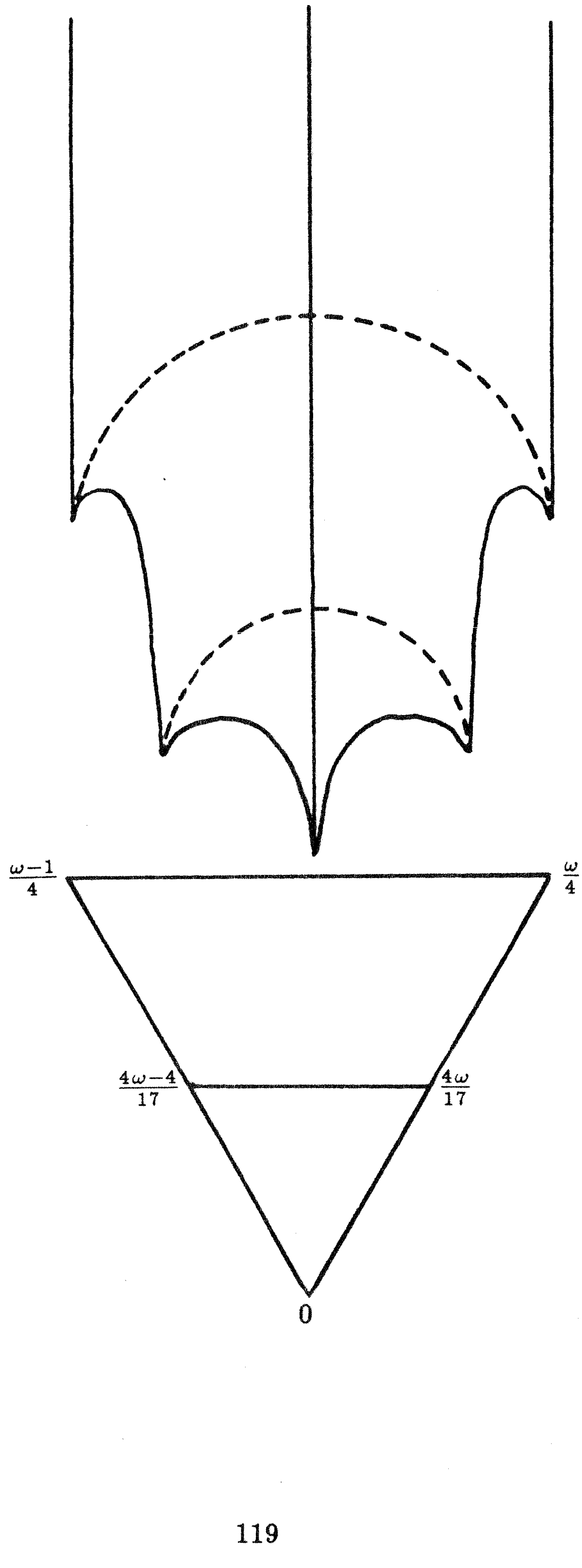
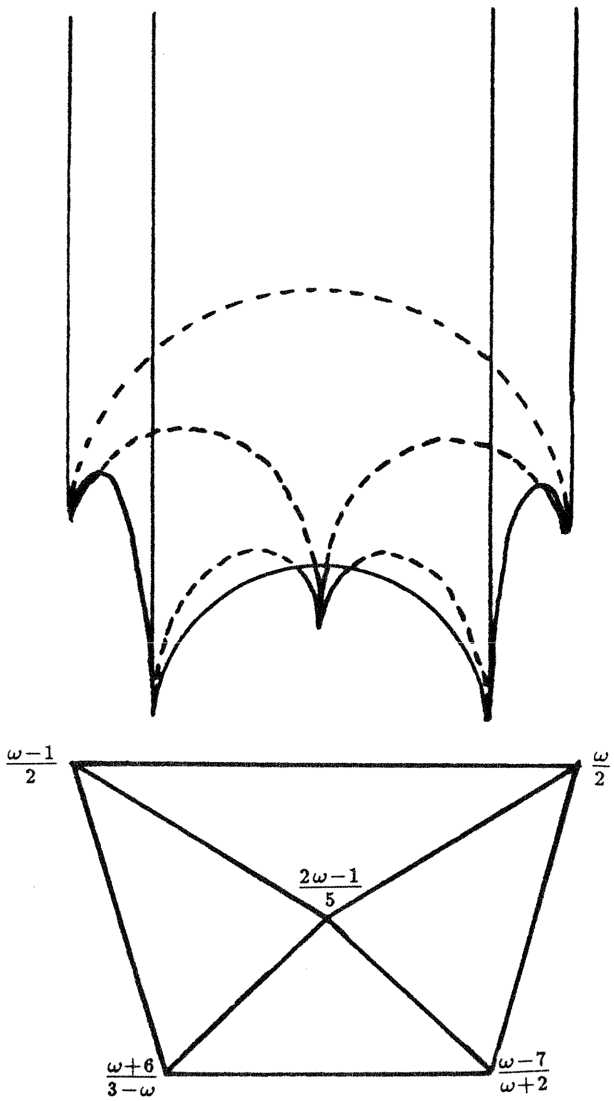


Figure (3.4.7)



$\frac{\omega-1}{2}$
$\frac{\omega}{2}$
$\frac{2\omega-1}{5}$
$\frac{\omega+6}{3-\omega}$
$\frac{\omega-7}{\omega+2}$

Figure (3.4.8)

§3.5 The Calculation for $\mathbf{Q}(\sqrt{-163})$

Recall the results of §2.6. Once again the projection of the partitioning of D onto F_0 is shown in Figure (3.5.1). Notice that in this case it is not always true that $v \in D_v$.

As in the previous cases, we begin by splitting V into equivalence classes according to the ζ co-ordinates of the elements. Thus we reduce V into 20 classes. However, unlike the previous fields, there are cases in $\mathbf{Q}(\sqrt{-163})$ where the equivalence of the ζ co-ordinate is not sufficient to determine equivalence under the action of $\Gamma^{\pm}$.

eg. Consider the set of points with ζ co-ordinate $\sqrt{\frac{66}{4075}}$. These are :

$$\{v_{12}, v_{19}, v_{37}, v_{43}\}$$

As before, we find

$$\begin{aligned}
T_{v_{12}} &= \{\frac{2\omega+2}{5}, \frac{2\omega+1}{5}, \frac{3\omega+2}{7}, \frac{\omega+18}{4-\omega}, \frac{2\omega-16}{\omega+3}\},\\
T_{v_{19}} &= \{\frac{\omega+1}{3}, \frac{2\omega+2}{5}, \frac{2\omega+1}{5}, \frac{15}{2-\omega}, \frac{\omega-15}{\omega+1}\},\\
T_{v_{37}} &= \{\frac{\omega}{4}, \frac{\omega+1}{4}, \frac{\omega}{5}, \frac{19}{1-\omega}, \frac{\omega-9}{\omega+3}\},\\
T_{v_{43}} &= \{\frac{\omega+1}{5}, \frac{\omega}{5}, \frac{\omega+1}{6}, \frac{8}{2-\omega}, \frac{\omega-7}{\omega+4}\}.
\end{aligned}$$

Consider v_{43}.

$$G_{v_{43}} = \{I, \begin{pmatrix} \omega-7 & \omega+8 \\ \omega+4 & 7-\omega \end{pmatrix}\},$$

and

$$H_{v_{19}v_{43}} = \{\begin{pmatrix} 8 & -3\omega-3 \\ 2-\omega & -16 \end{pmatrix}, \begin{pmatrix} \omega+1 & 16-\omega \\ 5 & -2\omega-1 \end{pmatrix}\}.$$

$H_{v_{19}v_{43}} \neq \phi$, so v_{19} and v_{43} are equivalent. However, $H_{v_{12}v_{43}} = H_{v_{37}v_{43}} = \phi$. So v_{12} and v_{37} are **not** equivalent to v_{43}, although they are equivalent to each other. Thus, there are 2 equivalence classes of points with ζ co-ordinate $\sqrt{\frac{66}{4075}}$:

1: $\{v_{19}, v_{43}\}$,

2: $\{v_{12}, v_{37}\}$.

In fact, there are 25 distinct equivalence classes of V and 5 types of polyhedra associated to them. These are as follows:

11 tetrahedrons associated to the classes:

$$\{v_6, v_{46}, v_{50}\}, \{v_7, v_{51}\}, \{v_8, v_{23}, v_{53}\}, \{v_9, v_{24}\}, \{v_{10}, v_{39}\},$$

$$\{v_{14}, v_{33}\}, \{v_{15}\}, \{v_{16}\}, \{v_{17}, v_{45}\}, \{v_{21}, v_{22}, v_{54}\}, \{v_{13}, v_{35}, v_{40}\};$$

8 triangular prisms associated to:

$$\{v_2, v_{29}, v_{55}\}, \{v_3, v_{30}, v_{25}\}, \{v_4, v_{27}\}, \{v_5, v_{32}\},$$

$$\{v_{12}, v_{37}\}, \{v_{19}, v_{43}\}, \{v_{28}, v_{56}\}, \{v_{42}, v_{47}\};$$

3 square pyramids associated to:

$$\{v_{11}, v_{34}, v_{38}\}, \{v_{18}, v_{44}, v_{52}\}, \{v_{20}, v_{26}, v_{31}\};$$

2 hexagonal caps associated to:

$$\{v_1, v_{49}\}, \{v_{41}, v_{48}\};$$

and a cuboctahedron associated to:

$$\{v_{36}\}.$$

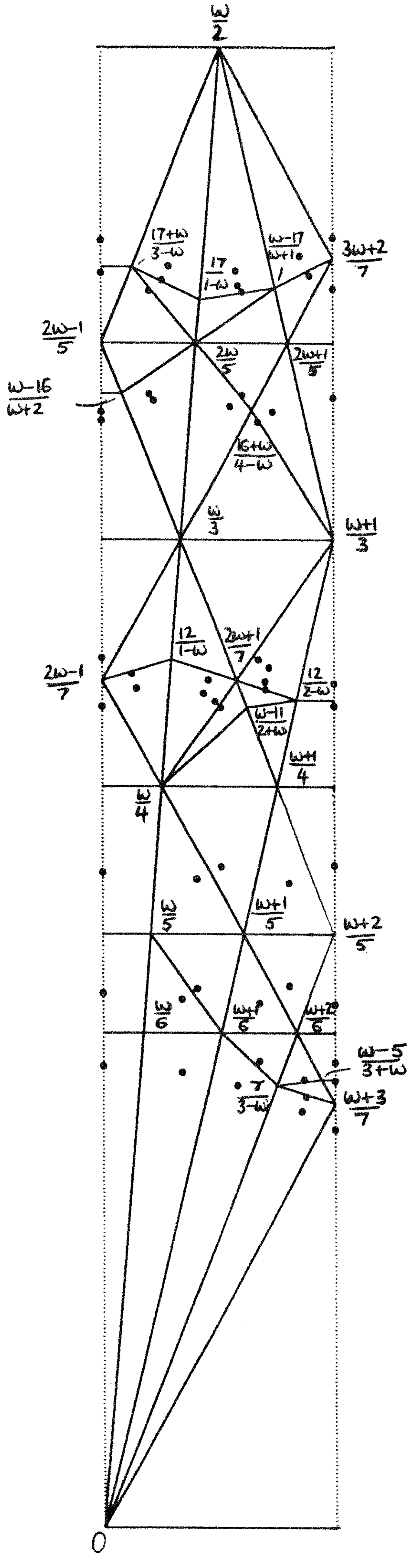

ω/2
17+ω/3−ω
17/1−ω
ω−17/ω+1
3ω+2/7
2ω−1/5
2ω/5
2ω+1/5
ω−16/ω+2
16+ω/4−ω
ω/3
ω+1/3
12/1−ω
2ω+1/7
12/2−ω
2ω−1/7
ω−11/2+ω
ω+1/4
ω/4
ω/5
ω+1/5
ω+2/5
ω/6
ω+1/6
ω+2/6
ω−5/3+ω
7/3−ω
ω+3/7
0

Figure (3.5.1)

CHAPTER 4: HOMOLOGY

In this chapter I will be concerned with the problem of determining $H_1(G\backslash\mathcal{H}_3^*,\mathbf{Q})$, where K is one of the four non-Euclidean, class number 1 fields. The tessellation of $\mathcal{H}_3^*$ by hyperbolic polyhedra, found in the previous chapter, will play an important part in the calculation.

The method used is based on that used by Cremona in [4], although the generalisation from Euclidean to non-Euclidean fields is not an immediate one. §4.1 will be an outline of the general algorithm, while explicit calculations for each of the four fields in question will be given in §4.2 – §4.5. §4.6 will be a description of M-symbols and the Hecke algebra in the non-Euclidean case.

§4.1 The General Method

Recall §1.3, the description of Cremona's method for determining $H_1(G\backslash\mathcal{H}_3^*,\mathbf{Q})$ in the case where K is a Euclidean field. He does this using the 1-skeleton of the tessellation of $\mathcal{H}_3^*$ by hyperbolic polyhedra. Similarly, we wish to determine $H_1(G\backslash\mathcal{H}_3^*,\mathbf{Q})$, using the tessellations found in Chapter 3, for each of the four non-Euclidean, class number 1 fields.

Because we are using the 1-skeleton of a given tessellation, we are interested in the vertices and edges of it, and in how they behave under the action of $\Gamma = SL(2,\vartheta_K)$. The vertices of the tessellation are simply the cusps of $\mathcal{H}_3^*$ and, as in the Euclidean case, Γ acts transitively on them. In the Euclidean case Γ also acts transitively on the edges of the tessellation (and on the polyhedra themselves) and so every edge has the form $\{\gamma(0), \gamma(\infty)\}, \gamma \in \Gamma$, ie the image of the basic edge $\{0, \infty\}$ under some $\gamma \in \Gamma$. In the non-Euclidean cases the action of Γ on the edges is not transitive. Instead, the edges split up into a finite number of orbits under Γ.

Let A be the set of centres of hemispheres in D defined in §2.2.

Definition 4.1.1:

Let $(\gamma)_\alpha$ denote the modular symbol $\{\gamma(\alpha), \gamma(\infty)\}$, where $\alpha \in A$.

We shall call this the **type-α symbol.**

We define an equivalence relation between the $(\gamma)_\alpha$ under the action of Γ as follows:

Definition 4.1.2:

$(\gamma)_\alpha \equiv (\gamma)_\beta$ *for* $\alpha, \beta \in A \Leftrightarrow \exists M \in \Gamma$ *such that* $(\gamma)_\alpha = \pm(\gamma M)_\beta, \forall \gamma \in \Gamma$.

We can determine **all** such symbols, $(\gamma)_\alpha$, and matrices, M, by means of the following:

Lemma 4.1.3:

If $(\gamma)_\alpha = \pm(\gamma M)_\beta$ *where* $\alpha = \frac{\lambda_1}{\mu_1} \neq \beta = \frac{\lambda_2}{\mu_2} \in A$, *then either*

(1) $M = J$ *and* $(\gamma)_\alpha = (\gamma J)_\beta$ *where* $\alpha = -\beta$

or

(2) $M = \begin{pmatrix} \lambda_1 & x \\ \mu_1 & -\lambda_2 \end{pmatrix}$ *where* $-\lambda_1\lambda_2 - x\mu = \pm 1$ *and* $\mu_1 = \mu_2$.

Then $(\gamma)_\alpha = -(\gamma M)_\beta$.

Proof:

(1) Suppose $(I)_\alpha = (M)_\beta$ with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Then $\{\alpha, \infty\} = \{M(\beta), M(\infty)\}$, so $\alpha = M(\beta)$ and $M(\infty) = \infty$.

$M(\infty) = \frac{a}{c}$. Thus, $c = 0$ and $\alpha = \frac{a\beta+b}{d}$.

Since $ad = \pm 1$ we may assume that $d = 1$, so $\alpha = \pm\beta + b$.

But, because $\alpha, \beta \in F$, this forces $b = 0$ and so either

$\alpha = \beta$ and $M = I$ - the trivial case

or

$\alpha = -\beta$ and $M = J$.

(2) Now suppose that $(I)_\alpha = -(M)_\beta, M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Then $\alpha = M(\infty)$ and $\infty = M(\beta)$.

So $\alpha = \frac{a}{c}$ and $\beta = -\frac{d}{c}$, in lowest terms.

If $\alpha = \frac{\lambda_1}{\mu_1}$ and $\beta = \frac{\lambda_2}{\mu_2}$ then $a = \lambda_1, c = \mu_1 = \mu_2$ and $d = -\lambda_2$.

Setting $x = b$ gives the result. ∎

Factoring out by this equivalence relation reduces the number of types by a factor of approximately 4. The resulting type-α_i symbols are the orbits of the edges under Γ. For convenience we will number the α_i; $\alpha_1, \ldots, \alpha_t$ where t is the number of orbits. Then $(\gamma)_i = \{\gamma(\alpha_i), \gamma(\infty)\}$, for $1 \leq i \leq t$.

Now suppose that G is a subgroup of finite index in Γ and let $\mathcal{G}$ be a complete set of coset representatives for G in Γ. Set $V(G) =$ the $\mathbf{Q}$-vector space spanned by the symbols $\{(\gamma)_i : 1 \leq i \leq t, \gamma \in \mathcal{G}\}$. It is sufficient to use $\gamma \in \mathcal{G}$ rather than

$\gamma \in G$ since $(g\gamma)_i = (\gamma)_i, \forall g \in G$ and $1 \leq i \leq t$. So we can generate $H_1(G\backslash\mathcal{H}_3^*, \mathbf{Q})$ by $\{(\gamma)_i : 1 \leq i \leq t, \gamma \in \mathcal{G}\}$.

Γ acts on $V(G)$, on the left, via

$$\gamma : (g)_i \mapsto (\gamma g)_i,$$

and so $V(G)$ is a $\mathbf{Z}\Gamma$ module.

Relations between the symbols $(\gamma)_i$ arise from the tessellation as in the Euclidean case; from gluing together two polyhedra along a common edge; and from adding together the edges of a face of a polyhedron to give zero. In the Euclidean case we define the relation ideal, $\mathcal{R}$, to be the ideal of $\mathbf{Z}\Gamma$ generated by all such relations. In the non-Euclidean case the relations do not generate an ideal of $\mathbf{Z}\Gamma$.

Instead, just as we obtained the generators of $V(G)$ from the tessellation by taking one type of symbol from each orbit and, for each type, one generator for each coset representative; so, we get a relation from each face, and also from certain edges, of the tessellation and these relations fall into **types**, one from each orbit of Γ acting on the faces and one from each orbit of Γ acting on the edges of interest. Within each type we then obtain one relation for each coset representative. Now, because Γ maps faces to faces and edges to edges, and because each relation arises from a single face or edge of the tessellation, we can see that the left action of Γ takes relations to relations. Thus the set of all such relations form a $\mathbf{Z}\Gamma$ submodule of $V(G)$ which we will call $R(G)$.

Algebraically, any relation in the homology has the general form:

$$\sum_{\gamma \in \mathcal{G}} \sum_{i=1}^{t} c(\gamma, i)(\gamma)_i = 0. \qquad (4.1.1)$$

Define

$$\delta : V(G) \to H_0(G)$$

via

$$\delta : (\gamma)_i \mapsto [\gamma(\infty)] - [\gamma(\alpha_i)]. \tag{4.1.2}$$

Then, because the relation (4.1.1) comes from a face of the tessellation, or from the self-identification of an edge, the symbols $c(\gamma, i)(\gamma)_i$ add together to give a closed path. Thus, each relation has:

$$\sum_{\gamma \in \mathcal{G}} \sum_{i=1}^{t} c(\gamma, i)\delta(\gamma)_i = 0.$$

Set $Z(G) = \mathrm{Ker}(\delta)$. If $\sum c(i)(\theta)_i \in Z(G)$ then, $\forall \gamma \in \Gamma, \sum c(i)(\gamma\theta)_i \in Z(G)$. So $Z(G)$ is also a $\mathbf{Z}\Gamma$ submodule of $V(G)$ and $R(G) \subset Z(G)$.

Finally we set $H(G) = Z(G)/R(G)$. Then there is an isomorphism,

$$\xi : H(G) \to H_1(G \backslash \mathcal{H}_3^*, \mathbf{Q}),$$

defined via:

$$\xi : (\gamma)_i \mapsto \{\gamma(\alpha_i), \gamma(\infty)\}. \tag{4.1.3}$$

The main problem now is that of determining $R(G)$. Each relation in $R(G)$ arises from a face or edge of the tessellation. However, by considering one face or edge in each Γ-orbit, we can find a finite generating set of relations for $R(G)$. So in the same way that we found the generating set of edges, we find the generating set of relations by taking one for each coset representative in $\mathcal{G}$ and one for each type.

As in the Euclidean case, we only need to calculate $R(G)$ once for each field. The remainder of this section will be concerned with determining the types of relation. The relations which arise from the faces depend on the exact nature of the geometry of the field in question and will be dealt with in the following sections. The two-term relations, which arise from the gluing together of two polyhedra along a common edge, can be determined purely algebraically according to:

Lemma 4.1.4:

Let $K = \mathbf{Q}(\sqrt{-d}), d \in \{19, 43, 67, 163\}$.

Then the two-term relations described above are exactly those of the form:

(1) $(\gamma)_0 = (\gamma J)_0, \quad J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$;

(2) $(\gamma)_0 = -(\gamma S)_0, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$;

(3) $(\gamma)_{\frac{\omega}{2}} = (\gamma K)_{\frac{\omega}{2}}, \quad K = \begin{pmatrix} -1 & \omega \\ 0 & 1 \end{pmatrix}$;

(4) $(\gamma)_\alpha = -(\gamma M)_\alpha$, *where* $\alpha = \frac{\lambda}{\mu} \in A$ *and* $M = \begin{pmatrix} \lambda & x \\ \mu & -\lambda \end{pmatrix} \in \Gamma^{\pm}$.

Note: $0, \frac{\omega}{2} \in A$, for all our fields of interest and so relations (1), (2) and (3) will always occur.

Proof:

The two-term relations represent the gluing together of two polyhedra along a common edge, ie the self-identification of an edge $\{\beta, \delta\}$.

All of our edges are of the form $(\gamma)_\alpha = \{\gamma(\alpha), \gamma(\infty)\}$, some $\gamma \in \mathcal{G}$ and $\alpha \in A$ and so the two-term relations will all be of the form:

$$\{\gamma(\alpha), \gamma(\infty)\} = \pm\{\gamma M(\alpha), \gamma M(\infty)\}, M \in \Gamma^{\pm}.$$

So, we wish to find all pairs (M, α), with $M \in \Gamma^{\pm}$ and $\alpha \in A$, such that $(I)_\alpha = \pm(M)_\alpha$.

Suppose $(I)_\alpha = (M)_\alpha$ with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^{\pm}$.

Then $\{\alpha, \infty\} = \{M(\alpha), M(\infty)\}$, ie $\alpha = M(\alpha)$ and $\infty = M(\infty)$.

$M(\infty) = \frac{a}{c}$ and so $c = 0$ and $\alpha = \frac{a\alpha + b}{d}$.

$ad = \pm 1$ and so we may assume that $d = 1$; ie $\alpha = \pm\alpha + b$, ie $\alpha = \frac{b}{(1\mp 1)} \neq \infty$.

So $\alpha = \frac{b}{2}$.

All of the $\alpha \in A$ are in lowest terms and so the only possible solutions are:

either $\alpha = b = 0$ and $M = J$,

or $\alpha = \frac{\omega}{2}$ and $M = K$.

Now suppose that $(I)_\alpha = -(M)_\alpha$ with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^{\pm}$.

Then $\{\alpha, \infty\} = \{M(\infty), M(\alpha)\}$, ie $\alpha = M(\infty)$ and $\infty = M(\alpha)$.

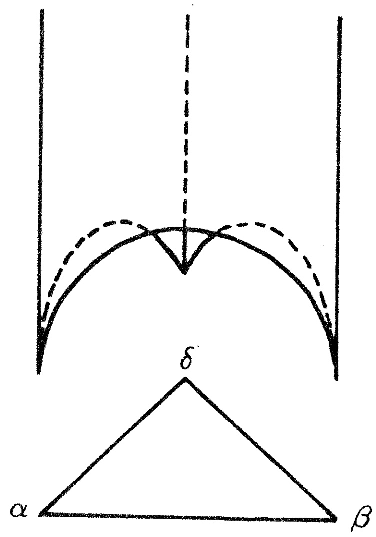
In lowest terms, $\alpha = \frac{a}{c} = -\frac{d}{c}$.

So if $\alpha = \frac{\lambda}{\mu}$ then $c = \mu$ and $a = -d = \lambda$.

Thus $M = \begin{pmatrix} \lambda & x \\ \mu & -\lambda \end{pmatrix} \in \Gamma^{\pm}$.

In particular, if $\alpha = 0$ then $M = S$. ∎

Note: Consider the tetrahedron



This gives 4 relations between the paths $\{A, B\}$, $A, B \in \{\alpha, \beta, \delta, \infty\}$, one from each face. These are:

(1) $\{\infty, \alpha\} + \{\alpha, \beta\} + \{\beta, \infty\} = 0,$

(2) $\{\infty, \beta\} + \{\beta, \delta\} + \{\delta, \infty\} = 0,$

(3) $\{\infty, \delta\} + \{\delta, \alpha\} + \{\alpha, \infty\} = 0,$

(4) $\{\alpha, \beta\} + \{\beta, \delta\} + \{\delta, \alpha\} = 0.$

Clearly, the fourth relation is a consequence of the first three. Similarly, for each polyhedron, there is always at least one relation which can be written in terms of the others. Generally, if a polyhedron has n faces then there are at most $n-1$ independent relations associated with it.

Finally, it is easier in computing terms if we reduce the number of types of generating symbols $(\gamma)_i$. We do this, at the cost of complicating the relations between them, by using the mixed type relations to express one type as a sum of the others, ie

$$(\gamma)_i = \sum_j c(j)(M_j\gamma)_j. \tag{4.1.4}$$

This reduction will be considered in more detail in the following sections.

§4.2 The Calculation for $\mathbf{Q}(\sqrt{-19})$

We begin by using Lemma 4.1.3 to reduce the number of types of modular symbol required to generate $H_1(G\backslash\mathcal{H}_3^*,\mathbf{Q})$.

Recall §2.3; $A = \{0, \frac{\omega}{2}, \frac{\omega-1}{2}\}$. So we begin with three types of symbol:

$$(\gamma)_0, (\gamma)_{\frac{\omega}{2}}, (\gamma)_{\frac{\omega-1}{2}}.$$

Set $M_{\frac{\omega}{2}} = \begin{pmatrix} \omega - 1 & 2 \\ 2 & -\omega \end{pmatrix}$. Then $M_{\frac{\omega}{2}}(\frac{\omega}{2}) = \infty$ and $M_{\frac{\omega}{2}}(\infty) = \frac{\omega-1}{2}$.

So we can replace

$$(\gamma)_{\frac{\omega-1}{2}} = \{\gamma(\frac{\omega-1}{2}), \gamma(\infty)\}$$

by

$$-(\gamma M_{\frac{\omega}{2}})_{\frac{\omega}{2}} = \{\gamma M_{\frac{\omega}{2}}(\infty), \gamma M_{\frac{\omega}{2}}(\frac{\omega}{2})\},$$

where appropriate.

Thus, we can reduce the number of generating symbols required to generate $H_1(G\backslash\mathcal{H}_3^*,\mathbf{Q})$ to two. This is summarised in Table (i):

α	$\{\alpha, \infty\}$
0	$(I)_0$
$\frac{\omega-1}{2}$	$-(M_{\frac{\omega}{2}})_{\frac{\omega}{2}}$
$\frac{\omega}{2}$	$(I)_{\frac{\omega}{2}}$

Set $(\gamma)_1 = (\gamma)_0$ and $(\gamma)_2 = (\gamma)_{\frac{\omega}{2}}$.

We determine the 2-term relations by means of Lemma 4.1.4. These are:

(1) $(I)_1 = (J)_1$,

(2) $(I)_1 = -(S)_1$,

(3) $(I)_2 = (K)_2$, where $K = \begin{pmatrix} -1 & \omega \\ 0 & 1 \end{pmatrix}$.

Finally we turn our attention to those relations which arise from the faces of the polyhedra in the tesselation found in §3.2. Recall that there are two polyhedra, P_{v_1} and P_{v_2}, which form the basis for D^+.

Consider P_{v_1}, the triangular prism shown in Figure (3.2.2). It has five faces, two triangular and three square, and so we can obtain at most four independent relations from it as follows:

Take the triangular face with vertices at ∞, $\frac{\omega-1}{2}$ and $\frac{\omega}{2}$. This generates the following relation between paths:

$$\{\infty, \frac{\omega-1}{2}\} + \{\frac{\omega-1}{2}, \frac{\omega}{2}\} + \{\frac{\omega}{2}, \infty\} = 0.$$

Now, $(I)_2 = \{\frac{\omega}{2}, \infty\}$,

$(M_{\frac{\omega}{2}})_2 = \{\infty, \frac{\omega-1}{2}\}$,

and $(M_{\frac{\omega}{2}}^{-1})_2 = \{\frac{\omega-1}{2}, \frac{\omega}{2}\}$.

Set $N_1 = M_{\frac{\omega}{2}}$. Then, in terms of symbols $(\gamma)_1$ and $(\gamma)_2$ we have:

(4) $(I)_2 + (N_1)_2 + (N_1^{-1})_2 = 0$.

Similarly, the triangular face with vertices $0, \frac{2}{1-\omega}$ and $-\frac{2}{\omega}$ gives the relation

$$(S)_2 + (SN_1)_2 + (SN_1^{-1})_2 = 0.$$

But this is what we get if we multiply relation (4) through on the left by S. Thus it is not independent of the previous relations and we do not include it.

The square face with vertices $\infty, 0, \frac{2}{1-\omega}$ and $\frac{\omega}{2}$ gives:

(5) $(S)_1 + (N_1^{-1})_2 + (N_1^{-1})_1 + (I)_1 = 0.$

The square face with vertices $\infty, 0, -\frac{2}{\omega}$ and $\frac{\omega-1}{2}$ gives the relation:

$$(S)_1 - (S)_2 - (SN_1^{-1})_1 - (N_1)_2 = 0.$$

Multiplying through on the left by S gives:

(6) $(I)_1 - (I)_2 - (N_1^{-1})_1 - (SN_1)_2 = 0.$

The relation which arises from the final square face can be deduced from the others.

So we have three independent relations arising from P_{v_1}.

Similarly for P_{v_2}, we find two further independent relations:

(7) $(I)_1 + (TS)_1 + ((TS)^2)_1 = 0,$

(8) $(I)_2 + (N_2)_2 + (N_2^2)_2 = 0, \quad$ where $N_2 = \begin{pmatrix} \omega+1 & 2-\omega \\ 2 & -\omega \end{pmatrix}.$

We cannot eliminate any other types of symbol using these relations, so $H_1(G\backslash\mathcal{H}_3^\star,\mathbf{Q})$ is generated by the two types of symbol, $(\gamma)_1$ and $(\gamma)_2$, and $R(G)$ is generated by relations (1)-(8).

§4.3 The Calculation for $\mathbf{Q}(\sqrt{-43})$

As before, we begin by reducing the set of generating symbols, using Lemma 4.1.3. The results of this reduction are shown in Table (i). To make the notation tidier we set:

$$(\gamma)_1 = (\gamma)_0,$$

$$(\gamma)_2 = (\gamma)_{\frac{\omega}{2}},$$

$$(\gamma)_3 = (\gamma)_{\frac{\omega}{3}},$$

$$(\gamma)_4 = (\gamma)_{\frac{\omega+1}{3}}.$$

Table (i):

α	$\{\alpha,\infty\}$
0	$(I)_1$
$\frac{\omega-1}{2}$	$-(M_{\frac{\omega}{2}})_2$
$\frac{\omega}{2}$	$(I)_2$
$\frac{\omega-1}{3}$	$-(JM_{\frac{\omega}{3}})_3$
$\frac{\omega}{3}$	$(I)_3$
$\frac{\omega+1}{3}$	$(I)_4$
$\frac{2-\omega}{3}$	$(TJ)_4$
$-\frac{\omega}{3}$	$(J)_3$
$\frac{1-\omega}{3}$	$-(M_{\frac{\omega}{3}})_3$

where $M_{\frac{\omega}{2}}$ and $M_{\frac{\omega}{3}}$ are defined in §2.4.

So we now work with the smaller set of generating symbols:

$$\{(\gamma)_1,(\gamma)_2,(\gamma)_3,(\gamma)_4\}.$$

The 2-term relations between these symbols are:

(a) $(I)_1 - (J)_1 = 0,$

(b) $(I)_1 + (S)_1 = 0,$

(c) $(I)_2 - (K)_2 = 0,$

(d) $(I)_4 + (L)_4 = 0, \quad \text{where } L = \begin{pmatrix} \omega+1 & 3-\omega \\ 3 & -\omega-1 \end{pmatrix}.$

From the geometry, we have four 3-term relations:

(e) $(I)_1 + (TS)_1 + ((TS)^2)_1 = 0;$

(f) $(I)_2 + (N_1)_2 + (N_1^2)_2 = 0, \quad \text{where } N_1 = M_{\frac{\omega}{2}};$

(g) $(I)_2 + (N_2)_2 + (N_2^2)_2 = 0, \quad \text{where } N_2 = \begin{pmatrix} \omega & 5-\omega \\ 2 & -\omega-1 \end{pmatrix};$

(h) $(I)_3 + (N_3)_3 - (I)_4 = 0, \quad \text{where } N_3 = \begin{pmatrix} \omega & 4 \\ 3 & 1-\omega \end{pmatrix};$

and two 4-term relations:

(i) $(I)_1 - (I)_3 + (N_4)_2 - (N_4)_3 = 0, \quad \text{where } N_4 = \begin{pmatrix} -3 & \omega \\ \omega-1 & 4 \end{pmatrix};$

(j) $(I)_2 - (I)_4 + (N_5)_2 - (N_5)_4 = 0, \quad \text{where } N_5 = \begin{pmatrix} \omega-4 & \omega+5 \\ \omega+1 & 4-\omega \end{pmatrix}.$

This time we can use these "mixed" relations, ie. relations involving more than one type of symbol, to further reduce the number of generating symbols which we require.

Relation (h) $\Rightarrow (I)_4 = (I)_3 + (N_3)_3$.

So we can replace $(\gamma)_4$ by $(\gamma)_3 + (\gamma N_3)_3$ where appropriate.

Thus (d) becomes:

$$(I)_3 + (N_3)_3 + (L)_3 + (LN_3)_3 = 0;$$

and (j) becomes:

$$(I)_2 - (I)_3 - (N_3)_3 + (N_5)_2 - (N_5)_3 - (N_5N_3)_3 = 0.$$

We can also eliminate the symbol $(\gamma)_1$, using (i). Thus:

$$(I)_1 = (I)_3 - (N_4)_2 + (N_4)_3.$$

Then (a) becomes:

$$(I)_3 - (N_4)_2 + (N_4)_3 - (J)_3 + (JN_4)_2 - (JN_4)_3 = 0,$$

(b) becomes:

$$(I)_3 - (N_4)_2 + (N_4)_3 + (S)_3 - (SN_4)_2 + (SN_4)_3 = 0,$$

and (e) becomes:

$$(I)_3 - (N_4)_2 + (N_4)_3 + (TS)_3 - (TSN_4)_2 + (TSN_4)_3$$

$$+((TS)^2)_3 - ((TS)^2N_4)_2 + ((TS)^2N_4)_3 = 0.$$

So, in $\mathbf{Q}(\sqrt{-43})$, $H_1(G\backslash\mathcal{H}_3^*,\mathbf{Q})$ is generated by two types of symbol, $(\gamma)_2 = (\gamma)_{\frac{\omega}{2}}$ and $(\gamma)_3 = (\gamma)_{\frac{\omega}{3}}$, and these symbols satisfy the relations:

(1) $(I)_3 - (N_4)_2 + (N_4)_3 - (J)_3 + (JN_4)_2 - (JN_4)_3 = 0,$

(2) $(I)_3 - (N_4)_2 + (N_4)_3 + (S)_3 - (SN_4)_2 + (SN_4)_3 = 0,$

(3) $(I)_2 - (K)_2 = 0,$

(4) $(I)_3 + (N_3)_3) + (L)_3 + (LN_3)_3 = 0,$

$$(5)\ (I)_3 - (N_4)_2 + (N_4)_3 + (TS)_3 - (TSN_4)_2 + (TSN_4)_3$$

$$+((TS)^2)_3 - ((TS)^2N_4)_2 + ((TS)^2N_4)_3 = 0,$$

$$(6)\ (I)_2 + (M_1)_2 + (N_1^2)_2 = 0,$$

$$(7)\ (I)_2 + (N_2)_2 + (N)_2^2)_2 = 0,$$

$$(8)\ (I)_2 - (I)_3 - (N_3)_3 + (N_5)_2 - (N_5)_3 - (N_5N_3)_3 = 0.$$

§4.4 The Calculations for $\mathbf{Q}(\sqrt{67})$

The results of the initial reduction in types, using Lemma 4.1.3, are given in Table (i).

Set $(\gamma)_1 = (\gamma)_0$, $(\gamma)_2 = (\gamma)_{\frac{\omega}{2}}$, $(\gamma)_3 = (\gamma)_{\frac{\omega}{3}}$, $(\gamma)_4 = (\gamma)_{\frac{\omega}{4}}$, $(\gamma)_5 = (\gamma)_{\frac{\omega+6}{3-\omega}}$, $(\gamma)_6 = (\gamma)_{\frac{\omega-7}{\omega+2}}$, $(\gamma)_7 = (\gamma)_{\frac{\omega+1}{3}}$ and $(\gamma)_8 = (\gamma)_{\frac{\omega+1}{4}}$.

Table (i):

α	$\{\alpha,\infty\}$
0	$(I)_1$
$\frac{\omega-1}{3}$	$-(M_{\frac{\omega}{2}})_2$
$\frac{\omega}{2}$	$(I)_2$
$\frac{\omega+6}{3-\omega}$	$(M)_5$
$\frac{\omega+2}{3-\omega}$	$-(M_{\frac{\omega+6}{3-\omega}})_5$
$\frac{-2-\omega}{3-\omega}$	$-(JM_{\frac{-6-\omega}{3-\omega}})_5$
$\frac{-6-\omega}{3-\omega}$	$(J)_5$
$\frac{\omega-7}{\omega+2}$	$(I)_6$
$\frac{\omega-3}{\omega+2}$	$-(M_{\frac{\omega-7}{\omega+2}})_6$
$\frac{7-\omega}{\omega+2}$	$(J)_6$
$\frac{3-\omega}{\omega+2}$	$-(JM_{\frac{7-\omega}{\omega+2}})_6$
$\frac{\omega-1}{3}$	$-(JM_{-\frac{\omega}{3}})_3$
$\frac{\omega}{3}$	$(I)_3$
$\frac{\omega+1}{3}$	$(I)_7$
$-\frac{\omega}{3}$	$(J)_3$
$\frac{1-\omega}{3}$	$-(M_{\frac{\omega}{3}})_3$
$\frac{2-\omega}{3}$	$(TJ)_7$

α	$\{\alpha,\infty\}$
$\frac{\omega-1}{4}$	$-(M_{\frac{\omega}{4}})_4$
$\frac{\omega}{4}$	$(I)_4$
$\frac{\omega+1}{4}$	$(I)_8$
$\frac{1-\omega}{4}$	$-(JM_{-\frac{\omega}{4}})_4$
$\frac{2-\omega}{4}$	$-(M_{\frac{\omega+1}{4}})_8$
$\frac{-1-\omega}{4}$	$(J)_8$
$\frac{\omega-2}{4}$	$-(JM_{\frac{-1-\omega}{4}})_8$
$-\frac{\omega}{4}$	$(J)_4$

The independent relations between these 8 types of symbol are:

$$(I)_1 - (J)_1 = 0;$$

$$(I)_1 + (S)_1 = 0;$$

$$(I)_2 - (K)_2 = 0;$$

$$(I)_7 + (N_1)_7 = 0, \quad N_1 = \begin{pmatrix} \omega + 1 & 5 - \omega \\ 3 & -\omega - 1 \end{pmatrix};$$

$$(I)_3 + (N_2)_3 - (I)_7 = 0, \quad N_2 = \begin{pmatrix} \omega & 6 \\ 3 & 1 - \omega \end{pmatrix};$$

$$(I)_2 + (N_3)_2 + (N_3^{-1})_2 = 0, \quad N_3 = \begin{pmatrix} \omega & 8 - \omega \\ 2 & -\omega - 1 \end{pmatrix};$$

$$(I)_3 + (N_2)_5 - (I)_8 = 0;$$

$$(I)_8 + (N_4)_8 - (I)_4 = 0, \quad N_4 = \begin{pmatrix} \omega + 1 & 4 - \omega \\ 4 & -\omega - 2 \end{pmatrix};$$

$$(I)_6 - (I)_3 + (N_5)_8 = 0, \quad N_5 = \begin{pmatrix} \omega - 7 & \omega + 6 \\ \omega + 2 & 3 - \omega \end{pmatrix};$$

$$(I)_4 + (N_6)_4 + (N_6^{-1})_4 = 0, \quad N_6 = \begin{pmatrix} \omega & 4 \\ 4 & 1 - \omega \end{pmatrix};$$

$$(I)_2 + (N_7)_2 + (N_7^{-1})_2 = 0, \quad N_7 = \begin{pmatrix} \omega & 8 \\ 2 & 1 - \omega \end{pmatrix};$$

$$(I)_6 - (I)_5 + (N_5)_1 = 0;$$

$$(I)_2 - (I)_6 + (N_7)_5 = 0;$$

$$(I)_1 + (TS)_1 + ((TS)^2)_1 = 0;$$

$$(I)_3 - (I)_2 + (N_8)_3 - (N_8)_4 = 0, \quad N_8 = \begin{pmatrix} 7 & -2\omega \\ 1 - \omega & -5 \end{pmatrix};$$

$$(I)_7 - (N_9)_8 + (N_9)_1 - (I)_8 = 0, \quad N_9 = \begin{pmatrix} 5 & -\omega - 1 \\ 2 - \omega & -4 \end{pmatrix};$$

$$(I)_1 - (I)_4 + (N_{10})_1 - (N_{10})_4 = 0, \quad N_{10} = \begin{pmatrix} 4 & -\omega \\ 1 - \omega & -4 \end{pmatrix}.$$

By means of the mixed relations, we can reduce the generating set of symbols to:

$$\{(\gamma)_3 = (\gamma)_{\frac{\omega}{3}}, (\gamma)_5 = (\gamma)_{\frac{\omega+6}{3-\omega}}\},$$

where the complete set of independent relations between these types is:

(1) $(I)_3 + (N_2)_3 + (N_1)_3 + (N_1N_2)_3 = 0;$

(2) $(I)_3 - (N_5^{-1})_3 + (N_2)_5 + (N_5^{-1})_5$

$+(S)_3 - (SN_2^{-1})_5 + (SN_2)_5 + (SN_5^{-1})_5 = 0;$

(3) $(I)_3 - (N_5)_3 - (N_5N_2)_5 - (N_7)_5$

$-(K)_3 + (KN_5)_3 + (KN_5N_2)_5 + (KN_7)_5 = 0;$

(4) $(I)_3 - (N_5^{-1})_3 + (N_2)_5 + (N_5^{-1})_5$

$-(J)_3 + (JN_5^{-1})_3 - (JN_2)_5 - (JN_5^{-1})_5 = 0;$

(5) $(I)_3 + (N_2)_5 + (N_4)_3 + (N_4N_2)_5$

$+(N_6)_3 + (N_6N_2)_5(N_6N_4)_3 + (N_6N_4N_2)_5$

$+(N_6^{-1})_3 + (N_6^{-1}N_2)_5 + (N_6^{-1}N_4)_3 + (N_6^{-1}N_2)_5 = 0;$

(6) $(I)_3 - (N_5)_3 - (N_5N_2)_5 - (N_7)_5$

$+(N_3)_3 - (N_3N_5)_3 - (N_3N_5N_2)_5 - (N_3N_7)_5$

$+(N_3^{-1})_3 - (N_3^{-1}N_5)_3 - (N_3^{-1}N_5N_2)_5 - (N_3^{-1}N_7)_5 = 0;$

(7) $(I)_3 - (N_5^{-1})_3 + (N_2)_5 + (N_5^{-1})_5$

$+(TS)_3 - (TSN_5^{-1})_3 + (TSN_2)_5 + (TSN_5^{-1})_5$

$+((TS)^2)_3 - ((TS)^2N_5^{-1})_3 + ((TS)^2N_2)_5 + ((TS)^2N_5^{-1})_5 = 0.$

§4.6 The Calculations for $\mathbf{Q}(\sqrt{-163})$

We use Lemma 4.1.3 to reduce the initial number of types from 99 to 27. These 27 types are shown in Table (i):

$(\gamma)_1$	$(\gamma)_0$
$(\gamma)_2$	$(\gamma)_{\frac{\omega}{2}}$
$(\gamma)_3$	$(\gamma)_{\frac{\omega}{3}}$
$(\gamma)_4$	$(\gamma)_{\frac{\omega+1}{3}}$
$(\gamma)_5$	$(\gamma)_{\frac{\omega}{4}}$
$(\gamma)_6$	$(\gamma)_{\frac{\omega+1}{4}}$
$(\gamma)_7$	$(\gamma)_{\frac{\omega}{5}}$
$(\gamma)_8$	$(\gamma)_{\frac{\omega+1}{5}}$
$(\gamma)_9$	$(\gamma)_{\frac{\omega+2}{5}}$
$(\gamma)_{10}$	$(\gamma)_{\frac{2\omega}{5}}$
$(\gamma)_{11}$	$(\gamma)_{\frac{3-2\omega}{5}}$
$(\gamma)_{12}$	$(\gamma)_{\frac{\omega}{6}}$
$(\gamma)_{13}$	$(\gamma)_{\frac{\omega+1}{6}}$
$(\gamma)_{14}$	$(\gamma)_{\frac{\omega+2}{6}}$

$(\gamma)_{15}$	$(\gamma)_{\frac{12}{1-\omega}}$
$(\gamma)_{16}$	$(\gamma)_{\frac{-17}{\omega}}$
$(\gamma)_{17}$	$(\gamma)_{\frac{12}{2-\omega}}$
$(\gamma)_{18}$	$(\gamma)_{\frac{\omega-17}{\omega+1}}$
$(\gamma)_{19}$	$(\gamma)_{\frac{\omega+17}{3-\omega}}$
$(\gamma)_{20}$	$(\gamma)_{\frac{\omega+15}{3-\omega}}$
$(\gamma)_{21}$	$(\gamma)_{\frac{\omega-16}{\omega+2}}$
$(\gamma)_{22}$	$(\gamma)_{\frac{\omega-18}{\omega+2}}$
$(\gamma)_{23}$	$(\gamma)_{\frac{\omega+3}{7}}$
$(\gamma)_{24}$	$(\gamma)_{\frac{2\omega+1}{7}}$
$(\gamma)_{25}$	$(\gamma)_{\frac{3\omega+2}{7}}$
$(\gamma)_{26}$	$(\gamma)_{\frac{\omega+16}{4-\omega}}$
$(\gamma)_{27}$	$(\gamma)_{\frac{\omega-5}{\omega+3}}$

There are five 2-term relations, thirty-eight 3-term relations and twelve 4-term relations between these types. Using these relations, we can reduce the number of types needed to generate homology to six. These types are:

$$(\gamma)_{12}, (\gamma)_{14}, (\gamma)_{18}, (\gamma)_{21}, (\gamma)_{24}, (\gamma)_{27}.$$

Between these types there are: one 4-term, one 6-term, two 12-term, one 16-term, three 20-term, two 24-term, one 26-term, one 28-term, one 32-term, one 40-term and one 48-term independent relations.

§4.6 M-Symbols and Hecke Operators

As in the Euclidean case, we are calculating homology in terms of right coset representatives for $\Gamma_0(\mathbf{a})$ in Γ. Recall the definition of M-symbols in §1.4. There is a one-to-one correspondence between right coset representatives of $\Gamma_0(\mathbf{a})$ in Γ and $P^1(\vartheta_K/\mathbf{a})$ given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow (c:d),$$

and the action of Γ on $P^1(\vartheta_K/\mathbf{a})$ is given by:

$$(c:d)\begin{pmatrix} p & q \\ r & s \end{pmatrix} = (cp+dr : cq+ds).$$

The main difference between the Euclidean and non-Euclidean cases is that the symbol $(c : d)$ no longer corresponds to just one symbol (γ), where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have seen in the previous sections that, generally, $H_1(G\backslash\mathcal{H}_3^*,\mathbf{Q})$ is generated by more than one type of symbol $(\gamma)_i$. Therefore any M-symbol $(c:d)$ corresponds to more than one type $(\gamma)_i = \{\gamma(\alpha_i),\gamma(\infty)\}$, where $(\gamma)_i$ is one of our generating types. So we represent the symbol $(\gamma)_i$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, by $(c:d)_i$.

Then, in terms of M-symbols, the boundary map (4.1.2) is:

$$\delta : (c:d)_i \mapsto [\frac{a}{c}] - [\frac{a\lambda_i + b\mu_i}{c\lambda_i + d\mu_i}], \tag{4.6.1}$$

where $\alpha_i = \frac{\lambda_i}{\mu_i}$, and the main involution J is:

$$(c:d)_i \mapsto (\varepsilon c : d)_i. \tag{4.6.2}$$

Once again, we will want to convert M-symbols to modular symbols and vice versa. To do this we will make use of the pseudo-Euclidean algorithm defined in §2.7.

To convert from M-symbols to modular symbols we use the isomorphism (4.1.3):

$$\xi : (\gamma)_i \mapsto \{\gamma(\alpha_i), \gamma(\infty)\}.$$

Thus, the symbol $(c:d)_i$ corresponds to $(\gamma)_i$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, ie

$$(c:d)_i \mapsto \{\gamma(\alpha_i), \gamma(\infty)\} = \{\frac{a\lambda_i + b\mu_i}{c\lambda_i + d\mu_i}, \frac{a}{c}\}. \tag{4.6.3}$$

To convert the modular symbol $\{\frac{\lambda}{\mu}, 0\}$ to M-symbols we use the continued fraction convergents to $\frac{\lambda}{\mu}$ as described in §2.7.

Recall that at each stage of the pseudo-Euclidean algorithm we invert in a hemisphere with centre α_k, some $\alpha_k \in A$. So, not only do we write down the continued fraction convergents:

$$\frac{\lambda}{\mu} = \frac{x_n}{y_n}, \ldots, \frac{x_0}{y_0} = \frac{1}{0}, \frac{0}{1},$$

we also write down the centres of the hemispheres which we inverted in at each stage:

$$\alpha_n, \ldots, \alpha_0,$$

where $\alpha_k \in A$, and the series of matrices:

$$M_n, \ldots, M_0,$$

defined in §2.7.

Then

$$\{0, \frac{\lambda}{\mu}\} = \sum_{k=0}^{n} (y_k : y_k{}')_{\alpha_k}, \tag{4.6.4}$$

where $(y_k : y_k{}')_{\alpha_k}$ corresponds to the symbol $\{M_k(\alpha_k), M_k(\infty)\}$, where $M_k = \begin{pmatrix} x_k & x_k{}' \\ y_k & y_k{}' \end{pmatrix}$.

Finally, recall that we are generating homology by a reduced set of symbols $(\gamma)_i, 1 \leq i \leq s$. So we must express each M-symbol $(c : d)_j$ on the right hand side of (4.6.4) in terms of these generating types. We do this using (4.1.4). So, if $(\gamma)_j = \sum_i c(i)(M_i\gamma)_i$, then we can write

$$(c : d)_j = \sum_i c(i)(cp + dr : cq + ds)_i, \tag{4.6.5}$$

where $M_i = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

We now turn our attention to the Hecke algebra acting on $V(\mathbf{a}) = H(\Gamma_0(\mathbf{a}))$. We define the Hecke operator, T_π, for prime ideals $\wp = (\pi)$ which do not divide $\mathbf{a}$, and the W-involution, W_π, for prime ideals $\wp = (\pi)$ which divide $\mathbf{a}$, as in the Euclidean case. The action of these operators on cusps is:

$$T_\pi : [\beta] \mapsto \sum_{x \bmod \pi} [\frac{\beta + x}{\pi}] + [\pi\beta],$$

and

$$W_\pi : [\beta] \mapsto [\frac{\pi^r x\beta + y}{\alpha z\beta + \pi^r w}],$$

where $\mathbf{a} = (\alpha), r$ is the highest power of π dividing α and x, y, z, w are chosen so that $\pi^{2r}xw - \alpha zy = \pi^r$.

To compute the action of the Hecke operators and the W-involution on M-symbols we convert to modular symbols using (4.6.3), use the actions defined in (0.6.4) and (0.6.5) respectively, and then convert back to M-symbols using (4.6.4).

Chapter 5: Results

For each of the 4 non-Euclidean class number 1 fields, computer programs have been written in Algol68 to carry out the calculations discussed in the previous chapters. These programs have been run on an ICL3980 computer at the South West Universities Regional Computing Centre. The results of these are given here.

§5.1 Introduction to the Tables

In each field K, for each non-zero ideal $\mathbf{a}$ in ϑ_K, with $N(\mathbf{a})$ less than some bound, we have calculated the dimensions of $V^+(\mathbf{a})$; the action of the J and W_π involutions, for primes π which divide $\mathbf{a}$; the action of the Hecke operators T_π, for primes π which do not divide $\mathbf{a}$; the splitting of $V^+(\mathbf{a})$ into one-dimensional spaces which are eigenspaces for all these operators, using a method described by Cremona [5] in the rational case; and the eigenvalues of each such subspace.

We have also made a search for elliptic curves with small conductor over each of the four fields. We did this by implementing Tate's algorithm [19] for determining the type of reduction of a curve E at prime $\wp$, given the coefficients a_1, a_2, a_3, a_4 and a_6 of (0.1.1). From [19], it can be seen that we may assume that a_1, a_2 and a_3 are reduced modulo 2, 3 and 2, respectively, and so the search consists of a systematic search through the pairs (a_4, a_6), with a_4 and a_6 within certain bounds, with all values of $a_1, a_3 (\bmod 2)$ and $a_2 (\bmod 3)$.

It should be noted that this search for curves is not an exhaustive one. Curves almost certainly exist with small conductor but with coefficients outside our search region. In fact there are ideals $\mathbf{a}$ in ϑ_K for which a rational newform in $V^+(\mathbf{a})$ exists but for which no elliptic curves with conductor $\mathbf{a}$ have been found. In view of this incompleteness, we will give a list of "missing conductors", ie ideals $\mathbf{a}$ in ϑ_K for which a rational newform exists but no corresponding curve has been found.

The limit for $N(\mathbf{a})$ in $K = \mathbf{Q}(\sqrt{-d})$, determined by the computer time and space available, is as follows:

d	Bound
19	500
43	230
67	265
163	100

It should be noticed at this point that, in the case $K = \mathbf{Q}(\sqrt{-163})$, only two ideals $\mathbf{a}$ of ϑ_K were found for which $\dim V^+(\mathbf{a}) > 0$. These ideals were:

(i) $\mathbf{a} = (7)$ where $\dim V^+(\mathbf{a}) = 2$ and

(ii) $\mathbf{a} = (9)$ where $\dim V^+(\mathbf{a}) = 52$.

In neither case was a rational newform found and, similarly, no elliptic curves with sufficiently small conductors were located. Therefore, there will be no tables for $\mathbf{Q}(\sqrt{-163})$.

For each of the three remaining fields we give the following tables of results:

(i) The dimension of $V^+(\mathrm{a})$ and the number of rational newforms in $V^{\mathrm{new}}(\mathbf{a})$. Levels, $\mathbf{a}$, with dim $V^+(\mathbf{a}) = 0$ have been omitted. Also, only one ideal for each conjugate pair $(\mathbf{a}, \overline{\mathbf{a}})$ is given, as conjugation induces an isomorphism from $V^+(\mathbf{a})$ to $V^+(\overline{\mathbf{a}})$.

(ii) For each newform in $V^+(\mathbf{a})$ we give the Hecke eigenvalues, and the eigenvalues of the W_π involutions, for the first 15 primes in K. The eigenvalues for the W_π involutions are either $+1$ or -1 and we shall denote them by $+$ and $-$ respectively.

(iii) A list of elliptic curves defined over K, with conductor $\mathbf{a}$; we give the conductor and coefficients of each curve.

(iv) For each curve in (iii) we give the trace of Frobenius for the first 15 primes in K.

In all cases we find that the eigenvalues of the T_π and W_π operators acting on a newform and the trace of Frobenius of the corresponding elliptic curve agree at the first 15 primes, according to Conjecture 0.7.1.

§5.2 Results for $\mathbf{Q}(\sqrt{-19})$

We give the following tables:

Table (5.2.1): Ideals $\mathbf{a}$ of $\mathbf{Z}[\omega]$ with $N(\mathbf{a}) < 500$ and $\dim V^+(\mathbf{a}) > 0$.

Table (5.2.2): Rational newforms in $V^+(\mathbf{a})$.

Table (5.2.3): Elliptic curves with conductor $\mathbf{a}$, $N(\mathbf{a}) < 500$.

Table (5.2.4): The trace of Frobenius for elliptic curves.

Table (5.2.1): Ideals a of $\mathbf{Z}[(\frac{\sqrt{-19}+1}{2})]$ with $N(\mathrm{a}) < 500$ and $\dim V^{+}(\mathrm{a}) > 0$

a	$N(\mathrm{a})$	$\dim V^{+}(\mathrm{a})$	$\dim V^{new}(\mathrm{a})$
$(1-2\omega)$	19	1	1
(2ω)	20	1	1
$(4+2\omega)$	44	1	1
$(2-4\omega)$	76	4	2
$(1-4\omega)$	77	2	0
(4ω)	80	2	0
(9)	81	4	0
$(5+3\omega)$	85	1	1
$(9+\omega)$	95	2	0
(10)	100	4	0
$(8+2\omega)$	100	3	1
(11)	121	3	1
$(8+3\omega)$	133	2	0
$(6+4\omega)$	140	2	0
$(10+2\omega)$	140	2	0
(12)	144	2	0
$(3-6\omega)$	171	5	3
$(1-6\omega)$	175	2	2
$(5+5\omega)$	175	1	1
$(8+4\omega)$	176	2	0
(6ω)	180	2	0
(14)	196	1	1
$(2+6\omega)$	196	1	1
$(3+6\omega)$	207	1	1
$(7+5\omega)$	209	2	0
$(4+6\omega)$	220	5	1
$(10+4\omega)$	220	5	1

Table (5.2.1) (continued)

a	$N(\mathbf{a})$	dim $V^+(\mathbf{a})$	dim $V^{new}(\mathbf{a})$
(15)	225	3	1
$(11+4\omega)$	245	3	1
(16)	256	6	2
$(3+7\omega)$	275	2	2
$(10+5\omega)$	275	2	2
(17)	289	1	1
$(4+7\omega)$	289	4	0
$(4-8\omega)$	304	12	1
$(6-8\omega)$	308	7	1
$(16+2\omega)$	308	3	1
$(15+3\omega)$	315	1	1
(8ω)	320	5	2
$(6+7\omega)$	323	2	0
(18)	324	18	0
$(2+8\omega)$	340	2	0
$(10+6\omega)$	340	5	1
$(7+7\omega)$	343	6	1
(19)	361	11	0
$(4+8\omega)$	368	1	1
$(18+2\omega)$	380	11	1
$(5+8\omega)$	385	1	1
$(17+3\omega)$	385	5	1
$(19+\omega)$	385	5	1
$(12+6\omega)$	396	2	0
$(16+4\omega)$	400	6	0
(20)	400	9	1

Table (5.2.1) (concluded)

$\mathbf{a}$	$N(\mathbf{a})$	dim $V^+(\mathbf{a})$	dim $V^{new}(\mathbf{a})$
(9ω)	405	8	0
$(7+8\omega)$	425	3	1
$(15+5\omega)$	425	6	2
$(17+4\omega)$	437	2	0
$(3+9\omega)$	441	2	0
(21)	441	3	3
$(14+6\omega)$	460	4	2
$(20+2\omega)$	460	4	0
$(5+9\omega)$	475	4	1
$(5-10\omega)$	475	11	0
$(18+4\omega)$	476	2	2
$(2-10\omega)$	484	3	1
(22)	484	10	0
(10ω)	500	10	1
$(10+8\omega)$	500	5	0

Table (5.2.2): **Rational Newforms in $V^+(\mathrm{a})$**

We give the eigenvalues of the T_π and W_π operators for the first 15 primes in $\mathbf{Q}(\sqrt{-19})$. These are: $\{2, \omega, 1-\omega, 2-\omega, 1+\omega, 3, 3-\omega, 2+\omega, 4-\omega, 3+\omega, 1-2\omega, 3-2\omega, 1+2\omega, 2-3\omega, 1-3\omega\}$.

a	Eigenvalues														
$(1-2\omega)$	−4	3	3	−1	−1	−2	3	3	−3	−3	−	0	0	−1	−1
(2ω)	−	+	0	−1	−1	1	6	0	−3	3	2	−3	−3	8	−10
$(4+2\omega)$	+	0	−2	−5	1	1	2	+	−5	1	−4	−1	−5	8	−4
$(2-4\omega)a$	−	0	0	−1	−1	−5	−6	−6	3	3	−	3	3	8	8
$(2-4\omega)b$	−	−4	−4	3	3	−5	2	2	3	3	+	−1	−1	4	4
$(5+3\omega)$	3	+	2	0	2	2	0	−6	−	−6	6	−2	−4	2	6
$(8+2\omega)$	+	0	+	−1	1	−1	0	−6	−3	−3	2	−3	3	−10	−8
(11)	0	1	1	−2	−2	−5	−	−	−2	−2	0	−1	−1	−6	−6
$(3-6\omega)a$	0	1	1	3	3	−	−3	−3	3	3	+	4	4	−1	−1
$(3-6\omega)b$	0	−3	−3	−5	−5	−	1	1	−1	−1	+	−4	−4	−1	−1
$(3-6\omega)c$	−3	−2	−2	0	0	−	0	0	−6	−6	+	4	4	−4	−4
$(1-6\omega)a$	3	1	−	0	+	0	2	−1	7	5	−3	−8	−2	−1	9
$(1-6\omega)b$	−3	1	−	0	−	0	−2	1	−7	5	−3	−8	2	−1	−9
$(5+5\omega)$	−1	−	−	4	−	−2	−4	4	−6	2	4	4	0	8	4
(14)	−	0	0	−	−	−2	0	0	6	6	2	0	0	8	8
$(2+6\omega)$	+	−2	2	−	3	3	2	2	1	−1	0	5	5	−10	−10
$(3+6\omega)$	0	1	−1	−3	−1	+	1	3	−3	7	−8	4	+	7	−11
$(4+6\omega)$	+	−2	+	−2	1	1	+	−1	−2	4	−1	4	8	2	−4
$(10+4\omega)$	+	−	−4	1	−1	1	−	6	−7	−5	−8	3	−1	8	−8
(15)	−3	−	−	0	0	−	−4	−4	2	2	4	0	0	4	4
$(11+4\omega)$	−2	−1	+	−4	−	−5	−5	−3	1	2	5	−3	−4	1	−7
$(16)a$	−	2	2	−3	3	3	2	−2	1	1	0	−5	5	−10	10
$(16)b$	−	2	2	3	−3	3	−2	2	1	1	0	5	−5	10	−10

Table (5.2.2) (continued)

a	Eigenvalues													
$(3+7\omega)a$	1	2	−	−4	−2	4	−6	−	−2	−1	−5	−1	6	−8	−4
$(3+7\omega)b$	−1	2	+	−4	2	−4	6	+	2	−1	−5	−1	−6	−8	4
$(10+5\omega)a$	3	−	−	4	0	−2	4	−	−6	2	−4	−4	−8	0	−4
$(10+5\omega)b$	−3	−	+	1	0	1	−5	−	0	8	−1	−4	−8	0	−4
(17)	−3	−2	−2	4	4	−6	0	0	−	−	−4	4	4	4	4
$(4-8\omega)$	+	−1	−1	−3	−3	−2	5	5	−3	−3	+	8	8	1	1
$(6-8\omega)$	+	1	−4	3	−	0	−3	−	−7	3	0	4	−6	4	4
$(16+2\omega)$	−	0	−3	−1	−	4	+	−3	6	0	−7	−9	−3	8	−1
$(15+3\omega)$	1	−	−2	+	0	−	0	0	2	−6	−4	−4	4	−12	−4
$(8\omega)a$	−	+	−2	2	4	2	−2	−2	6	2	0	6	4	−2	−8
$(8\omega)b$	−	+	4	−1	−5	5	−2	4	−3	−1	6	−3	1	4	−2
$(10+6\omega)$	−	+	0	4	4	−4	−4	0	+	8	2	2	−8	8	10
$(7+7\omega)$	2	−1	−2	−	−	−1	−1	5	−7	−2	−3	−3	0	1	1
$(4+8\omega)$	+	3	1	−1	5	2	−3	3	−1	−3	4	0	−	−11	3
$(18+2\omega)$	−	0	−	2	−4	4	3	0	3	6	−	9	−3	5	−4
$(5+8\omega)$	−1	−	3	+	2	−2	−5	−	−1	−7	1	−6	−4	1	−11
$(17+3\omega)$	2	0	+	2	−	−1	6	+	8	−3	−4	5	4	−8	13
$(19+\omega)$	−1	0	−	−	−1	1	+	−3	0	0	−7	6	6	−4	−4
(20)	+	+	+	2	2	−2	0	0	−6	−6	−4	6	6	−10	−10
$(7+8\omega)$	−3	2	+	2	0	−2	6	0	6	−	6	−4	2	6	−2
$(15+5\omega)a$	1	−	−	0	0	2	4	4	−6	−	−4	8	0	4	4
$(15+5\omega)b$	−1	+	+	2	−4	−2	−6	0	6	−	2	0	−6	−10	−10
$(21)a$	3	0	2	+	−	+	4	−6	6	0	−4	2	8	4	8
$(21)b$	3	2	0	−	+	+	−6	4	0	6	−4	8	2	8	4
$(21)c$	−3	−2	−2	+	+	−	4	4	−6	−6	4	0	0	−4	−4
$(14+6\omega)a$	+	1	−	−1	2	2	−3	−3	−3	2	−8	−	−3	0	−8
$(14+6\omega)b$	+	3	+	−1	−2	−2	1	−3	−1	−2	6	+	3	4	−4

Table (5.2.2) (concluded)

a	Eigenvalues														
$(5+9\omega)$	4	+	3	1	−1	2	−3	−3	−3	3	−	0	0	1	−1
$(18+4\omega)a$	−	0	0	−1	−	−2	3	−3	+	−6	2	−6	6	−10	−1
$(18+4\omega)b$	−	−2	0	−3	+	4	−3	3	−	−2	0	0	0	−6	−3
$(2-10\omega)$	−	−2	−2	−3	3	−3	2	−	−1	1	0	5	5	−10	10
(10ω)	+	+	+	1	−1	−1	0	−6	3	3	2	3	−3	10	8

Table (5.2.3): **Elliptic curves over Q($\sqrt{-19}$) with conductor a, $N(\mathrm{a}) < 500$**

a	a_1	a_2	a_3	a_4	a_6
$(1-2\omega)$	0	1	1	1	0
(2ω)	ω	$1-\omega$	1	-1	0
$(4+2\omega)$	1	0	$-1+\omega$	ω	0
$(2-4\omega)a$	1	0	1	-16	22
$(2-4\omega)b$	1	1	1	0	1
$5+3\omega)$	1	$-\omega$	0	-1	0
$(8+2\omega)$	1	$-\omega$	0	$-3+\omega$	ω
(11)	0	-1	1	0	0
$(3-6\omega)a$	0	1	1	20	-32
$(3-6\omega)b$	0	-1	1	-2	2
$(3-6\omega)c$	1	0	1	-7	5
$(1-6\omega)a$	$-1+\omega$	$-2+\omega$	$-1+\omega$	$3+4\omega$	$-11-7\omega$
$(1-6\omega)b$	ω	$-1-\omega$	1	0	0
$(5+5\omega)$	ω	0	ω	$3-\omega$	1
(14)	1	0	1	-1	0
$(2+6\omega)$	$-1+\omega$	$-2+\omega$	$-1+\omega$	$4-\omega$	2ω
$(3+6\omega)$	0	ω	1	$-2+\omega$	-1
$(4+6\omega)$	1	$1-\omega$	0	-1	1
$(10+4\omega)$	$-1+\omega$	ω	ω	5	0
(15)	1	1	1	0	0
$(11+4\omega)$	0	$-1-\omega$	$-1+\omega$	$-6+\omega$	$8+2\omega$
$(16)a$	0	0	0	1	$-2+4\omega$
$(16)b$	0	0	0	1	$2-4\omega$
$(3+7\omega)a$	ω	$-2+\omega$	$-1+\omega$	$4-\omega$	ω
$(3+7\omega)b$	1	$-2+\omega$	0	$-1-\omega$	1
$(10+5\omega)$	1	$1-\omega$	1	$-5+2\omega$	$2-\omega$
(17)	1	-1	1	-1	0

Table (5.2.3) (concluded)

a	a_1	a_2	a_3	a_4	a_6
$(4-8\omega)$	0	-1	0	-21	-31
$(6-8\omega)$	ω	$-1-\omega$	$-1+\omega$	ω	ω
$(16+2\omega)$	ω	0	1	0	0
$(15+3\omega)$	ω	$-\omega$	ω	$3+\omega$	$1-\omega$
$(8\omega)a$	0	$-1+\omega$	0	-1	0
$(7+7\omega)$	0	$-1+\omega$	$-1+\omega$	-1	0
$(4+8\omega)$	0	$1-\omega$	0	$-4-3\omega$	$-7+4\omega$
$(18+2\omega)$	$-1+\omega$	ω	1	0	-1
$(5+8\omega)$	ω	$1-\omega$	$-1+\omega$	$5+\omega$	$4-2\omega$
$(19+\omega)$	ω	$-2+\omega$	0	$3-2\omega$	$-2+2\omega$
(20)	0	1	0	-1	0
$(7+8\omega)$	ω	$-\omega$	ω	$1-\omega$	2
$(15+5\omega)a$	$-1+\omega$	1	$-1+\omega$	0	0
$(15+5\omega)b$	$-1+\omega$	ω	$-1+\omega$	$1+3\omega$	$2+2\omega$
$(21)a$	ω	$-1+\omega$	ω	$2-2\omega$	3
$(21)b$	$-1+\omega$	$-\omega$	$-1+\omega$	2ω	3
$(21)c$	1	0	0	1	0
$(14+6\omega)a$	$-1+\omega$	$-\omega$	$-1+\omega$	$2+3\omega$	0
$(14+6\omega)b$	1	0	ω	1	2
$(5+9\omega)$	0	$-\omega$	1	$-5+\omega$	$1+\omega$
$(18+4\omega)a$	$-1+\omega$	0	1	0	0
$(18+4\omega)b$	1	1	1	$1+\omega$	2
$(2-10\omega)$	$-1+\omega$	$-2+\omega$	0	$2-\omega$	$-4-\omega$
(10ω)	ω	-1	1	2	$-2-\omega$

Missing Conductors: $(10+5\omega)b, (8\omega)b, (10+6\omega), (17+3\omega)$

Table (5.2.4): Trace of Frobenius for Elliptic Curves over $\mathbf{Q}(\sqrt{-19})$

For each curve in Table (5.2.3) we give the trace of Frobenius for the first 15 primes in $\mathbf{Q}(\sqrt{-19})$. These are: $\{2, \omega, 1-\omega, 2-\omega, 1+\omega, 3, 3-\omega, 2+\omega, 4-\omega, 3+\omega, 1-2\omega, 3-2\omega, 1+2\omega, 2-3\omega, 1-3\omega\}$.

a	Trace of Frobenius														
$(1-2\omega)$	−4	3	3	−1	−1	−2	3	3	−3	−3	1	0	0	−1	−1
(2ω)	1	−1	0	−1	−1	1	6	0	−3	3	2	−3	−3	8	−10
$(4+2\omega)$	−1	0	−2	−5	1	1	2	−1	−5	1	−4	−1	−5	8	−4
$(2-4\omega)a$	1	0	0	−1	−1	−5	−6	−6	3	3	1	3	3	8	8
$(2-4\omega)b$	1	−4	−4	3	3	−5	2	2	3	3	−1	−1	−1	4	4
$(5+3\omega)$	3	−1	2	0	2	2	0	−6	1	−6	6	−2	−4	2	6
$(8+2\omega)$	−1	0	0	−1	1	−1	0	−6	−3	−3	2	−3	3	−10	−8
(11)	0	1	1	−2	−2	−5	1	1	−2	−2	0	−1	−1	−6	−6
$(3-6\omega)a$	0	1	1	3	3	1	−3	−3	3	3	−1	4	4	−1	−1
$(3-6\omega)b$	0	−3	−3	−5	−5	1	1	1	−1	−1	−1	−4	−4	−1	−1
$(3-6\omega)c$	−3	−2	−2	0	0	1	0	0	−6	−6	−1	4	4	−4	−4
$(1-6\omega)a$	3	1	0	0	−1	0	2	−1	7	5	−3	−8	−2	−1	9
$(1-6\omega)b$	−3	1	0	0	1	0	−2	1	−7	5	−3	−8	2	−1	−9
$(5+5\omega)$	−1	1	1	4	1	−2	−4	4	−6	2	4	4	0	8	4
(14)	1	0	0	1	1	−2	0	0	6	6	2	0	0	8	8
$(2+6\omega)$	−1	−2	2	0	3	3	2	2	1	−1	0	5	5	−10	−10
$(3+6\omega)$	0	1	−1	−3	−1	−1	1	3	−3	7	−8	4	−1	7	−11
$(4+6\omega)$	−1	−2	−1	−2	1	1	−1	−1	−2	4	−1	4	8	2	−4
$(10+4\omega)$	−1	1	−4	1	−1	1	1	6	−7	−5	−8	3	−1	8	−8
(15)	−3	1	1	0	0	1	−4	−4	2	2	4	0	0	4	4
$(11+4\omega)$	−2	−1	−1	−4	0	−5	−5	−3	1	2	5	−3	−4	1	−7
$(16)a$	0	2	2	−3	3	3	2	−2	1	1	0	−5	5	−10	10
$(16)b$	0	2	2	3	−3	3	−2	2	1	1	0	5	−5	10	−10

Table (5.2.4) (concluded)

a	Trace of Frobenius													
$(3+7\omega)a$	1	2	0	−4	−2	4	−6	1	−2	−1	−5	−1	6	−8	−4
$(3+7\omega)b$	−1	2	0	−4	2	−4	6	−1	2	−1	−5	−1	−6	−8	4
$(10+5\omega)a$	3	1	1	4	0	−2	4	1	−6	2	−4	−4	−8	0	−4
(17)	−3	−2	−2	4	4	−6	0	0	1	1	−4	4	4	4	4
$(4-8\omega)$	0	−1	−1	−3	−3	−2	5	5	−3	−3	−1	8	8	1	1
$(6-8\omega)$	−1	1	−4	3	1	0	−3	1	−7	3	0	4	−6	4	4
$(16+2\omega)$	1	0	−3	−1	1	4	−1	−3	6	0	−7	−9	−3	8	−1
$(15+3\omega)$	1	1	−2	−1	0	1	0	0	2	−6	−4	−4	4	−12	−4
$(8\omega)a$	0	−1	−2	2	4	2	−2	−2	6	2	0	6	4	−2	−8
$(7+7\omega)$	2	−1	−2	1	0	−1	−1	5	−7	−2	−3	−3	0	1	1
$(4+8\omega)$	0	3	1	−1	5	2	−3	3	−1	−3	4	0	1	−11	3
$(18+2\omega)$	1	0	1	2	−4	4	3	0	3	6	1	9	−3	5	−4
$(5+8\omega)$	−1	1	3	−1	2	−2	−5	1	−1	−7	1	−6	−4	1	−11
$(19+\omega)$	−1	0	1	1	−1	1	−1	−3	0	0	−7	6	6	−4	−4
(20)	0	−1	−1	2	2	−2	0	0	−6	−6	−4	6	6	−10	−10
$(7+8\omega)$	−3	2	0	2	0	−2	6	0	6	1	6	−4	2	6	−2
$(15+5\omega)a$	1	1	1	0	0	2	4	4	−6	1	−4	8	0	4	4
$(15+5\omega)b$	−1	−1	−1	2	−4	−2	−6	0	6	1	2	0	−6	−10	−10
$(21)a$	3	0	2	−1	1	−1	4	−6	6	0	−4	2	8	4	8
$(21)b$	3	2	0	1	−1	−1	−6	4	0	6	−4	8	2	8	4
$(21)c$	−3	−2	−2	−1	−1	1	4	4	−6	−6	4	0	0	−4	−4
$(14+6\omega)a$	−1	1	1	−1	2	2	−3	−3	−3	2	−8	1	−3	0	−8
$(14+6\omega)b$	−1	3	−1	−1	−2	−2	1	−3	−1	−2	6	−1	3	4	−4
$(18+4\omega)a$	1	0	0	−1	1	−2	3	−3	−1	−6	2	−6	6	−10	−1
$(18+4\omega)b$	1	−2	0	−3	−1	4	−3	3	1	−2	0	0	0	−6	−3
$(2-10\omega)$	1	−2	−2	−3	3	−3	2	0	−1	1	0	5	5	−10	10
(10ω)	−1	0	−1	1	−1	−1	0	−6	3	3	2	3	−3	10	8

§5.3 Results for $\mathbf{Q}(\sqrt{-43})$

We give the following tables:

Table (5.3.1): Ideals $\mathbf{a}$ of $\mathbf{Z}[\omega]$ with $N(\mathbf{a}) < 230$ and dim $V^+(\mathbf{a}) > 0$.

Table (5.3.2): Rational newforms in $V^+(\mathbf{a})$.

Table (5.3.3): Elliptic curves with conductor $\mathbf{a}$, $N(\mathbf{a}) < 230$.

Table (5.3.4): The trace of Frobenius for elliptic curves.

Table (5.3.1): Ideals a of $\mathbf{Z}[(\frac{\sqrt{-43}+1}{2})]$ with $N(\mathrm{a}) < 230$ and dim $V^{+}(\mathrm{a}) > 0$

a	$N(\mathrm{a})$	dim $V^{+}(\mathrm{a})$	dim $V^{new}(\mathrm{a})$
(4)	16	2	0
(6)	36	2	0
$(1-2\omega)$	43	5	1
(8)	64	4	0
(9)	81	12	0
(10)	100	2	0
(11)	121	1	1
$(10+\omega)$	121	1	1
$(8+2\omega)$	124	1	1
(12)	144	8	0
$(2-4\omega)$	172	14	0
(4ω)	176	4	0
$(13+\omega)$	193	1	1
(14)	196	1	1
$(4+4\omega)$	208	4	0
(15)	225	1	1

Table (5.3.2): Rational Newforms in $V^{+}(\mathrm{a})$

We give the eigenvalues of the T_π and W_π operators for the first 15 primes in $\mathbf{Q}(\sqrt{-43})$. These are: $\{2, 3, 1-\omega, \omega, 2-\omega, 1+\omega, 3-\omega, 2+\omega, 4-\omega, 3+\omega, 5, 5-\omega, 4+\omega, 6-\omega, 5+\omega\}$.

a	Eigenvalues														
$(1-2\omega)$	0	−2	3	3	−5	−5	−3	−3	−1	−1	6	−1	−1	5	5
(11)	0	−5	–	–	4	4	−2	−2	−1	−1	−9	7	7	−8	−8
$(10+\omega)$	2	0	–	1	−3	3	7	−7	1	1	4	3	3	5	−5
$(8+2\omega)$	–	1	−3	0	−4	5	−6	−3	3	0	−1	−4	–	3	−9
$(13+\omega)$	2	2	−3	−6	2	−3	−2	0	−8	8	−4	11	8	10	−3
(14)	–	−2	0	0	−4	−4	6	6	0	0	−10	−4	−4	6	6
(15)	−3	–	−4	−4	−2	−2	2	2	0	0	–	0	0	10	10

Table (5.3.3): Elliptic curves over $\mathbf{Q}(\sqrt{-43})$ with conductor a, $N(\mathrm{a}) < 230$

a	a_1	a_2	a_3	a_4	a_6
$(1-2\omega)$	0	1	1	0	0
(11)	0	-1	1	-10	-20
$(8+2\omega)$	$-1+\omega$	ω	1	$-\omega$	0
$(13+\omega)$	0	$-\omega$	$-1+\omega$	-2	1
(14)	1	0	1	-1	0
(15)	1	1	1	0	0

Missing Conductor: $(10+\omega)$

Table (5.3.4): Trace of Frobenius for Elliptic Curves over $\mathbf{Q}(\sqrt{-43})$

For each curve in Table (5.3.3) we give the trace of Frobenius for the first 15 primes in $\mathbf{Q}(\sqrt{-43})$. These are: $\{2, 3, 1-\omega, \omega, 2-\omega, 1+\omega, 3-\omega, 2+\omega, 4-\omega, 3+\omega, 5, 5-\omega, 4+\omega, 6-\omega, 5+\omega\}$.

a	Trace of Frobenius														
$(1-2\omega)$	0	−2	3	3	−5	−5	−3	−3	−1	−1	6	−1	−1	5	5
(11)	0	−5	1	1	4	4	−2	−2	−1	−1	−9	7	7	−8	−8
$(8+2\omega)$	1	1	−3	0	−4	5	−6	−3	3	0	−1	−4	1	3	−9
$(13+\omega)$	2	2	−3	−6	2	−3	−2	0	−8	8	−4	11	8	10	−3
(14)	1	−2	0	0	−4	−4	6	6	0	0	−10	−4	−4	6	6
(15)	−3	1	−4	−4	−2	−2	2	2	0	0	1	0	0	10	10

§5.4 Results for $\mathbf{Q}(\sqrt{-67})$

We give the following tables:

Table (5.4.1): Ideals $\mathbf{a}$ of $\mathbf{Z}[\omega]$ with $N(\mathbf{a}) < 265$ and $\dim V^+(\mathbf{a}) > 0$.

Table (5.4.2): Rational newforms in $V^+(\mathbf{a})$.

Table (5.4.3): Elliptic curves with conductor $\mathbf{a}$, $N(\mathbf{a}) < 265$.

Table (5.4.4): The trace of Frobenius for elliptic curves.

Table (5.4.1): Ideals a of $\mathbf{Z}[(\frac{\sqrt{-67}+1}{2})]$ with $N(\mathrm{a}) < 265$ and $\dim V^{+}(\mathrm{a}) > 0$

a	$N(\mathrm{a})$	$\dim V^{+}(\mathrm{a})$	$\dim V^{new}(\mathrm{a})$
(ω)	17	3	0
$(1-2\omega)$	67	5	1
(2ω)	68	6	0
(9)	81	20	0
(11)	121	1	1
(12)	144	2	0
(3ω)	153	7	1
(14)	196	5	1
(15)	225	1	1
$(12-3\omega)$	261	3	1

Table (5.4.2): Rational Newforms in $V^+(\mathrm{a})$

We give the eigenvalues of the T_π and W_π operators for the first 15 primes in $\mathbf{Q}(\sqrt{-67})$. These are: $\{2, 3, 1-\omega, \omega, 2-\omega, 1+\omega, 3-\omega, 2+\omega, 5, 4-\omega, 3+\omega, 5-\omega, 4+\omega, 6-\omega, 5+\omega\}$.

a	Eigenvalues														
$(1-2\omega)$	0	−2	3	3	7	7	9	9	−6	−5	−5	−1	−1	−1	−1
(11)	0	−5	−2	−2	0	0	−1	−1	−9	0	0	3	3	8	8
(3ω)	−1	−	1	+	6	0	4	−3	−1	5	−10	−6	−3	−7	8
(14)	−	−2	6	6	2	2	0	0	−10	−6	−6	2	2	−12	−12
(15)	−3	−	2	2	4	4	0	0	−	−2	−2	−10	−10	8	8
$(12-3\omega)$	1	+	5	0	8	4	−4	−6	−1	−	−2	1	−8	−1	−6

Table (5.4.3): Elliptic curves over Q($\sqrt{-67}$) with conductor a, $N(\mathrm{a}) < 265$

a	a_1	a_2	a_3	a_4	a_6
$(1-2\omega)$	0	1	1	-12	-21
(11)	0	-1	1	0	0
(14)	1	0	1	-1	0
(15)	1	1	1	0	0
$(12-3\omega)$	$-1+\omega$	$1-\omega$	ω	$3+3\omega$	$13-\omega$

Missing Condctor: (3ω)

Table (5.4.4): Trace of Frobenius for Elliptic Curves over $\mathbf{Q}(\sqrt{-67})$

For each curve in Table (5.3.3) we give the trace of Frobenius for the first 15 primes in $\mathbf{Q}(\sqrt{-67})$. These are: $\{2, 3, 1-\omega, \omega, 2-\omega, 1+\omega, 3-\omega, 2+\omega, 5, 4-\omega, 3+\omega, 5-\omega, 4+\omega, 6-\omega, 5+\omega\}$.

a	Trace of Frobenius														
$(1-2\omega)$	0	−2	3	3	7	7	9	9	−6	−5	−5	−1	−1	−1	−1
(11)	0	−5	−2	−2	0	0	−1	−1	−9	0	0	3	3	8	8
(14)	1	−2	6	6	2	2	0	0	−10	−6	−6	2	2	−12	−12
(15)	−3	1	2	2	4	4	0	0	1	−2	−2	−10	−10	8	8
$(12-3\omega)$	1	−1	5	0	8	4	−4	−6	−1	1	−2	1	−8	−1	−6

PART II

SOME WORK ON THE BIRCH, SWINNERTON – DYER CONJECTURE

Chapter 1: The Birch, Swinnerton-Dyer Conjecture

In this chapter I will describe joint work carried out with Cremona [7] to provide numerical evidence to support the Birch, Swinnerton-Dyer conjecture over the imaginary quadratic number fields with class number 1.

In §1.1 I will give a restatement of the conjecture, along with details of the quantities involved and the exact nature of the calculations carried out. In §1.2 and §1.3 I will give a synopsis of Cremona's work in calculating the value of the L-series, $L(F, s)$, at $s = 1$ and the period, $\pi(F)$, associated with some cusp form F. This exposition will be brief; for more detail see [7]. In §1.4 I will show how to find the torsion points of an elliptic curve, $E(\mathrm{K})$, defined over K and in §1.5 I will describe how to find curves isogenous to E given the torsion points, $E_{tors}(\mathrm{K})$, using a method due to Vélu.

The results of this work are given in Chapter 2. Here, and throughout, K is one of the 9 imaginary quadratic fields with class number 1, ie $\mathrm{K} = \mathbf{Q}(\sqrt{-d})$ where $d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$.

§1.1 General Overview

Recall from Part I of this thesis that, for some non-zero ideal $\mathbf{a}$ of ϑ_K, there is a correspondence (with certain restrictions) between elliptic curves defined over K with conductor $\mathbf{a}$ and the rational newforms in $S(\mathbf{a})$, the space of cusp forms of weight 2 for $\Gamma_0(\mathbf{a})$. We now explore the arithmetic correspondence between such pairs E and F further; in particular in relation to the Birch, Swinnerton-Dyer Conjecture.

First we consider the L-series associated with E and F. Both $L(E,s)$ and $L(F,s)$ have the same type of Euler product expansion. They agree in at least the first 15 Euler factors in all cases computed. This is demonstrated for the 5 Euclidean fields in [4] and for the 4 non-Euclidean fields in Part I of this thesis.

The computation of $L(F,1)$, for given F, is dealt with in §1.2. By an analogue of the Manin-Drinfeld Theorem [10],[11], the quantity $L(F,1)$ is a rational multiple of $\pi(F)$, the real period of F. We can determine this ratio using modular symbols and in particular we can determine whether or not $L(F,1) = 0$. The determination of $\pi(F)$ will concern us in §1.3.

The value of $L(E,1)$, for an elliptic curve E defined over K is predicted by the Birch, Swinnerton-Dyer conjectures, (0.7.2), as follows:

(i) $L(E,1) = 0 \Leftrightarrow \text{rank}(E(\text{K})) > 0$

(ii) If $\text{rank}(E(\text{K})) = 0$ then

$$L(E,1) = \frac{\alpha(E)\prod c_{\mathfrak{p}}|\text{III}|}{|E_{tors}(\text{K})|^2}, \tag{1.1.1}$$

where:

Ш is the Tate-Shaferavitch group of E over K;

$c_\wp$ is the local index $[E(\mathrm{K}_\wp) : E^0(\mathrm{K}_\wp)]$, for each prime $\wp$ of ϑ_K, where $E(\mathrm{K}_\wp)$ is the group of points of E defined over the $\wp$-adic completion of K and $E^0(\mathrm{K}_\wp)$ is the subgroup of points whose reduction mod $\wp$ is non-singular (if $\wp$ doesn't divide the conductor of E then there are no singular points and so $c_\wp = 1$);

$\alpha(E)$ is the "complex period", or volume, of $E(\mathrm{K})$:

$$\alpha(E) = |\Im(\omega_1\overline{\omega_2})|,$$

where ω_1 and ω_2 generate the period lattice of E,

(alternatively

$$\alpha(E) = \int_{E(\mathbf{C})} \omega \wedge \overline{\omega},$$

where ω is the Néron differential defined in §0.1);

and

$|E_{tors}(\mathrm{K})|$ is the order of the group of torsion points in $E(\mathrm{K})$.

We calculate $c_\wp$ using Tate's algorithm [19]. $\alpha(E)$ is calculated in terms of ω_1 and ω_2 using a method similar to that used by Cremona in [5]. The calculation of $E_{tors}(\mathrm{K})$ is the subject of §1.4.

We investigate the conjecture in two ways; first we compare the values of $\alpha(E)$ and $\pi(F)$, computed numerically to within a given accuracy, and, secondly, when $L(F,1) \neq 0$ we compare the rational numbers $\frac{L(F,1)}{\pi(F)}$ and $\frac{\prod c_\wp}{|E_{tors}(\mathrm{K})|^2}$. We do this by computing the non-zero rational number

$$S := \frac{L(F,1)}{\pi(F)} \Big/ \frac{\prod c_\wp}{|E_{tors}(\mathrm{K})|^2}. \tag{1.1.2}$$

If $L(F,s) = L(E,s)$ and if Part(ii) holds for E, then S should equal the order of the Tate-Shaferavitch group. In most cases we find $S = 1$ with $S = 4$ occasionally. In addition, we have found two possible occurrences of $S = 9$ and one of $S = 16$ in the case where $\mathrm{K} = \mathbf{Q}(\sqrt{-19})$. This is consistent with a theorem of Cassels [3] which states that if **III** is finite then |**III**| is a perfect square. In all the Euclidean cases and some non-Euclidean cases, we find that, if **III** has order S, then $L(F,1) = L(E,1)$ (to within computational accuracy) as predicted by Birch and Swinnerton-Dyer.

Note: $L(E,s)$ is invariant under isogeny and hence so is the left hand side of (1.1.1). However, the factors of the left hand side of (1.1.1) are not. Therefore, given an elliptic curve E with conductor $\mathbf{a}$, we first compute the isogeny class of curves of conductor $\mathbf{a}$ which contains E. In each class of curves in the Euclidean fields and some in the non-Euclidean fields there is one, or more, curve for which the ratio

$$\frac{S\prod c_p}{|E_{tors}(\mathrm{K})|^2} = \frac{L(F,1)}{\pi(F)}. \tag{1.1.3}$$

For each such curve, E, we compare the area $\alpha(E)$ with $\pi(F)$ to find agreement to within computational accuracy.

Note: This problem does not arise in the rational case as one can single out the "strong Weil curve," in a given isogeny class of curves with conductor $N \in \mathbf{Z}$, to be the one whose period lattice consists precisely of the $X_0(N)$ periods of the associated cusp form f for $\Gamma_0(N)$. In our case, the cusp form F has only the single real period $\pi(F)$. The main stumbling block in moving from $\mathbf{Q}$ to K is that there is no analogue to Shimura's construction [15], [18] of modular elliptic curves over $\mathbf{Q}$ from cusp forms using periods.

The results of these investigations are given in Chapter 2.

§1.2 The Calculation of $L(F,1)$

Recall from §0.4, that a plusform, F, in $S(\mathrm{a})$ with Fourier expansion

$$F(z,\zeta) = \sum_{\alpha\in\vartheta_K} c(\alpha)H(\eta^{-1}\alpha\zeta)\psi(\eta^{-1}\alpha z),$$

has an L-series associated with it which has Euler product expansion:

$$L(F,s) = \prod_{\wp\in\vartheta_K} (1 - c(\wp)N(\wp)^{-s} + \chi(\wp)N(\wp)^{1-2s})^{-1},$$

where

$$\chi(\wp) = \begin{cases} 0 & \text{if } \wp|\mathrm{a} \\ 1 & \text{if } \wp \not|\mathrm{a}. \end{cases}$$

We have the estimate $|c(\wp)| \leq 2N(\wp)^{\frac{1}{2}}$ and so we see that the series $L(F,s)$ converges for $\Re(s) > \frac{3}{2}$.

We consider an analytic continuation of $L(F,s)$, associated with the Mellin transform of F, which satisfies a functional equation, as follows:

If $F \in S(\mathrm{a})$ we can form the Mellin transform of F by multiplying by t^{2s-2} and integrating along the imaginary axis. Then we define:

$$Z(F,s) := a_K \int_0^\infty t^{2s-2} F.\beta, \tag{1.2.1}$$

where the normalising constant,

$$a_{\mathrm{K}} := \frac{(4\pi)^2}{|\vartheta^*_{\mathrm{K}}||D|},$$

depends only on the field K and not on the form F or its level.

$F(0,t)$ decays rapidly as $t \to \infty$ so $Z(F,s)$ is an entire function of s. In particular

$$Z(F,1) = a_{\mathrm{K}} \int_0^\infty F.\beta \tag{1.2.2}$$

The following is proved by Cremona in [7]:

Lemma 1.2.1:

(i) For $\Re(s) > \frac{3}{2}$ we have:

$$Z(F,s) = (2\pi)^{2-2s}|D|^{s-1}\Gamma(s)^2 L(F,s). \tag{1.2.3}$$

(ii) $Z(F,s)$ satisfies the functional equation:

$$Z(F,s) = \pm N(\mathbf{a})^{1-s} Z(F, 2-s). \tag{1.2.4}$$

We therefore abuse notation and write $L(F,1) = Z(F,1)$ although, strictly, $L(F,s)$ does not converge at $s = 1$.

We now wish to compute $Z(F,1)$ numerically. We do this by splitting the range of integration at the point $(z,\zeta) = (0, \frac{1}{\sqrt{|\nu|}})$, where ν is a generator of $\mathbf{a}$. Then, changing variables t and $\frac{1}{|\nu|t}$ in the integral from 0 to $\frac{1}{\sqrt{|\nu|}}$, using Fourier expansions where possible, and integrating term by term, we obtain [7]:

$$Z(F,1) = \frac{4\pi(1-\epsilon)}{w\sqrt{|\nu D|}} \sum_\alpha \frac{c(\alpha)}{|\alpha|} K_1\left(\frac{4\pi|\alpha|}{\sqrt{|\nu D|}}\right), \tag{1.2.5}$$

where $\epsilon = \pm 1$ is the eigenvalue of the Fricke involution: if $\epsilon = +1$ then, trivially,

$Z(F, 1) = 0$ so we will assume $\epsilon = -1$;

$w = |\vartheta_{\mathrm{K}}^{*}|$;

D is the discriminant of K;

$c(\alpha)$ is the Fourier coefficient, $\alpha \in \vartheta_{\mathrm{K}}$ non-zero;

K_1 is the K-Bessel function [1].

§1.3 The Calculation of $\pi(F)$

Let F be a newform in $S(\mathrm{a})$. Then F induces a map

$$I_{\mathrm{F}} : H_1(X_{\Gamma}, \mathbf{C}) \to \mathbf{C},$$

via

$$\{A, B\} \mapsto a_{\mathrm{K}} \int_A^B F.\beta. \tag{1.3.1}$$

I_F is a homomorphism and has a kernel of codimension 1. The restriction of I_F to $H_1(X_{\Gamma}, \mathbf{Z})$, has an image which is a non-trivial, discrete subgroup of $\mathbf{R}$. It has the form $\pi(F)\mathbf{Z}$, for some unique positive real number $\pi(F)$. This $\pi(F)$ is the **real period** of F.

We will give two methods for calculating $\pi(F)$ numerically; a direct method and a more indirect method in the case where $L(F, 1) \neq 0$.

Method 1:

To each rational newform $F \in S(\mathrm{a})$ there is associated a one-dimensional eigenspace of $H_1(X_{\Gamma_0(\mathbf{a})}, \mathbf{Q})$, given explicitly in terms of modular symbols, which we shall denote V_F. Hence we may determine $\gamma \in \Gamma_0(\mathbf{a})$ such that the symbol $\{A, \gamma(A)\}$ generates $V_F \cap H_1(X_{\Gamma_0(\mathbf{a})}, \mathbf{Z})$.

Then $\pi(F) = I_F(\{A, \gamma(A)\})$.

Since $\{A, \gamma(A)\} = \{A, \infty\} - \{\gamma(A), \infty\}$ we consider paths of the form $\{A, \infty\}$ for $A = (z_0, \zeta_0) \in \mathcal{H}_3$. Then

$$I_F(A) = \frac{4\pi\zeta_0}{\sqrt{|D|}} {\sum_\alpha}' \frac{c(\alpha)}{|\alpha|} \tilde{\psi}(\frac{\alpha z_0}{\sqrt{D}}) K_1(\frac{4\pi\zeta_0}{\sqrt{|D|}}|\alpha|), \tag{1.3.2}$$

where $\tilde{\psi}(z) = \sum_{\varepsilon \in \vartheta_K^*} \psi(\varepsilon z)$ and the restricted sum, $\sum'$, is taken over all non-zero $\alpha \in \vartheta_K$, modulo units.

This sum converges fairly fast and to aid this convergence we choose A so that ζ_0 is as large as possible. Let ν be a generator of the ideal $\mathbf{a}$ of ϑ_K. Then, if $\gamma = \begin{pmatrix} a & b \\ \nu c & d \end{pmatrix} \in \Gamma_0(\mathbf{a})$ is chosen with $|c|$ as small as possible, we have:

$$A = (\frac{-d}{\nu c}, \frac{1}{|\nu c|})$$

so that

$$\gamma(A) = (\frac{a}{Nc}, \frac{1}{|\nu c|}).$$

Thus

$$\begin{aligned} \pi(F) &= I_F(\{A, \gamma(A)\}) \\ &= \frac{4\pi}{|\nu c|\sqrt{|D|}} {\sum_\alpha}' \frac{c(\alpha)}{|\alpha|} K_1(\frac{4\pi}{|\nu c|\sqrt{|D|}}|\alpha|) \\ &\quad (\tilde{\psi}(\frac{-\alpha d}{\nu c\sqrt{D}}) - \tilde{\psi}(\frac{\alpha a}{\nu c\sqrt{D}})). \end{aligned} \tag{1.3.3}$$

Method 2:

If $Z(F,1) \neq 0$ then we may compute $\pi(F)$ indirectly by computing $Z(F,1)$ and dividing by the ratio $\frac{Z(F,1)}{\pi(F)}$. We know that $Z(F,1) = I_F(\{0,\infty\})$. Now let q be any prime not dividing the level **a**. Then F is an eigenform for the Hecke operator with eigenvalue $c(\mathrm{q})$, so

$$c(\mathrm{q})I_F(\{0,\infty\}) = I_F(T_{\mathrm{q}}(\{0,\infty\}),$$

where

$$T_{\mathrm{q}}(\{0,\infty\}) = (N(\mathrm{q})+1)\{0,\infty\} - \sum_{\alpha \bmod \mathrm{q}} I_F(\{0,\frac{\alpha}{\mathrm{q}}\}).$$

Thus

$$(1+N(\mathrm{q})-c(\mathrm{q}))I_F(\{0,\infty\}) = \sum_{\alpha \bmod \mathrm{q}} I_F(\{0,\frac{\alpha}{\mathrm{q}}\}). \tag{1.3.4}$$

Each cycle $\{0,\frac{\alpha}{\mathrm{q}}\} \in H_1(X_{\Gamma_0(\mathbf{a})},\mathbf{Z})$ since $\mathrm{q} \not| \mathbf{a}$. Thus the right hand side of (1.3.4) has the form $n_{\mathrm{q}}\pi(F)$ with $n_{\mathrm{q}} \in \mathbf{Z}$. Thus we have the result

$$\frac{Z(F,1)}{\pi(F)} = \frac{n_{\mathrm{q}}}{1+N(\mathrm{q})-c(\mathrm{q})}. \tag{1.3.5}$$

In practise it is easy to compute the integer n_{q} using modular symbols (See [12] and [5], §2.8).

§1.4 Torsion Points

Recall the general form (0.1.1) of an elliptic curve, E, defined over K:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in \vartheta_{\mathrm{K}}.$$

We wish to determine the group of torsion points, $E_{tors}(\mathrm{K})$, of $E(\mathrm{K})$ explicitly. We do this, using the following:

Theorem 1.4.1: (Chapter VIII, Theorem 7.1, [16])

Let E be an elliptic curve defined over K *with equation* (0.1.1). *Let $P \in E$ be a point of exact order $m \geq 2$. Then*

(i) If m is not a prime power then $x(P), y(P) \in \vartheta_{\mathrm{K}}$.

(ii) If $m = p^n$ is a prime power then for each prime π in ϑ_{K} let

$$r_\pi = [\frac{\mathrm{ord}_\pi(p)}{p^n - p^{n-1}}], \quad where\ [\quad] = greatest\ integer.$$

Then $\mathrm{ord}_\pi(x(P)) \geq -2r_\pi$ and $\mathrm{ord}_\pi(y(P)) \geq -3r_\pi$.

In particular, $x(P)$ and $y(P)$ are π-integral if $\mathrm{ord}_\pi(p) = 0$.

Using this theorem, we can make a change of variables, $(x, y) \mapsto (X, Y)$ to produce an elliptic curve $E'(\mathrm{K})$ for which **all** points of finite order are π-integral, $\forall$ primes π in ϑ_{K}.

Definition 1.4.2:

Let E be an elliptic curve defined over K, *where $h(\mathrm{K}) = 1$, given by* (0.1.1).

Set $X = 4x$,

$Y = 8y + 4x + 4.$

Then E' is the elliptic curve defined over K *with equation*

$$Y^2 = X^3 + AX^2 + BX + C,$$

where $A, B, C \in \vartheta_{\mathrm{K}}$ *are given by:*

$$A = 4a_2 + a_1^2,$$

$$B = 16a_4 + 8a_1a_3,$$

$$C = 64a_6 + 16a_3^2.$$

If $\mathrm{K} = \mathbf{Q}(\sqrt{-3})$ *then set*

$$\overline{X} = -3X - A,$$

$$\overline{Y} = (\sqrt{-3})^3 Y.$$

Then E' *is the elliptic curve defined over* K *with equation:*

$$\overline{Y}^2 = \overline{X}^3 + D\overline{X} + E,$$

where $D, E \in \vartheta_{\mathrm{K}}$ *are given by*

$$D = 9B - 27A^2,$$

$$E = 27C - 9AB + 2A^3.$$

Then we have the following:

Theorem 1.4.3:

Let E' *be as defined in* Definition 1.4.2. *Then if* P *is a point on* E' *with finite order we have:*

(i) If $\mathrm{K} \neq \mathbf{Q}(\sqrt{-3})$ *then* $X(P), Y(P) \in \vartheta_{\mathrm{K}}$.

(ii) If $\mathrm{K} = \mathbf{Q}(\sqrt{-3})$ *then* $\overline{X}(P), \overline{Y}(P) \in \vartheta_{\mathrm{K}}$.

Proof:

By Theorem 1.4.1, the torsion point $P \in E$ is π-integral if $r_\pi = 0$. So we need only consider the cases where $r_\pi \neq 0$.

If $\text{ord}_\pi(p) = 2$ then p is ramified in K and $(p) = (\pi)^2$. Otherwise, $\text{ord}_\pi(p) = 0$ or 1 according to whether or not π divides p.

Moreover $p^n - p^{n-1} = \begin{cases} 1 & \text{if } p = 2, n = 1 \\ 2 & \text{if } p = 2, n = 2 \text{ or } p = 3, n = 1 \\ \geq 4 & \text{otherwise.} \end{cases}$

Thus $r_\pi \neq 0$ when

(1) $\text{ord}_\pi(p) = 2; p = 2, n = 1,$

(2) $\text{ord}_\pi(p) = 2; p = 2, n = 2,$

(3) $\text{ord}_\pi(p) = 2; p = 3, n = 1,$

(4) $\text{ord}_\pi(p) = 1; p = 2, n = 1.$

Cases (1) and (2) occur in $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{-2})$ while case (3) occurs in $\mathbf{Q}(\sqrt{-3})$. Case (4) occurs in all fields other than $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{-2})$.

Consider case (1):

Let $\mathrm{K} = \mathbf{Q}(\sqrt{-1})$. Then $(2) = (1+i)^2$.

By Theorem 1.4.1 we can see that the "worst" possible case which can arise is:

$$P = (x, y) = \left(\frac{x_0}{(1+i)^4}, \frac{y_0}{(1+i)^6}\right), \; x_0, y_0 \in \vartheta_{\mathrm{K}} \text{ such that } (1+i) \nmid x_0, y_0.$$

Now, using Definition 1.4.2, we set:

$X(P) = 4x = x_0;$

$Y(P) = 8y + 4x + 4 = y_0 + x_0 + 4.$

Thus $X, Y \in \vartheta_{\mathrm{K}}$.

Similarly in the other 3 cases; in (2) and (4) the change of variable $(x, y) \mapsto (X, Y)$ is sufficient to ensure that torsion points on E' are π-integral, $\forall \pi$ in ϑ_K, while in case (3) it is necessary to make the further change $(X, Y) \mapsto (\overline{X}, \overline{Y})$. ∎

So, by considering E', rather than E, we can reduce the problem of finding torsion points to one of finding **integral** points. We can determine all points P of E', such that P and $2P$ are integral, explicitly using the following theorem. This will include all torsion points of E', by Theorem 1.4.3, and may also include some points of infinite order.

Theorem 1.4.4:

Let E' be as defined in Definition 1.4.2. *Then, if $P \in E$ has exact order $n < \infty$, we have*

(i) If $n = 2$ *then* $\begin{cases} Y(P) = 0 & \text{if } K \neq \mathbf{Q}(\sqrt{-3}) \\ \overline{Y}(P) = 0 & \text{if } K = \mathbf{Q}(\sqrt{-3}). \end{cases}$

(ii) If $n > 2$ *then* $\begin{cases} Y(P)^2 | \Delta & \text{if } K \neq \mathbf{Q}(\sqrt{-3}) \\ \overline{Y}(P)^2 | \Delta & \text{if } K = \mathbf{Q}(\sqrt{-3}). \end{cases}$

Proof:

Suppose $K \neq \mathbf{Q}(\sqrt{-3})$.

Then E' is given by

$$Y^2 = f(X) = X^3 + AX^2 + BX + C, \quad A, B, C \in \vartheta_K.$$

Set $h_1 = -27(X^3 + AX^2 + BX - C) + 4A^3 - 18AB$,

and $h_2 = 3X^2 + 2AX + 4B - A^2$.

The discriminant of E',

$$\begin{aligned}\Delta &= \gcd(f(X), f'(X)) \\ &= h_1 f(X) + h_2(f'(X))^2 \\ &= 27C^2 + 4A^3C - 18ABC + 4B^3 - A^2B^2.\end{aligned}$$

$f(X)$ has no repeated roots so Δ is a constant.

Let $P \in E'$. By Theorem 1.4.3, $X(P), Y(P) \in \vartheta_{\mathrm{K}}$.

Suppose P has order 2. Then $P = -P$.

So $Y(P) = Y(-P) = -Y(P)$, ie $Y(P) = 0$.

Now let $P = (X_0, Y_0) \in E'$ have order $n > 2$. We find $2P = (U, V)$ using simple geometry:

The gradient of the tangent to E' at P is given by $m = \frac{f'(X_0)}{2Y_0}$.

Then the equation of the tangent is:

$$Y = Y_0 + m(X - X_0).$$

So we have

$$(Y_0 + m(X - X_0))^2 = f(X) = X^3 + AX^2 + BX + C.$$

Now

$$\begin{aligned}2X_0 + U &= \sum \text{ roots of } f(X) \\ &= \text{ coefficient of } X^2 \text{ term} \\ &= m^2 - A \\ &= \frac{(f'(X_0))^2}{4Y_0^2} - A \\ &= \frac{(f'(X_0))^2}{4f(X_0)} - A.\end{aligned}$$

By Theorem 1.4.3, X_0 and $U \in \vartheta_K$, and $A \in \vartheta_K$ by definition.

So $m^2 = \frac{(f'(X_0))^2}{4f(X_0)} \in \vartheta_K$.

But $m \in$ K and $m^2 \in \vartheta_K \Rightarrow m \in \vartheta_K$.

ie $\frac{f'(X_0)}{2Y_0} \in \vartheta_K \Rightarrow Y_0 | f'(X_0)$ and $Y_0^2 = f(X_0) \Rightarrow Y_0^2 | f(X_0)$.

Thus, $Y_0^2 | \Delta$.

If K=$\mathbf{Q}(\sqrt{-3})$ then we apply the same arguments to the curve

$$E' : \overline{Y}^2 = f(\overline{X}) = \overline{X}^3 + D\overline{X} + E, \quad D, E \in \vartheta_K.$$

Hence, the result. ∎

So, given E with equation (0.1.1), we proceed as follows:

[**1**] Form E' according to Definition 1.4.2.

[**2**] If K $\neq \mathbf{Q}(\sqrt{-3})$ then find all pairs $(X, Y) \in \vartheta_K \times \vartheta_K$ which satisfy **either**

(i) $Y = 0;\ X^3 + AX^2 + BX + C = 0$

or

(ii) $Y^2 | \Delta;\ X^3 + AX^2 + BX + C = Y^2$.

If K $= \mathbf{Q}(\sqrt{-3})$ then find all pairs $(\overline{X}, \overline{Y}) \in \vartheta_K \times \vartheta_K$ which satisfy **either**

(i) $\overline{Y} = 0;\ \overline{X}^3 + D\overline{X} + E = 0$

or

(ii) $\overline{Y}^2 | \Delta;\ \overline{X}^3 + D\overline{X} + E = \overline{Y}^2$.

[3] Convert integral points on E' back to rational torsion points on E according to:

$$\begin{cases} x = \dfrac{X}{4} & \\ y = \dfrac{1}{8}(Y - X - 4) & \text{if K} \neq \mathbf{Q}(\sqrt{-3}) \\ x = -\dfrac{1}{12}(\overline{X} + A) & \\ y = \dfrac{\overline{Y}}{8(\sqrt{-3})^3} + \dfrac{\overline{X}}{24} + \dfrac{1}{2}(\dfrac{A}{12} - 1) & \text{if K} = \mathbf{Q}(\sqrt{-3}). \end{cases}$$

§1.5 Isogeny Classes

As stated in §1.1, it is not always enough to consider just one curve of conductor a. Instead we must consider the isogeny class of curves of conductor **a** in order to find the curve which agrees most strongly with the plusform $F \in S(\text{a})$.

In fact, what we will calculate is not necessarily the complete isogeny class. There is no known bound on the degrees of isogenies possible over number fields K $\neq$ **Q**; we will only look for those isogenies whose kernels are pointwise defined over K. In the 5 Euclidean cases, this search is sufficient to obtain at least one curve in each class for which $\alpha(E) = \pi(F)$. This is not always true in the non-Euclidean cases where there may be curves isogenous to those we have, which cannot be found using the method described here.

We begin with a single curve of conductor **a**. In the Euclidean case such curves are listed in [4]; in the non-Euclidean case they are found in Part I of this thesis. We then "fill out" the isogeny class of curves of conductor **a** using a method due to Vélu [20].

Before we can proceed we must determine $E_{tors}(\mathrm{K})$ according to the method given in §1.4. Then, for each point $P \in E_{tors}(\mathrm{K})$ of order m, we proceed as follows to find the equation of the m-isogenous curve, if it exists.

Set $F =< P >=$ finite subgroup of $E(\mathrm{K})$, of order m, generated by P, and $F_2 =$ the points of order 2 in F.

Let R be part of $F \backslash \{F_2 \cup \{0\}\}$ such that:

$$F \backslash \{F_2 \cup \{0\}\} = R \cup (-R) \text{ and } R \cap (-R) = \phi.$$

Set $S = F_2 \cup R$.

Then, $\forall Q = (x_Q, y_Q) \in S$, define

$$t_Q := \begin{cases} 3x_Q^2 + 2a_2 x_Q + a_4 - a_1 y_Q & \text{if } Q \in F_2 \\ 6x_Q^2 + b_2 x_Q + b_4 & \text{if } Q \notin F_2. \end{cases}$$

$$u_Q := 4x_Q^3 + b_2 x_Q^2 + 2b_4 x_Q + b_6.$$

Let $t := \sum_{Q \in S} t_Q$ and $w := \sum_{Q \in S} (u_Q + x_Q t_Q)$.

Then the m-isogenous curve is given by:

$$Y^2 + A_1 XY + A_3 Y = X^3 + A_2 X^2 + A_4 X + A_6,$$

where

$$A_1 = a_1,$$

$$A_2 = a_2,$$

$$A_3 = a_3,$$

$$A_4 = a_4 - 5t,$$

$$A_6 = a_6 - b_2 t - 7w.$$

Chapter 2: Results

Computer programs have been written in Algol68 to compute the numbers discussed in the previous Chapter for the imaginary quadratic number fields with class number one for which elliptic curves have been found. These programs have again been run on an ICL980 computer at SWURCC. The results of these calculations are given in this chapter.

§2.1 Introduction to the Tables

Let K be one of the 8 imaginary quadratic number fields $\mathbf{Q}(\sqrt{-d})$ where $d \in \{1,2,3,7,11,19,43,67\}$. For each elliptic curve, E, listed in Chapter 5 of [4] and Part I, Chapter 5 of this thesis we have found (at least part of) the isogeny class of curves containing E using Vélu's method as described in §1.5. For every curve, E, with conductor $\mathbf{a}$, we have calculated $c_{\wp}$ for all primes $\wp$ in ϑ_{K} which divide $\mathbf{a}$; the order of $E_{tors}(\mathrm{K})$; and, for curves without complex multiplication, the complex period, $\alpha(E)$, of E.

To each of our curves with conductor $\mathbf{a}$ in ϑ_{K} there corresponds a newform in $V^{+}(\mathbf{a})$. In the non-Euclidean cases there are some additional newforms for which no corresponding curves have yet been found. We have calculated the ratio

$$\frac{L(F,1)}{\pi(F)},$$

for each form F, except in those cases where the corresponding curve has complex multiplication. In addition, for those forms, F, for which $\frac{L(F,1)}{\pi(F)} \neq 0$, we have also calculated $L(F,1)$ and $\pi(F)$ using **Method 2** of §1.3.

In all the cases where K is Euclidean it can be seen that to each form F in $V^+(\mathbf{a})$, where $\frac{L(F,1)}{\pi(F)} \neq 0$, there corresponds at least one curve, E, with conductor $\mathbf{a}$, such that $\alpha(E) = \pi(F)$. However, this is not always the case when K is not Euclidean.

In the non-Euclidean cases there may be some isogenous curves, which we have not found using Vélu's method, which have the required properties. Alternatively there may be errors in some of the quantities calculated, due to (as yet) untraced bugs in the programs. However it is worth noting that, in the majority of cases where the ratios do not agree, $\alpha(E)$ and $\pi(F)$ differ by some **rational** factor to within the required computational accuracy.

For each field we give the following tables:

(i) The coefficients of the elliptic curves with conductor $\mathbf{a}$ along with an identifying letter for each. We also give a picture of the isogenies between the curves as in Table 1 of [2]. In the cases where the corresponding newform, F, is such that $\frac{L(F,1)}{\pi(F)} \neq 0$, those curves, E, for which $\alpha(E) = \pi(F)$ are circled.

(ii) For each curve, E, in (i) we list the numbers defined in (1.1.1); namely the local index, $c_\wp$, for primes $\wp$ which divide $\mathbf{a}$, the conductor of E; the order of the torsion group $T = E_{tors}(\mathrm{K})$; the quotient $\frac{\prod c_\wp}{|T|^2}$; and, for those curves which do not have complex multiplication, the complex period $\alpha(E)$ (rounded to 8 decimal places). Those curves, for which the ratio S (defined in (1.1.2)) is 4, are marked with an asterix. In the $\mathbf{Q}(\sqrt{-19})$ tables there are two cases where $S = 9$ (marked with a dagger) and one where $S = 16$ (marked with a double dagger). These are the curves (14) A, (14) B and (15) D respectively. No other values of S have been found.

(iii) For each newform, F, in $V^{+}(\mathbf{a})$, which does not correspond to a curve with complex multiplication, we give the quotient $\frac{L(F,1)}{\pi(F)}$. In those cases where this quotient is non-zero we also list the values of $L(F,1)$ and $\pi(F)$ (again rounded to 8 decimal places).

§2.2 The Results for $\mathbf{Q}(\sqrt{-1})$

We give the following tables:

Table (2.2.1): Isogeny classes of elliptic curves with conductor $\mathbf{a}$, $N(\mathbf{a}) < 500$.

Table (2.2.2): The Birch, Swinnerton-Dyer Numbers.

Table (2.2.3): Newforms in $V^{+}(\mathbf{a})$, $N(\mathbf{a}) < 500$.

Table (2.2.1): Elliptic Curves with Small Conductor over $Q(\sqrt{-1})$

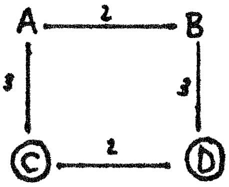
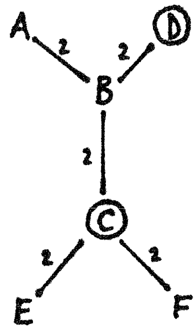
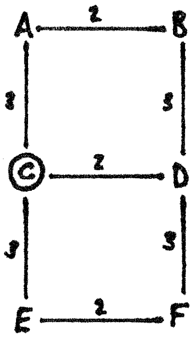
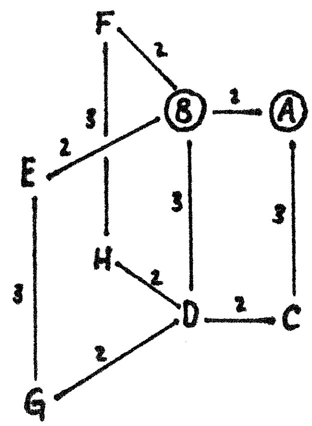
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(3-4i)$	A	$1+i$	i	i	0	0	A
(8)	A	0	0	0	-1	0	
	B	0	0	0	-11	14	
$(7+4i)$	A	$1+i$	0	1	0	0	
	B	$1+i$	$1+i$	i	$-1+i$	-1	
	C	$-1-i$	0	-1	$-5-5i$	$-5-2i$	
	D	$-1-i$	$1+i$	$-i$	$4+16i$	$-29-3i$	
$(6+6i)$	A	0	1	0	1	0	
	B	$1+i$	$-i$	$1+i$	$1-i$	$-i$	
	C	$1+i$	0	$1+i$	$6-i$	$-5i$	
	D	$1+i$	0	$1+i$	$16-i$	$27i$	
	E	$1+i$	$-i$	$1+i$	$-4-i$	$22i$	
	F	$1+i$	$-i$	$1+i$	$96-i$	$346i$	
$(7+7i)$	A	i	0	i	0	0	
	B	1	0	1	-11	12	
	C	i	0	i	5	6	
	D	1	0	1	-36	-70	
	E	i	0	i	-170	874	
	F	1	0	1	-2731	-55146	
(10)	A	0	1	0	-1	0	
	B	$1+i$	$-i$	$1+i$	$-1-i$	0	
	C	0	1	0	-41	-116	
	D	$-1-i$	0	$-1-i$	$9-i$	$17i$	
	E	$-1-i$	$-i$	$-1-i$	$-11+4i$	$-12+11i$	
	F	$-1+i$	i	$-1+i$	$-11-4i$	$-12-11i$	
	G	$1+i$	0	$1+i$	$-1+54i$	$118+118i$	
	H	$1-i$	0	$1-i$	$-1-54i$	$118-118i$	

Table (2.2.1) (continued)

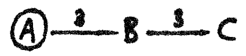
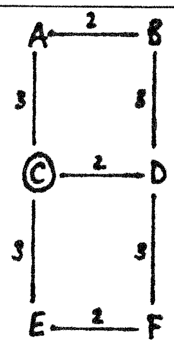
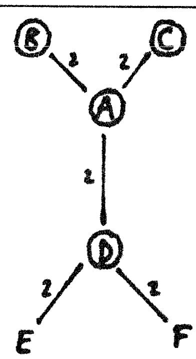
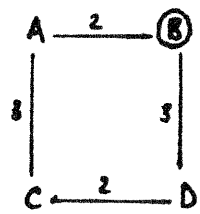
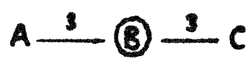
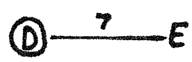
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(9+5i)$	A	1	$-1+i$	$1+i$	$-1-i$	0	
	B	-1	$-1+i$	$-1-i$	$-31-51i$	$30+174i$	
	C	-1	$-1+i$	$-1-i$	$14-76i$	$345+225i$	
(11)	A	0	-1	1	0	0	
	B	0	-1	1	-10	-20	
	C	0	-1	1	-7820	-263580	
$(9+7i)$	A	i	$1-i$	i	$-i$	0	
	B	-1	$-1+i$	-1	$-6-6i$	$-8i$	
	C	$-i$	$1-i$	$-i$	$5+9i$	$18+2i$	
	D	-1	$-1+i$	-1	$14-i$	$-30-30i$	
	E	$-i$	$1-i$	$-i$	$-50+89i$	$14-386i$	
	F	$-i$	$1-i$	$-i$	$-130+9i$	$-882-688i$	
$(12+4i)$	A	0	$-1-i$	0	i	$-1-i$	
	B	0	$-1+i$	0	$1+2i$	$-3-i$	
	C	$1-i$	$1+i$	$1-i$	$-5+4i$	$2-3i$	
	D	$-1-i$	1	0	$-2i$	$1-2i$	
	E	$-1-i$	1	0	$-5-37i$	$53-88i$	
	F	$-1-i$	1	0	$5-7i$	$-7-4i$	
$(10+8i)$	A	$1+i$	$1+i$	0	i	0	
	B	$-1-i$	$1+i$	0	$-4i$	$6-4i$	
	C	$-1-i$	$1+i$	0	$-10+21i$	$-18-38i$	
	D	$-1-i$	$1+i$	0	$-10+26i$	$-10-20i$	
$(13+5i)$	A	1	$-1+i$	$-i$	$-4-3i$	$-4i$	
	B	-1	$-1+i$	$-i$	$1-24i$	$-29+33i$	
	C	$-i$	$1-i$	-1	$306+56i$	$640-2126i$	
	D	$-i$	i	$-i$	$-i$	1	
	E	i	i	i	$-35+24i$	$-19-180i$	

Table (2.2.1) (continued)

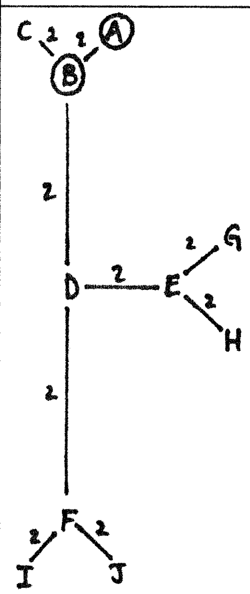
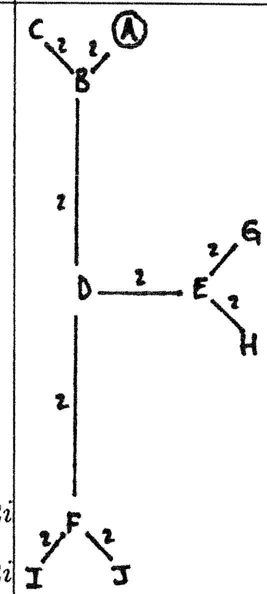
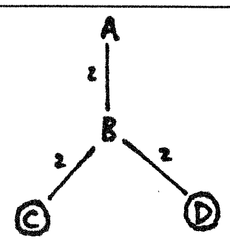
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(10+10i)$	A	0	0	0	-2	-1	
	B	$-1-i$	i	$-1-i$	$1-i$	i	
	C	$-1-i$	i	$-1-i$	$26-i$	$66i$	
	D	$-1-i$	i	$-1-i$	$-4-i$	$2i$	
	E	$-1-i$	i	$-1-i$	$-44-31i$	$106+94i$	
	F	$1+i$	i	0	$-44+30i$	$76-138i$	
	G	$-1-i$	i	$-1-i$	$-34-61i$	$240-48i$	
	H	$-1-i$	i	$-1-i$	$-694-481i$	$6036+7084i$	
	I	$1+i$	i	$1+i$	$-34+59i$	$-240-48i$	
	J	$1+i$	i	$1+i$	$-694+479i$	$-6036+7084i$	
(15)	A	1	1	1	0	0	
	B	i	-1	i	-4	-2	
	C	1	1	1	-80	242	
	D	1	1	1	-10	-10	
	E	i	-1	i	-134	660	
	F	i	-1	0	36	28	
	G	1	1	1	-2160	-39540	
	H	1	1	1	-110	-880	
	I	1	1	1	$395-105i$	$-1054-2982i$	
	J	1	1	1	$395+105i$	$-1054+2982i$	
$(13+8i)$	A	0	$1-i$	i	i	0	A
(16)	A	0	0	0	11	$14i$	A —2— B —2— C
	B	0	0	0	1	0	
	C	0	0	0	11	$-14i$	
$(16+i)$	A	1	i	$-i$	i	0	
	B	i	$-i$	1	5	$-1+3i$	
	C	1	i	i	$85+5i$	$34-274i$	
	D	1	i	i	$5-5i$	$12-4i$	

Table (2.2.1) (continued)

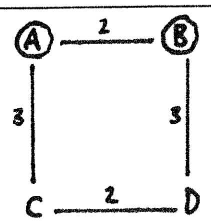
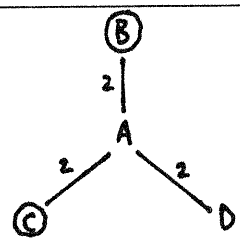
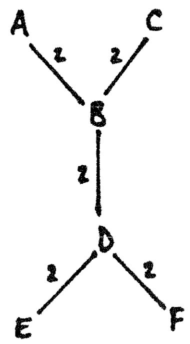
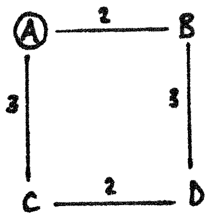
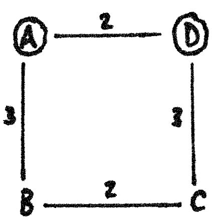
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(16+3i)$	A	$1-i$	0	$-i$	$-1+i$	$-i$	
	B	$1-i$	$-1-i$	1	$-4+2i$	$3i$	
	C	$1+i$	$-1+i$	i	$22-26i$	$69+41i$	
	D	$1+i$	$-i$	i	$-86+95i$	$192-520i$	
$(16+4i)$	A	$1-i$	$1+i$	$1-i$	$-1+i$	-1	
	B	$1+i$	$-1-i$	$1+i$	$-21-6i$	$43+19i$	
	C	$1+i$	$-1-i$	$1+i$	$-1+4i$	$3+3i$	
	D	0	$-1-i$	0	$2i$	1	
$(14+9i)$	A	1	$1+i$	$1+i$	i	0	A
(17)	A	i	1	i	0	0	
	B	1	-1	1	-6	-4	
	C	i	1	i	-90	310	
	D	i	1	i	0	14	
	E	1	-1	1	$39+75i$	$-304-38i$	
	F	1	-1	1	$39-75i$	$-304+38i$	
$(17+i)$	A	1	1	$-i$	1	1	
	B	i	-1	1	$11-11i$	$-25-4i$	
	C	i	-1	1	$21+29i$	$-91-2i$	
	D	1	1	i	$16+24i$	$121-10i$	
$(17+5i)$	A	1	$1+i$	i	0	0	A
(18)	A	0	0	0	0	1	
	B	0	0	0	0	-27	
	C	$1+i$	i	$1+i$	$33-i$	$-58i$	
	D	$1+i$	i	0	3	$-i$	

Table (2.2.1) (continued)

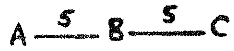
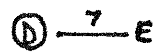
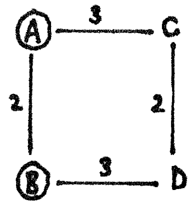
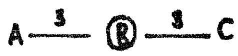
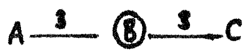
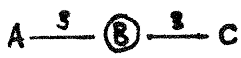
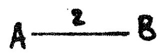
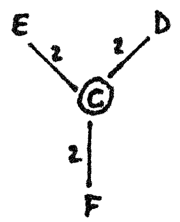
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(13+13i)$	A	1	0	1	0	0	
	B	i	0	i	-4	8	
	C	-1	0	-1	-460	-3830	
	D	i	1	i	-2	-3	
	E	i	1	i	-212	1257	
$(14+12i)$	A	0	i	0	$-1+2i$	$-1-i$	
	B	$1+i$	$-1+i$	0	$1-4i$	$-1+2i$	
	C	0	i	0	$19-18i$	$-41-13i$	
	D	$1+i$	$1+i$	$1+i$	$-19+7i$	$18-17i$	
(19)	A	0	1	1	1	0	
	B	0	-1	i	-9	15	
	C	0	1	-1	-769	-8470	
$(19+i)$	A	1	$-1+i$	1	i	0	
	B	i	$1-i$	i	$-4-9i$	$-4-2i$	
	C	-1	$-1+i$	-1	$-220-449i$	$646-4232i$	
$(16+11i)$	A	0	$-i$	i	i	1	
	B	0	i	1	$10-9i$	$-14-6i$	
	C	0	$-i$	i	$-70+i$	$183+84i$	
$(14+14i)$	A	$1+i$	$-i$	$1+i$	$-i$	0	
	B	$1+i$	0	$1+i$	$10-i$	$-11i$	
	C	$1+i$	i	$1+i$	$4-i$	$-2i$	
	D	$1+i$	i	$1+i$	$-1-i$	$-i$	
	E	$1+i$	i	0	14	$-10i$	
	F	$1+i$	i	0	74	$286i$	

Table (2.2.1) (continued)

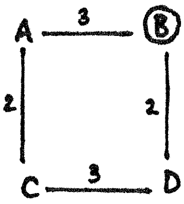
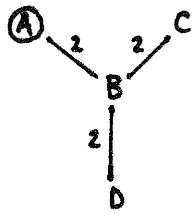
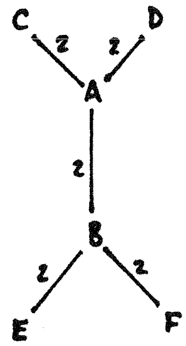
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(17+11i)$	A	1	$1-i$	1	$-i$	0	
	B	i	$-1+i$	i	$-4-6i$	$2-6i$	
	C	-1	$-1-i$	$-i$	$55-67i$	$-31+57i$	
	D	$-i$	$1+i$	-1	$-10+13i$	$-2-26i$	
	E	$-i$	$1+i$	-1	$695-707i$	$-10721-4409i$	
	F	-1	$-1-i$	$-i$	$-30-7i$	$82+82i$	
$(19+7i)$	A	1	$-i$	$-i$	$1+i$	$-i$	
	B	i	i	1	21	$-4+45i$	
	C	1	$-i$	i	$341+10i$	$146-2539i$	
	D	1	$-i$	i	$21-10i$	$38-47i$	
$(19+8i)$	A	$1+i$	$-1-i$	1	$-19+4i$	$-4+13i$	
	B	$-1-i$	$1-i$	-1	$61-20i$	$104+161i$	
(21)	A	1	0	0	-4	-1	
	B	i	0	0	-49	136	
	C	i	0	0	-39	-90	
	D	1	0	0	1	0	
	E	1	0	0	-784	-8515	
	F	1	0	0	-34	-217	
$(19+9i)$	A	i	-1	0	$2-i$	i	
	B	1	1	0	$12-16i$	$38-2i$	
	C	i	-1	0	$7+74i$	$-257-60i$	

Table (2.2.1) (concluded)

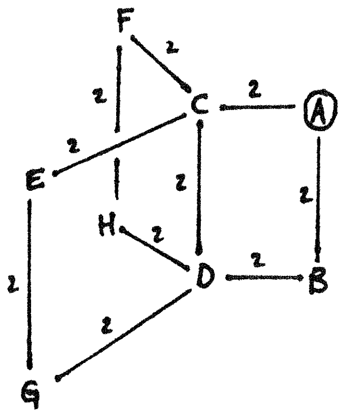
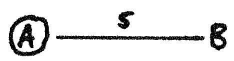
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(15+15i)$	A	1	0	1	1	2	
	B	i	0	i	-13	64	
	C	i	0	i	-18	-26	
	D	1	0	1	-334	-2368	
	E	1	0	1	-289	1862	
	F	1	0	1	-69	-194	
	G	$-i$	0	$-i$	-453	544	
	H	$-i$	0	$-i$	-5333	150368	
$(21+3i)$	A	$-i$	$-1+i$	$1-i$	0	0	
	B	i	$-1+i$	$1+i$	$15+24i$	$35+14i$	
(22)	A	0	$-i$	$1+i$	-1	0	
	B	0	i	$1+i$	19	$-27i$	

Table (2.2.2): The Birch, Swinnerton-Dyer Numbers

$\mathfrak{a}$		$c_{\mathfrak{p}}$	$\mid T \mid$	$\prod c_{\mathfrak{p}}/ \mid T \mid^2$	$\alpha(E)$
$(3-4i)$	A	2	10	1/50	
$(8)=(1+i)^2$	A	4	8	1/16	
	B	1	4	1/16	
$(7+4i)=(2-i)(3-2i)$	A	1, 2	6	1/18	7.65392980
	B	2, 1	6	1/18	7.65392980
	C	1, 6	6	1/6	2.55130993
	D	2, 3	6	1/6	2.55130993
$(6+6i)=(1+i)^3(-3)$	A	4, 1	8	1/16	7.27069404
	B	2, 2	8	1/16	7.27069404
	C	2, 4	8	1/8	3.63534702
	D	2, 1	4	1/8	3.63534702
	E	2, 8	8	1/4	1.81767351
	F	2, 2	4	1/4	1.81767351
$(7+7i)=(1+i)(7)$	A	2, 1	6	1/18	7.87875422
	B	2, 2	6	1/9	3.93937711
	C	2, 3	6	1/6	2.62625141
	D	2, 6	6	1/3	1.31312570
	E	2, 1	2	1/2	0.87541714
	F	2, 2	2	1	0.43770857
$(10)=(1+i)^2(2+i)(2-i)$	A	3, 1, 1	6	1/12	6.42309566
	B	3, 2, 2	12	1/12	6.42309566
	C	1, 1, 1	2	1/4	2.141031886
	D	1, 2, 2	4	1/4	2.141031886
	E	3, 2, 1	6	1/6	3.211547828
	F	3, 1, 2	6	1/6	3.211547828
	G	1, 2, 1	2	1/2	1.070515943
	H	1, 1, 2	2	1/2	1.070515943

Table (2.2.2) (continued)

a		$c_\wp$	$\mid T \mid$	$\prod c_\wp / \mid T \mid^2$	$\alpha(E)$
$(9+5i) = (1+i)(7-2i)$	A	9, 1	9	1/9	5.98534333
	B	3, 1	3	1/3	1.99511444
	C	1, 1	1	1	0.66503815
(11)	A	1	5	1/25	9.25771812
	B	5	5	1/5	1.85154362
	C	1	1	1	0.37030872
$(9+7i) = (1+i)(2+i)(2+3i)$	A	2, 1, 1	6	1/18	8.64653751
	B	1, 2, 2	6	1/9	4.32326875
	C	2, 1, 3	6	1/6	2.88217917
	D	1, 2, 6	6	1/3	1.44108958
	E	2, 1, 1	2	1/2	0.96072639
	F	1, 2, 2	2	1	0.48036139
$(12+4i) = (1+i)^5(2-i)$	A	4, 2	8	1/8	5.25049399
	B	2, 1	4	1/8	5.25049399
	C	2, 1	4	1/8	5.25049399
	D	2, 4	8	1/8	5.25049399
	E	2, 2	4	1/4	2.62524699
	F	2, 8	8	1/4	2.62524699
$(10+8i) = (1+i)^2(5+4i)$	A	3, 1	6	1/12	8.06203675
	B	3, 2	6	1/6	4.03101838
	C	1, 1	2	1/4	2.68734558
	D	1, 2	2	1/2	1.34367292
$(13+5i)$	A	1, 1	3	1/9	5.06720709
	B	1, 3	3	1/3	1.68906903
	C	1, 1	1	1	0.56302301
	D	7, 1	7	1/7	6.31413337
	E	1, 1	1	1	0.90201905

Table (2.2.2) (continued)

a		$c_\wp$	$\mid T \mid$	$\prod c_\wp / \mid T \mid^2$	$\alpha(E)$
$(10+10i) = (1+i)^3(2+i)(2-i)$	A	2, 1, 1	4	1/8	5.99377796
	B	2, 2, 2	8	1/8	5.99377796
	C	4, 1, 1	4	1/4	2.99688898
	D	4, 4, 4	16	1/4	2.99688898
	E	2, 2, 8	8	1/2	1.49844449
	F	2, 2, 8	8	1/2	1.49844449
	G	1, 16, 1	4	1	0.74922225
	H*	1, 4, 1	4	1/4	0.74922225
	I	1, 1, 16	4	1	0.74922225
	J*	1, 1, 4	4	1/4	0.74922225
$(15) = (2+i)(2-i)(3)$	A	1, 1, 1	4	1/16	8.94280685
	B*	2, 2, 2	8	1/8	4.47140343
	C	1, 1, 1	4	1/16	2.23570171
	D	4, 4, 4	16	1/4	2.23570171
	E	2, 2, 8	8	1/2	1.11785086
	F	8, 8, 2	16	1/2	1.11785086
	G*	1, 1, 4	4	1/4	0.55892543
	H	1, 1, 16	4	1	0.55892543
	I	16, 4, 1	8	1	0.55892543
	J	4, 16, 1	8	1	0.55892543
$(13+8i)$	A	1	1	1	8.80694161
$(16) = (1+i)^8$	A	4	4	1/4	3.43759291
	B	2	4	1/8	6.87518582
	C	4	4	1/4	3.43759291
$(16+i)$	A	1	4	1/16	8.70416074
	B	2	4	1/8	4.35208037
	C	1	2	1/4	2.17604018
	D	4	4	1/4	2.17604018

Table (2.2.2) (continued)

$\mathbf{a}$		$c_{\mathfrak{p}}$	$\mid T \mid$	$\prod c_{\mathfrak{p}} / \mid T \mid^2$	$\alpha(E)$
$(16+3i)=(2+i)(-7+2i)$	A	6, 1	6	1/6	4.46467996
	B	3, 2	6	1/6	4.46467996
	C	1, 2	2	1/2	1.48822665
	D	2, 1	2	1/2	1.48822665
$(16+4i)=(1+i)^4(1-4i)$	A	1, 2	4	1/8	6.46558815
	B	1, 1	2	1/4	3.23279407
	C	1, 4	4	1/4	3.23279407
	D	2, 1	4	1/8	6.46558815
$(14+9i)$	A	1	1	1	8.42019371
$(17)=(4+i)(4-i)$	A	1, 1	4	1/16	8.49575480
	B	2, 2	8	1/16	4.24787740
	C	1, 1	4	1/16	2.12393870
	D	4, 4	8	1/4	2.12393870
	E	2, 8	4	1	1.06196935
	F	8, 2	4	1	1.06196935
$(17+i)=(1+i)(2+i)(2-5i)$	A	6, 1, 1	6	1/6	6.44651933
	B	3, 2, 2	6	1/3	3.22325966
	C	2, 1, 1	2	1/2	2.14883978
	D	1, 2, 2	2	1	1.07441989
$(17+5i)=(1+i)(6+11i)$	A	2, 1	1	2	8.14466140
$(18)=(1+i)^2(3)^2$	A	3, 2	6	1/6	5.10811572
	B	1, 2	2	1/2	1.70270524
	C	1, 2	2	1/2	1.70270524
	D	3, 2	6	1/6	5.10811572

Table (2.2.2) (continued)

a		$c_\wp$	$\mid T \mid$	$\prod c_\wp / \mid T \mid^2$	$\alpha(E)$
$(13+13i) = (1+i)(3+2i)(2+3i)$	A	$2,1,1$	3	2/9	8.07240718
	B	$2,3,3$	3	2	2.69080239
	C	$2,1,1$	1	2	0.89693413
	D	$14,1,1$	7	2/7	3.92089952
	E	$2,1,1$	1	2	0.56012850
$(14+12i) = (1+i)^2(2+i)(-1+4i)$	A	$3,2,1$	6	1/6	5.08070712
	B	$3,1,2$	6	1/6	5.08070712
	C	$1,2,1$	2	1/2	1.69356904
	D	$1,1,2$	2	1/2	1.69356904
(19)	A	1	3	1/9	8.41778108
	B	3	3	1/3	2.80592703
	C	1	1	1	0.93530901
$(19+i) = (1+i)(9+10i)$	A	$1,1$	3	1/9	6.68341583
	B	$1,3$	3	1/3	2.22780528
	C	$1,1$	1	1	0.74260176
$(16+11i) = (3-2i)(-5+2i)$	A	$1,1$	3	1/9	7.77446571
	B	$3,1$	3	1/3	2.59148857
	C	$1,1$	1	1	0.86382952
$(14+14i) = (1+i)^3(7)$	A	$4,1$	2	1	6.39573929
	B	$2,2$	2	1	3.19786964
	C	$2,2$	4	1/4	4.01629572
	D	$2,1$	4	1/8	8.03259144
	E	$2,4$	4	1/2	2.00814786
	F	$2,1$	2	1/2	2.00814786

Table (2.2.2) (continued)

a		$c_\wp$	$\mid T \mid$	$\prod c_\wp / \mid T \mid^2$	$\alpha(E)$
$(17+11i)=(1+i)(2+i)(-5+4i)$	A	2, 2, 1	2	1	6.98271032
	B	1, 2, 2	2	1	3.49135516
	C	2, 2, 1	2	1	0.73471898
	D	2, 6, 1	6	1/3	2.20415695
	E	1, 4, 2	2	2	0.36735949
	F	1, 12, 2	6	2/3	1.10207847
$(19+7i)=(1+i)(2+i)(5+4i)$	A	8, 2, 1	8	1/4	4.89867831
	B	4, 4, 2	8	1/2	2.44933915
	C*	2, 2, 1	4	1/4	1.22466958
	D	2, 8, 4	8	1	1.22466958
$(19+8i)=(2-i)(4-i)$	A	2, 2	2	1	1.58750807
	B	2, 2	2	1	1.58750807
$(21)=(3)(7)$	A	4, 2	8	1/8	3.44830772
	B	2, 4	8	1/8	1.72415386
	C	8, 1	8	1/8	1.72415386
	D	2, 1	4	1/8	6.89661544
	E	1, 2	4	1/8	0.86207693
	F	1, 8	4	1/2	0.86207693
$(19+9i)=(1+i)(3-2i)(4+i)$	A	1, 1, 1	3	1/9	7.00216530
	B	1, 3, 1	4	1/3	2.33405510
	C	1, 1, 1	1	1	0.77801837

Table (2.2.2) (concluded)

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp} / \mid T \mid^2$	$\alpha(E)$
$(15+15i) = (1+i)(2+i)(2-i)(3)$	A	$2,1,1,3$	6	1/6	3.88287042
	B	$2,1,1,1$	2	1/2	1.29429014
	C	$2,2,2,6$	12	1/3	1.94143521
	D	$2,2,2,2$	4	1	0.64714507
	E	$2,2,2,3$	6	2/3	0.97071761
	F	$2,1,1,12$	6	2/3	0.97071761
	G	$2,2,2,1$	2	2	0.32357254
	H	$2,1,1,4$	2	2	0.32357254
$(21+3i) = (1+i)(2+i)^2(3)$	A	$5,1,1$	5	1/5	6.61905689
	B	$1,1,1$	1	1	1.32381138
$(22) = (1+i)^2(-11)$	A	$3,1$	3	1/3	7.37233759
	B	$1,3$	1	3	2.45744586

Table (2.2.3): Rational Newforms in $V^{+}(\mathrm{a})$; $\mathbf{K} = \mathbf{Q}(\sqrt{-1})$

a	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
$(7+4i)$	1/6	0.42521832	2.55130993
$(6+6i)$	1/8	0.45441838	3.63534702
$(7+7i)$	1/6	0.43770857	2.62625141
(10)	1/12	0.53525797	6.42309566
$(9+5i)$	1/9	0.66503815	5.98534333
(11)	1/5	0.37030872	1.85154362
$(9+7i)$	1/6	0.48036319	2.88217917
$(12+4i)$	1/8	0.65631175	5.25049399
$(10+8i)$	1/6	0.67183640	4.03101838
$(13+5i)a$	1/3	0.56302301	1.68906903
$(13+5i)b$	1/7	0.90201905	6.31413337
$(10+10i)$	1/8	0.74922225	5.99377796
(15)	1/16	0.55892543	8.94280685
$(13+8i)$	0		
(16)	0		
$(16+i)$	1/4	0.54401000	2.17604018
$(16+3i)$	1/6	0.74411333	4.46467996
$(16+4i)$	1/4	0.80819852	3.23279407
$(14+9i)$	0		
(17)	0		
$(17+i)$	1/6	1.07441989	6.44651933
$(17+5i)$	0		
(18)	1/6	0.85135262	5.10811572
$(13+13i)a$	0		
$(13+13i)b$	2/7	1.12025701	3.92089952

Table (2.2.3) (concluded)

a	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
$(14+12i)$	$1/6$	0.84678452	5.08070712
(19)	$1/3$	0.93530901	2.80592703
$(19+i)$	$1/3$	0.74260176	2.22780528
$(16+11i)$	$1/3$	0.86382952	2.59148857
$(14+14i)a$	0		
$(14+14i)b$	$1/4$	1.00407393	4.01629572
$(17+11i)a$	0		
$(17+11i)b$	$1/3$	0.73471898	2.20415695
$(19+7i)$	$1/4$	1.22466958	4.89867831
$(19+8i)$	0		
(21)	0		
$(19+9i)$	$1/3$	0.77801837	2.33405510
$(15+15i)$	$1/6$	0.64714507	3.88287042
$(21+3i)$	$1/5$	1.32381138	6.61905689
(22)	0		

§2.3 The Results for $\mathbf{Q}(\sqrt{-2})$

We give the following tables:

Table (2.3.1): Isogeny classes of elliptic curves with conductor $\mathbf{a}$, $N(\mathbf{a}) < 300$.

Table (2.3.2): The Birch, Swinnerton-Dyer Numbers.

Table (2.3.3): Newforms in $V^+(\mathbf{a})$, $N(\mathbf{a}) < 300$.

Table (2.3.1): Elliptic Curves with Small Conductor over $\mathbb{Q}(\sqrt{-2})$

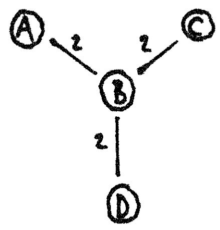
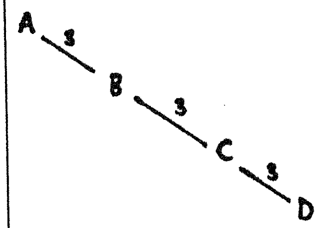
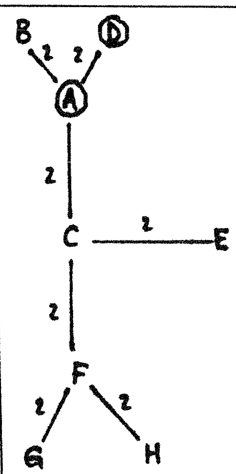
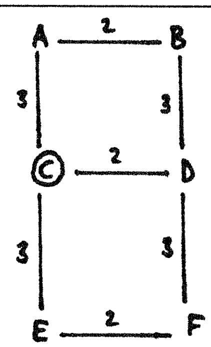
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(1-2\theta)$	A	θ	$1-\theta$	1	-1	0	A
(4θ)	A	0	0	0	1	0	
	B	0	0	0	-1	0	
	C	θ	-1	0	-2	3	
	D	θ	-1	θ	-1	0	
$(7+\theta)$	A	$-\theta$	$-1+\theta$	1	0	0	
	B	θ	$-1+\theta$	$1+\theta$	$2-2\theta$	0	
$(2+5\theta)$	A	$1-\theta$	θ	1	-1	0	
	B	1	$1-\theta$	0	$-1-\theta$	-1	
	C	1	$1-\theta$	0	$4-11\theta$	-26	
	D	1	$1-\theta$	0	$69-51\theta$	$339+62\theta$	
(6θ)	A	θ	1	θ	0	0	
	B	θ	1	θ	-15	-27	
	C	θ	1	θ	-5	5	
	D	0	-1	0	1	0	
	E	θ	1	θ	-95	347	
	F	θ	1	θ	5	23	
	G	θ	1	θ	$85-70\theta$	$559+98\theta$	
	H	θ	1	θ	$85+70\theta$	$559-98\theta$	
(7θ)	A	1	0	1	-1	0	
	B	1	0	1	-11	12	
	C	1	0	1	4	-6	
	D	1	0	1	-36	-70	
	E	-1	0	-1	-171	-874	
	F	-1	0	-1	-2731	-55146	

Table (2.3.1) (continued)

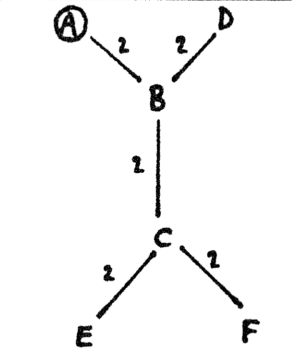
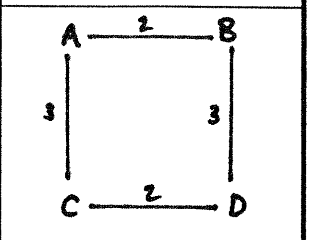
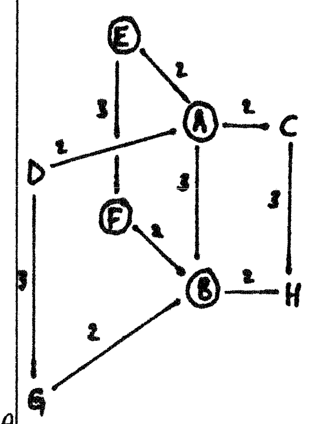
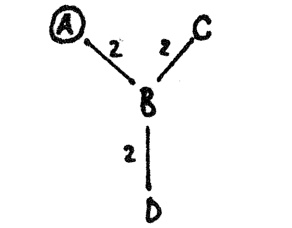
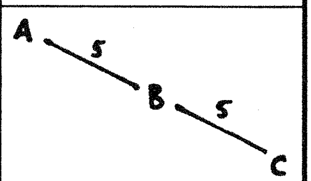
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(9+3\theta)$	A	$1+\theta$	$1-\theta$	$1+\theta$	-2θ	$-\theta$	
	B	$1+\theta$	$1-\theta$	$1+\theta$	$-5-2\theta$	$4+\theta$	
	C	$1+\theta$	$1-\theta$	$1+\theta$	8θ	$22+19\theta$	
	D	$1+\theta$	$1-\theta$	$1+\theta$	$-90-12\theta$	$302+71\theta$	
	E	$1+\theta$	$1-\theta$	$1+\theta$	$-15+173\theta$	$1006+859\theta$	
	F	$1+\theta$	$1-\theta$	$1+\theta$	$95+3\theta$	$-30+251\theta$	
(10)	A	0	1	0	-1	0	
	B	θ	0	θ	2	0	
	C	0	1	0	-41	-116	
	D	θ	0	θ	-8	18	
$(6+6\theta)$	A	θ	$1-\theta$	θ	$4-3\theta$	$4-2\theta$	
	B	θ	$1-\theta$	0	$-1-4\theta$	$-7+\theta$	
	C	θ	$1-\theta$	θ	$-21+2\theta$	$37+13\theta$	
	D	θ	$1-\theta$	θ	$49-48\theta$	$265+7\theta$	
	E	0	$-1-\theta$	0	9	$-2+5\theta$	
	F	0	$-1-\theta$	0	$-7-4\theta$	$4+11\theta$	
	G	θ	$1-\theta$	0	$14+11\theta$	$-19+16\theta$	
	H	θ	$1-\theta$	0	$4-59\theta$	$-261+122\theta$	
$(4+7\theta)$	A	1	$-\theta$	θ	$-\theta$	0	
	B	1	$-\theta$	$10-\theta$	$2+6\theta$	0	
	C	1	$-\theta$	θ	$170-6\theta$	$68+549\theta$	
	D	1	$-\theta$	θ	$10+4\theta$	$24+7\theta$	
(11)	A	0	-1	1	0	0	
	B	0	-1	1	-10	-20	
	C	0	-1	1	-7820	-26358	
$(7-6\theta)$	A	θ	1	1	$3-3\theta$	$-3-\theta$	A

Table (2.3.1) (continued)

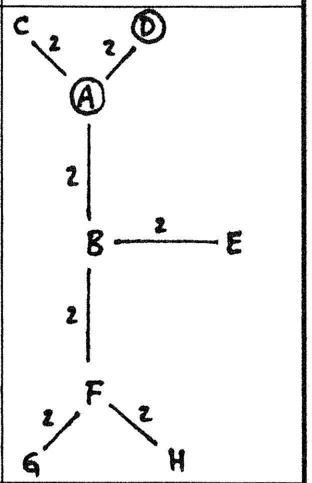
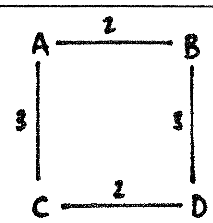
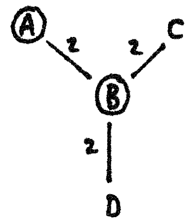
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
(12)	A	θ	0	θ	0	1	
	B	θ	0	θ	-5	-4	
	C	θ	0	θ	-15	28	
	D	0	1	0	1	0	
	E	θ	0	θ	-95	-346	
	F	θ	0	θ	5	-22	
	G	θ	0	θ	$85+70\theta$	$-558+98\theta$	
	H	θ	0	θ	$85-70\theta$	$-558-98\theta$	
$(6+8\theta)$	A	$-\theta$	$1-\theta$	$-\theta$	2	$1-\theta$	
	B	0	$-1-\theta$	0	$1+2\theta$	$2-\theta$	
$(4+9\theta)$	A	$1-\theta$	$-1-\theta$	$1-\theta$	$-1+2\theta$	$1+\theta$	
	B	$1+\theta$	$-1+\theta$	1	$8-17\theta$	$55+6\theta$	
$(12+5\theta)$	A	1	θ	1	-1	0	A
$(10+7\theta)$	A	1	$1-\theta$	$-\theta$	$2+\theta$	$2-2\theta$	
	B	1	$1-\theta$	θ	$42+10\theta$	$86-80\theta$	
	C	$1+\theta$	0	$1+\theta$	$1+\theta$	$2+\theta$	
	D	$1+\theta$	0	$1+\theta$	$-19-4\theta$	$38+\theta$	
(10θ)	A	0	0	0	-2	1	
	B	θ	-1	θ	0	2	
	C	θ	-1	θ	-25	67	
	D	θ	-1	θ	5	3	
$(11+7\theta)$	A	θ	θ	$1+\theta$	θ	0	
	B	θ	θ	$1+\theta$	$-\theta$	$-\theta$	

Table (2.3.1) (continued)

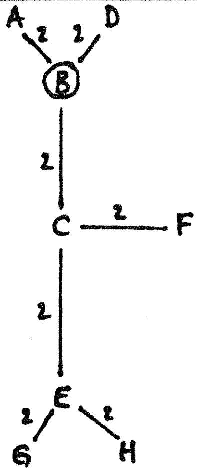
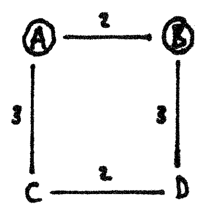
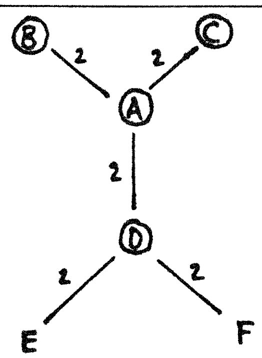
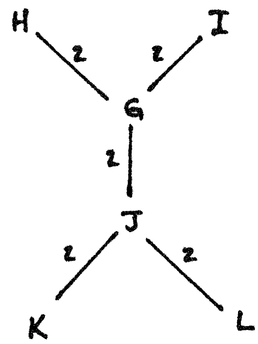
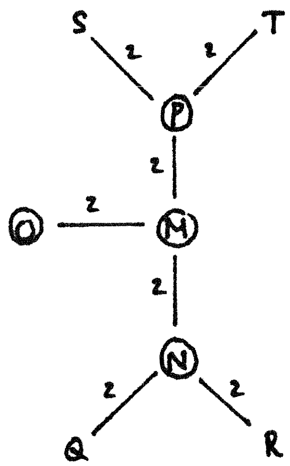
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
(15)	A	1	1	1	0	0	
	B	1	1	1	−5	2	
	C	1	1	1	−10	−10	
	D	1	1	1	−80	242	
	E	1	1	1	−135	−660	
	F	1	1	1	35	−28	
	G	1	1	1	−2160	−39540	
	H	1	1	1	−110	−880	
$(5+10\theta)$	A	θ	−1	$1+\theta$	$2-3\theta$	$5-\theta$	A —2— B
	B	θ	−1	1	$8+2\theta$	$-9+4\theta$	
$(12+7\theta)$	A	$-1-\theta$	θ	−1	$13+9\theta$	$40+10\theta$	Ⓐ
$(2+11\theta)$	A	$1-\theta$	0	$1-\theta$	$-\theta$	θ	A
$(7+10\theta)$	A	1	$-1+\theta$	1	$-1+\theta$	$-2-\theta$	A
(16)	A	0	θ	0	1	θ	A
	B	0	$-\theta$	0	1	$-\theta$	B
	C	0	−1	0	−2	2	Ⓒ —2— Ⓓ
	D	0	−1	0	1	$-\theta$	
	E	0	1	0	−2	−2	E —2— F
	F	0	1	0	1	θ	
$(16+\theta)$	A	$1-\theta$	$-\theta$	$-\theta$	1	$-1-\theta$	Ⓐ —2— B
	B	$1+\theta$	θ	0	4θ	$-12-\theta$	
$(16+2\theta)$	A	$-\theta$	1	$-\theta$	$-1-2\theta$	$-1-2\theta$	Ⓐ —2— B
	B	θ	1	θ	$-1+8\theta$	$-1-12\theta$	
$(5+11\theta)$	A	$-\theta$	0	$1-\theta$	$-\theta$	$-1-2\theta$	
	B	θ	0	$1+\theta$	$8-4\theta$	$7+4\theta$	
	C	θ	0	$1+\theta$	$-170-157\theta$	$-332-1222\theta$	
	D	θ	0	$1+\theta$	$-42-39\theta$	$48+165\theta$	

Table (2.3.1) (continued)

a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
(12θ)	A	0	1	0	-2	0	
	B	0	1	0	-4	2	
	C	θ	0	θ	-7	-7	
	D	θ	0	0	2	-1	
	E	θ	0	0	$22+5\theta$	$-23+26\theta$	
	F	θ	0	0	$22-5\theta$	$-23-26\theta$	
	G	0	-1	0	-2	0	
	H	θ	1	θ	-7	8	
	I	θ	-1	0	-4	-2	
	J	θ	1	0	2	1	
	K	θ	1	0	$22-5\theta$	$23+26\theta$	
	L	θ	1	0	$22+5\theta$	$23-26\theta$	
	M	0	θ	0	1	3θ	
	N	θ	$-1+\theta$	θ	$-1-6\theta$	$8-3\theta$	
	O	0	θ	0	11	-7θ	
	P	θ	$-1+\theta$	0	$-2+4\theta$	$-5-8\theta$	
	Q	θ	$-1+\theta$	θ	$-31-81\theta$	$248-294\theta$	
	R	θ	$-1+\theta$	θ	$-11-11\theta$	$28+16\theta$	
	S	θ	$-1+\theta$	0	$-12+9\theta$	$-15+6\theta$	
	T	θ	$-1+\theta$	0	$-32+79\theta$	$-215-374\theta$	
	U	0	$-\theta$	0	1	-3θ	
	V	θ	$-1-\theta$	0	$-2-4\theta$	$-5+8\theta$	
	W	θ	$-1-\theta$	θ	$-1+6\theta$	$8+3\theta$	
	X	0	$-\theta$	0	11	7θ	
	Y	θ	$-1-\theta$	0	$-12-9\theta$	$-15-6\theta$	
	Z	θ	$-1-\theta$	0	$-32-79\theta$	$-215+374\theta$	
	AA	θ	$-1-\theta$	θ	$-31+81\theta$	$248+294\theta$	
	BB	θ	$-1-\theta$	θ	$-11+11\theta$	$28-16\theta$	







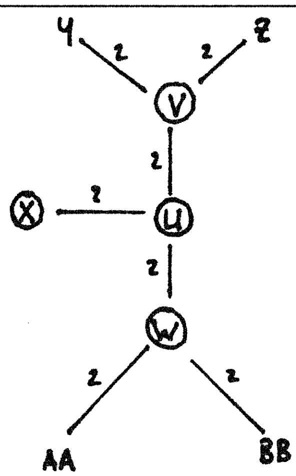


Table (2.3.1) (concluded)

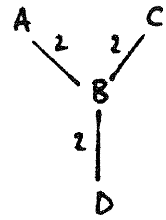
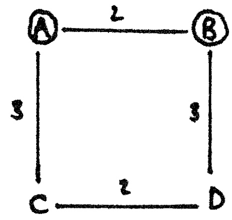
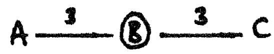
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
(17)	A	1	-1	1	-1	0	
	B	1	-1	1	-6	-4	
	C	1	-1	1	-91	-310	
	D	1	-1	1	-1	-14	
$(15+6\theta)$	A	θ	$1-\theta$	$1+\theta$	$-1-\theta$	1	
	B	θ	$1-\theta$	$1+\theta$	$1+\theta$	$-\theta$	
	C	θ	$1-\theta$	$1+\theta$	$-1-46\theta$	$104-109\theta$	
	D	θ	$1-\theta$	$1+\theta$	$1-14\theta$	$-17+7\theta$	
$(13+8\theta)$	A	0	$-1+\theta$	1	$1-\theta$	θ	
	B	0	$-1+\theta$	1	$11-11\theta$	$-27+3\theta$	
	C	0	$-1+\theta$	1	$-9+39\theta$	$-210-39\theta$	
	D	0	$1-\theta$	1	$2-2\theta$	1	
$(3+12\theta)$	A	θ	$1-\theta$	$1+\theta$	-3θ	$1-2\theta$	
	B	θ	$1-\theta$	1	$-6-2\theta$	-7	

Table (2.3.2): The Birch, Swinnerton-Dyer Numbers

$\mathfrak{a}$		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp}/\mid T\mid^2$	$\alpha(E)$
$(1-2\theta)=(1+\theta)^2$	A	2	6	1/18	
$(4\theta)=(\theta)^5$	A	2	4	1/8	6.87518582
	B	2	4	1/8	6.87518582
	C	2	4	1/8	6.87518582
	D	2	4	1/8	6.87518582
$(7+\theta)=(1+\theta)(3-2\theta)$	A	2,2	2	1	7.12241132
	B	2,1	2	1/2	7.12241132
$(2+5\theta)=(\theta)(1+\theta)^3$	A	1,1	3	1/9	8.20656212
	B	1,1	3	1/9	8.20656212
	C	1,3	3	1/3	2.73552071
	D	1,1	1	1	0.91184024
$(6\theta)=(\theta)^3(1+\theta)(1-\theta)$	A	2,2,2	8	1/8	7.27069403
	B	4,1,1	4	1/4	3.63534702
	C	4,2,2	8	1/4	3.63534702
	D	2,1,1	4	1/8	7.27069404
	E	2,2,2	4	1/2	1.81767351
	F	2,2,2	4	1/2	1.81767351
	G	1,2,2	2	1	0.90883675
	H	1,2,2	2	1	0.90883675
$(7\theta)=(7)(\theta)$	A	2,1	6	1/18	7.87875422
	B	2,2	6	1/9	3.93937711
	C	2,3	6	1/6	2.62625141
	D	2,6	6	1/3	1.31312570
	E	2,1	2	1/2	0.87541735
	F	2,2	2	1	0.43770857

Table (2.3.2) continued

a		$c_{\mathfrak{p}}$	$\mid T \mid$	$\prod c_{\mathfrak{p}} / \mid T \mid^2$	$\alpha(E)$
$(9+3\theta) = (1+\theta)(1-\theta)(3+\theta)$	A	$1,2,1$	4	1/8	6.78927365
	B	$2,4,2$	8	1/4	3.39463683
	C	$2,2,2$	4	1/2	1.69731841
	D	$1,8,1$	4	1/2	1.69731841
	E	$2,1,2$	2	1	0.84865921
	F	$2,1,2$	2	1	0.84865921
$(10) = (\theta)^2(5)$	A	$3,1$	6	1/12	6.42309566
	B	$3,2$	6	1/6	6.42309566
	C	$1,3$	2	3/4	2.14103189
	D	$1,6$	2	3/2	2.14103189
$(6+6\theta) = (\theta)^2(1+\theta)^2(1-\theta)$	A	$3,4,6$	12	1/2	2.97663388
	B	$1,4,2$	4	1/2	2.97663387
	C	$3,4,12$	12	1	1.48831694
	D	$3,4,3$	6	1	1.48831694
	E	$3,2,3$	6	1/2	2.97663387
	F	$1,2,1$	2	1/2	2.97663387
	G	$1,4,1$	2	1	1.48831694
	H	$1,4,4$	4	1	1.48831694
$(4+7\theta) = (\theta)(1+\theta)(1-3\theta)$	A	$4,1,1$	4	1/4	6.54173722
	B	$2,2,2$	4	1/2	3.27086861
	C*	$1,1,1$	2	1/4	1.63543430
	D	$1,2,2$	2	1	1.63543430
$(11) = (3+\theta)(-3+\theta)$	A	$1,1$	5	1/25	9.25771812
	B	$5,5$	5	1	1.85154362
	C	$1,1$	1	1	0.37030872
$(7-6\theta) = (3-\theta)^2$	A	2	2	1/2	3.82617502

Table (2.3.2) (continued)

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp} / \mid T \mid^2$	$\alpha(E)$
$(12) = (\theta)^4(1+\theta)(1-\theta)$	A	$1,2,2$	4	1/4	7.27069404
	B	$2,4,4$	8	1/2	3.63534702
	C	$2,1,1$	2	1/2	3.63534702
	D	$1,1,1$	2	1/4	7.27069404
	E	$4,2,2$	4	1	1.81767351
	F	$4,8,8$	16	1	1.81767351
	G	$2,16,4$	8	2	0.90883675
	H	$2,4,16$	8	2	0.90883675
$(6+8\theta) = (\theta)^2(3+4\theta)$	A	$3,2$	2	3/2	5.66201784
	B	$3,1$	2	3/4	5.66201784
$(4+9\theta) = (\theta)(9-2\theta)$	A	$3,1$	3	1/3	7.85487316
	B	$1,1$	1	1	2.61829105
$(12+5\theta) = (\theta)(-5+6\theta)$	A	$1,1$	1	1	8.96753450
$(10+7\theta) = (\theta)(1+\theta)^2(3+\theta)$	A	$2,4,1$	2	2	3.51379034
	B	$1,4,2$	2	2	1.75689517
	C	$2,4,3$	6	2/3	3.51379034
	D	$1,4,6$	6	2/3	1.75689517
$(10\theta) = (\theta)^3(5)$	A	$4,1$	4	1/4	5.99377796
	B	$2,2$	4	1/4	5.99377796
	C	$2,1$	2	1/2	2.99688898
	D	$2,4$	4	1/2	2.99688898
$(11+7\theta) = (1-\theta)(1-6\theta)$	A	$2,1$	2	1/2	5.69524643
	B	$2,2$	2	1	5.69524643

Table (2.3.2) (continued)

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp} / \mid T \mid^2$	$\alpha(E)$
$(15) = (1+\theta)(1-\theta)(5)$	A	$1,1,1$	4	1/16	8.94280685
	B	$2,2,2$	8	1/8	4.47140343
	C	$2,2,4$	8	1/4	2.23570171
	D$*$	$1,1,1$	4	1/16	2.23570171
	E	$2,2,2$	4	1/2	1.11785086
	F	$2,2,8$	8	1/2	1.11785086
	G	$2,2,1$	2	1	0.55892543
	H	$2,2,1$	2	1	0.55892543
$(5+10\theta) = (1-\theta)^2(5)$	A	$4,1$	2	1	2.92476321
	B	$2,2$	2	1	2.92476321
$(12+7\theta) = (\theta)(3-\theta)^2$	A	$1,1$	1	1	1.36404910
$(2+11\theta) = (\theta)(1+\theta)(3-4\theta)$	A	$2,9,1$	1	18	3.94641060
$(7+10\theta) = (1+\theta)(9+\theta)$	A	$2,1$	1	2	5.87867009
$(16) = (\theta)^8$	A	1	2	1/4	6.34499347
	B	1	2	1/4	6.34499347
	C	2	2	1/2	5.54479401
	D	2	2	1/2	5.54479401
	E	2	2	1/2	5.54479401
	F	4	2	1	5.54479401
$(16+\theta) = (\theta)(1+\theta)(5+3\theta)$	A	$2,1,1$	2	1/2	5.41235865
	B	$1,2,2$	2	1	2.70617933
$(16+2\theta) = (\theta)^3(1-\theta)(3-\theta)$	A	$2,1,1$	2	1/2	3.85581905
	B	$1,2,2$	2	1	1.92790953
$(5+11\theta) = (1+\theta)(9+2\theta)$	A	$12,1$	6	1/3	3.18586463
	B	$6,2$	6	1/3	3.18586463
	C	$4,1$	2	1	1.06195488
	D	$2,2$	2	1	1.06195488

Table (2.3.2) continued

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp} / \mid T \mid^2$	$\alpha(E)$
$(12\theta) = (\theta)^5(1+\theta)(1-\theta)$	A	2, 2, 2	4	1/2	4.69072860
	B	2, 1, 1	2	1/2	4.69072860
	C	2, 1, 1	2	1/2	4.69072860
	D	2, 4, 4	8	1/2	4.69072860
	E	1, 2, 8	4	1	2.34536430
	F	1, 8, 2	4	1	2.34536430
	G	4, 2, 2	8	1/4	4.69072860
	H	2, 1, 1	4	1/8	4.69072860
	I	2, 1, 1	4	1/8	4.69072860
	J	2, 2, 2	4	1/2	4.69072860
	K	2, 2, 2	2	2	2.34536430
	L	2, 2, 2	2	2	2.34536430
	M	4, 2, 4	8	1/2	3.35349283
	N	2, 2, 2	4	1/2	3.35349283
	O	2, 2, 2	4	1/2	3.35349283
	P	2, 2, 8	8	1/2	3.35349283
	Q	2, 2, 1	2	1	1.67674642
	R	2, 2, 1	2	1	1.67674642
	S	1, 1, 16	4	1	1.67674642
	T*	1, 1, 4	4	1/4	1.67674642
	U	4, 4, 2	8	1/2	3.35349283
	V	2, 8, 2	8	1/2	3.35349283
	W	2, 2, 2	4	1/2	3.35349283
	X	2, 2, 2	4	1/2	3.35349283
	Y	1, 16, 1	4	1	1.67674642
	Z*	1, 4, 1	4	1/4	1.67674642
	AA	2, 1, 2	2	1	1.67674642
	BB	2, 1, 2	2	1	1.67674642

Table (2.3.2) concluded

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp}/ \mid T \mid^2$	$\alpha(E)$
$(17) = (3+2\theta)(3-2\theta)$	A	$1,1$	4	1/16	8.49575480
	B	$2,2$	4	1/4	4.24787740
	C	$1,1$	2	1/4	2.12393870
	D	$4,4$	4	1	2.12393870
$(15+6\theta) = (1+\theta)^2(1-\theta)(3-\theta)$	A	$2,6,1$	6	1/3	4.91720118
	B	$2,3,2$	6	1/3	4.91720118
	C	$2,2,1$	2	1	1.63906706
	D	$2,1,2$	2	1	1.63906706
$(13+8\theta) = (1-\theta)^2(3+\theta)$	A	$1,1$	3	1/9	7.48086553
	B	$3,1$	3	1/3	2.49362184
	C	$1,1$	1	1	0.83120728
	D	$1,1$	1	1	3.99896687
$(3+12\theta) = (1+\theta)^2(1-\theta)(3+\theta)$	A	$2,1,2$	2	1	3.63844332
	B	$2,2,1$	2	1	3.63844332

Table (2.3.3): Rational Newforms in $V^+(\mathrm{a})$; $\mathbf{K} = \mathbf{Q}(\sqrt{-2})$

a	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
(4θ)	1/8	0.85939823	6.87518582
$(7+\theta)$	0		
$(2+5\theta)$	1/3	0.91184024	2.73552071
(6θ)	1/8	0.90883675	7.27069403
(7θ)	1/6	0.43770857	2.62625141
$(9+3\theta)$	1/8	0.84865921	6.78927365
(10)	0		
$(6+6\theta)$	1/2	1.48831694	2.97663388
$(4+7\theta)$	1/4	1.63543430	6.54173722
(11)	0		
$(7-6\theta)$	0		
(12)	1/4	1.81767351	7.27069404
$(6+8\theta)$	0		
$(4+9\theta)$	1/3	2.61829105	7.85487316
$(12+5\theta)$	0		
$(10+7\theta)$	0		
(10θ)	1/4	1.49844449	5.99377796
$(11+7\theta)$	0		
(15)	1/8	0.55892543	4.47140343
$(5+10\theta)$	1	2.92476321	2.92476321
$(12+7\theta)$	1	1.36404910	1.36404910
$(2+11\theta)$	0		
$(7+10\theta)$	0		

Table (2.3.3) (concluded)

a	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
$(16)c$	1/2	2.77239700	5.54479401
$(16)d$	0		
$(16+\theta)$	1/2	2.70617933	5.41235865
$(16+2\theta)$	1/2	1.92790953	3.85581905
$(5+11\theta)$	1/3	1.06195485	3.18586463
$(12\theta)a$	1/2	2.34536430	4.69072860
$(12\theta)b$	0		
$(12\theta)c$	1/2	1.67674642	3.35349283
$(12\theta)d$	1/2	1.67674642	3.35349283
(17)	0		
$(15+6\theta)$	1/3	1.63906706	4.91720118
$(13+8\theta)a$	1/3	0.83120728	2.49362184
$(13+8\theta)b$	1	3.99896687	3.99896687
$(3+12\theta)$	1	3.63844332	3.63844332

§2.4 The Results for $\mathbf{Q}(\sqrt{-3})$

We give the following tables:

Table (2.4.1): Isogeny classes of elliptic curves with conductor $\mathbf{a}$, $N(\mathbf{a}) < 500$.

Table (2.4.2): The Birch, Swinnerton-Dyer Numbers.

Table (2.4.3): Newforms in $V^+(\mathbf{a})$, $N(\mathbf{a}) < 500$.

Table (2.4.1): Elliptic Curves with Small Conductor over $\mathbf{Q}(\sqrt{-3})$

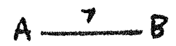
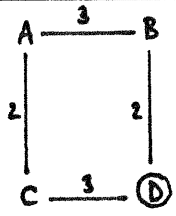
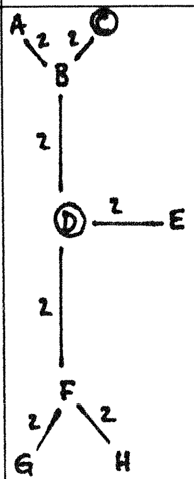
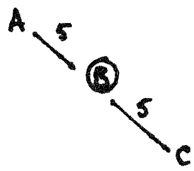
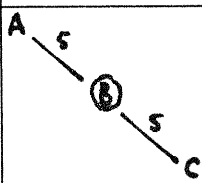
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(3+5\rho)$	A	0	$1+\rho$	ρ	ρ	0	
	B	0	$-2+\rho$	ρ	$1-\rho$	0	
$(8+\rho)$	A	1	$1+\rho$	0	ρ	0	
	B	1	$-2+\rho$	1	$10+4\rho$	$10-16\rho$	
	C	ρ	$-2+\rho$	0	$-4+4\rho$	$4-9\rho$	
	D	ρ	$-2+\rho$	0	$11-16\rho$	$11-20\rho$	
$(5+5\rho)$	A	$-1+\rho$	$-\rho$	1	0	0	
	B	1	1	1	-5	2	
	C	ρ	$-1+\rho$	1	79ρ	242	
	D	ρ	$-1+\rho$	1	9ρ	-10	
	E	$-1+\rho$	$-\rho$	1	$-35+35\rho$	-28	
	F	$-1+\rho$	$-\rho$	1	$135-135\rho$	-660	
	G	-1	1	-1	-110	-880	
	H	-1	1	-1	-2160	-39540	
(9)	A	0	0	1	0	0	
	B	0	0	1	30ρ	63	
(11)	A	0	-1	1	0	0	
	B	0	-1	1	-10	-20	
	C	0	-1	1	-7820	-26358	
$(10+2\rho)$	A	$-\rho$	$-1+\rho$	$1-\rho$	0	0	
	B	ρ	$-1+\rho$	$-1+\rho$	$10+5\rho$	$6+3\rho$	
	C	ρ	$-1+\rho$	$-1+\rho$	$-750-550\rho$	$-5980-10550\rho$	
(12)	A	0	0	0	0	1	
	B	0	0	0	$15-15\rho$	22	

Table (2.4.1) (continued)

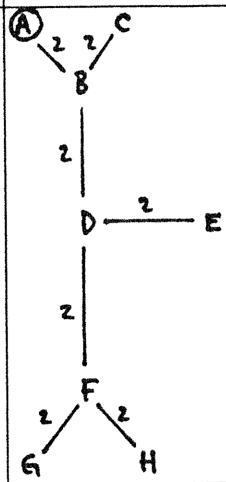
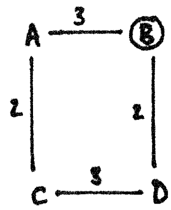
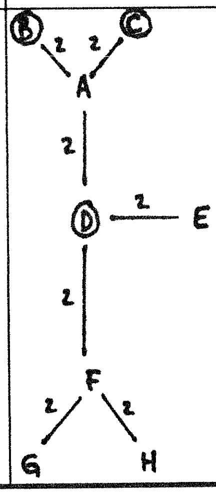
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(7+7\rho)$	A	$-1+\rho$	0	0	$-1+\rho$	0	
	B	1	0	0	-4	-1	
	C	ρ	0	0	39ρ	90	
	D	ρ	0	0	49ρ	-136	
	E	$-1+\rho$	0	0	$784-784\rho$	-8515	
	F	$-1+\rho$	0	0	$34-34\rho$	-217	
	G	-1	0	0	$-149+470\rho$	$-1906-1866\rho$	
	H	-1	0	0	$321-470\rho$	$-3772+1866\rho$	
$(9+6\rho)$	A	$-1+\rho$	ρ	ρ	ρ	0	
	B	$-1+\rho$	ρ	ρ	$5-9\rho$	$13-10\rho$	
	C	1	-1	ρ	$11-6\rho$	$1+10\rho$	
	D	1	-1	ρ	$6+19\rho$	$67-19\rho$	
$(8+8\rho)$	A	0	ρ	0	$-1+\rho$	0	
	B	0	-1	0	$6+5\rho$	$10-11\rho$	
	C	0	-1	0	$11-5\rho$	$-1+11\rho$	
	D	0	-1	0	-4	4	
	E	0	$1-\rho$	0	64ρ	220	
	F	0	$1-\rho$	0	24ρ	-36	
	G	0	ρ	0	$-16+16\rho$	-180	
	H	0	ρ	0	$384-384\rho$	-2772	

Table (2.4.1) (continued)

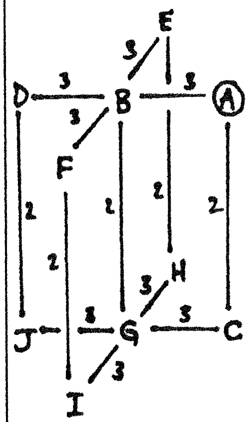
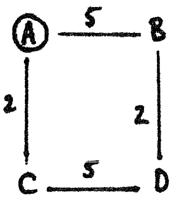
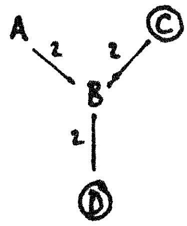
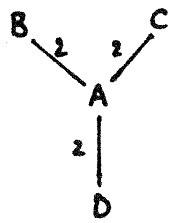
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
(14)	A	ρ	0	1	0	0	
	B	ρ	0	1	-5ρ	-6	
	C	$-1+\rho$	0	1	$11-11\rho$	12	
	D	$-\rho$	0	-1	$220-405\rho$	$-2854+1920\rho$	
	E	$-\rho$	0	-1	170ρ	-874	
	F	$-\rho$	0	-1	$-220-185\rho$	$-934-1920\rho$	
	G	$1-\rho$	0	-1	$36-36\rho$	-70	
	H	$1-\rho$	0	-1	$2731-2731\rho$	-55146	
	I	$1-\rho$	0	-1	$-184+414\rho$	$-806-1880\rho$	
	J	$1-\rho$	0	-1	$-414+184\rho$	$-2686+1880\rho$	
$(14+2\rho)$	A	$-1+\rho$	$1+\rho$	1	0	$-\rho$	
	B	$-1+\rho$	$-2+\rho$	ρ	$-34+54\rho$	$-18+112\rho$	
	C	1	$-2+\rho$	1	-10ρ	$8-9\rho$	
	D	1	$-2+\rho$	1	$-105+140\rho$	$266+615\rho$	
$(15+\rho)$	A	$-1+\rho$	$1-\rho$	ρ	0	0	
	B	1	ρ	ρ	$5-5\rho$	$-2-\rho$	
	C	ρ	-1	ρ	$-9+4\rho$	$9-13\rho$	
	D	ρ	-1	ρ	$-79-6\rho$	$-241-33\rho$	
(16)	A	0	$2-\rho$	0	$1-\rho$	0	
	B	0	$-1-\rho$	0	$-5+6\rho$	$1+3\rho$	
	C	0	$-1-\rho$	0	$5+\rho$	$-4+3\rho$	
	D	0	$-1-\rho$	0	-4ρ	$-4+8\rho$	

Table (2.4.1) (continued)

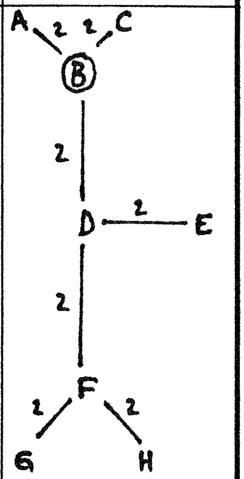
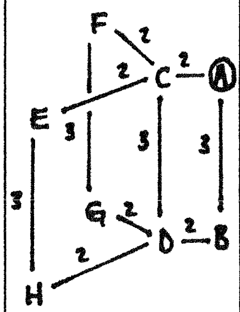
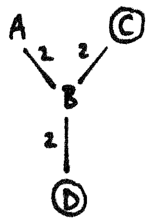
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(16+\rho)$	A	$-1+\rho$	1	$-1+\rho$	0	0	
	B	1	$-1+\rho$	$-1+\rho$	5ρ	$3+\rho$	
	C	ρ	$-\rho$	$-1+\rho$	$91-85\rho$	$274+35\rho$	
	D	ρ	$-\rho$	$-1+\rho$	$1-5\rho$	$4+11\rho$	
	E	$-1+\rho$	1	$-1+\rho$	$-80+125\rho$	$-11+476\rho$	
	F	$-1+\rho$	1	$-1+\rho$	$70-35\rho$	$63+126\rho$	
	G	-1	$-1+\rho$	$1-\rho$	$-170-80\rho$	$-702-558\rho$	
	H	-1	$-1+\rho$	$1-\rho$	$-540-550\rho$	$3564+8950\rho$	
$(11+8\rho)$	A	1	$1+\rho$	1	$-1+\rho$	-1	
	B	1	$-2+\rho$	0	$10-36\rho$	$-56+85\rho$	
	C	ρ	$-2+\rho$	1	$6-6\rho$	$-5+2\rho$	
	D	ρ	$-2+\rho$	1	$-39+24\rho$	$-17+38\rho$	
	E	$-1+\rho$	$-2+\rho$	$-1+\rho$	$-63-16\rho$	$-147-86\rho$	
	F	$-1+\rho$	$-2+\rho$	$-1+\rho$	$-23+14\rho$	$51-50\rho$	
	G	$1-\rho$	$-2+\rho$	$1-\rho$	$112+194\rho$	$1806-1031\rho$	
	H	$1-\rho$	$-2+\rho$	$1-\rho$	$-58-86\rho$	$-260+491\rho$	
$(13+6\rho)$	A	ρ	ρ	ρ	0	0	A
(17)	A	ρ	$1-\rho$	1	0	0	
	B	$-1+\rho$	ρ	1	$6-6\rho$	-4	
	C	1	-1	1	-1	-14	
	D	1	-1	1	-91	-310	

Table (2.4.1) (continued)

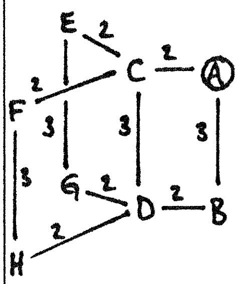
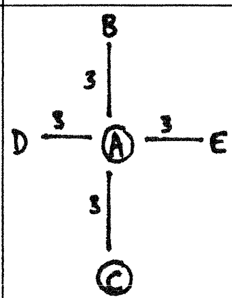
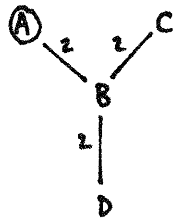
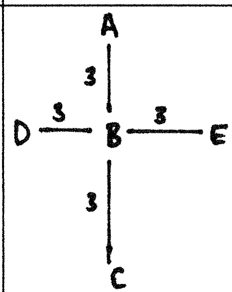
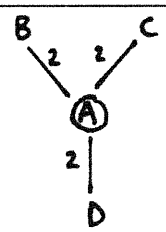
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(10+10\rho)$	A	1	0	1	1	2	
	B	-1	0	-1	-14	-64	
	C	ρ	0	1	18ρ	26	
	D	ρ	0	1	333ρ	-2368	
	E	$1-\rho$	0	-1	$289-289\rho$	1862	
	F	$1-\rho$	0	-1	$69-69\rho$	-194	
	G	$1-\rho$	0	-1	$454-454\rho$	-544	
	H	$1-\rho$	0	-1	$5334-5334\rho$	-150368	
(18)	A	$-1+\rho$	ρ	1	$-1+\rho$	-1	
	B	$1-\rho$	ρ	-1	$-61+91\rho$	$-13-276\rho$	
	C	-1	-1	0	-3	3	
	D	$1-\rho$	ρ	-1	$14-14\rho$	29	
	E	$1-\rho$	ρ	-1	$-91+61\rho$	$-289+276\rho$	
$(14+7\rho)$	A	$1-\rho$	$1-\rho$	$-\rho$	$11-7\rho$	$-5-9\rho$	
	B	1	ρ	ρ	$-22-29\rho$	$19+80\rho$	
	C	$-\rho$	-1	$-\rho$	$-463+795\rho$	$1082+6180\rho$	
	D	$-\rho$	-1	$-\rho$	$7-55\rho$	$152+416\rho$	
(19)	A	0	$-\rho$	1	$-1+\rho$	0	
	B	0	$-\rho$	1	$9-9\rho$	-15	
	C	0	$-\rho$	-1	$769-769\rho$	-8470	
	D	0	$-\rho$	-1	$99+31\rho$	$-424+498\rho$	
	E	0	$-\rho$	-1	$-31-99\rho$	$74-498\rho$	
$(11+11\rho)$	A	1	1	0	-11	0	
	B	ρ	$-1+\rho$	0	-44ρ	55	
	C	ρ	$-1+\rho$	0	146ρ	621	
	D	ρ	$-1+\rho$	0	6ρ	-9	
$(15+7\rho)$	A	$-\rho$	$1+\rho$	1	ρ	0	A

Table (2.4.1) (concluded)

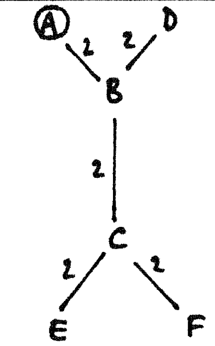
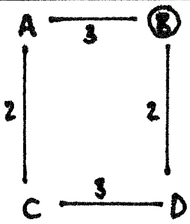
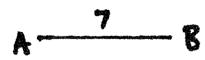
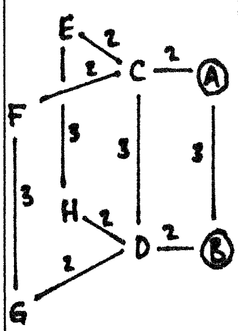
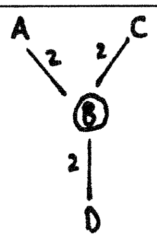
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(17+5\rho)$	A	$1-\rho$	$-1+\rho$	$1-\rho$	$-4+2\rho$	$-2+\rho$	
	B	1	$-\rho$	$-1+\rho$	$2+7\rho$	$9-2\rho$	
	C	ρ	1	$-1+\rho$	$-17+31\rho$	$42+4\rho$	
	D	ρ	1	$-1+\rho$	$113-129\rho$	$600-140\rho$	
	E	$1-\rho$	$-1+\rho$	$1-\rho$	$266+17\rho$	$1075-1961\rho$	
	F	$1-\rho$	$-1+\rho$	$1-\rho$	$436-283\rho$	$1581+1813\rho$	
(20)	A	0	$-1+\rho$	0	ρ	0	
	B	0	$-\rho$	0	$-4+4\rho$	4	
	C	0	$-1+\rho$	0	41ρ	-116	
	D	0	$-\rho$	0	$36-36\rho$	-140	
$(18+4\rho)$	A	ρ	$1-\rho$	ρ	0	0	A
$(16+7\rho)$	A	0	$-\rho$	$-\rho$	-2ρ	$-1+\rho$	
	B	0	$-\rho$	ρ	$-140+68\rho$	$-400+400\rho$	
(21)	A	-1	-1	$-\rho$	$-3+4\rho$	$1-4\rho$	
	B	ρ	$1-\rho$	$-1+\rho$	$-4+3\rho$	$-3+4\rho$	
	C	ρ	$1-\rho$	ρ	$20-2\rho$	$-11+29\rho$	
	D	$-1+\rho$	ρ	$-1+\rho$	$18+2\rho$	$18-29\rho$	
	E	$-1+\rho$	ρ	ρ	$14+12\rho$	$-50+161\rho$	
	F	$-1+\rho$	ρ	ρ	$-16-288\rho$	$-920+2189\rho$	
	G	1	-1	$-1+\rho$	$-13+25\rho$	$111-161\rho$	
	H	1	-1	$-1+\rho$	$17-305\rho$	$1269-2189\rho$	
$(15+10\rho)$	A	$-\rho$	ρ	$-\rho$	$-2+2\rho$	$-1+\rho$	
	B	$-1+\rho$	-1	ρ	$3+5\rho$	$7-4\rho$	
	C	1	$1-\rho$	ρ	$5+17\rho$	$62-34\rho$	
	D	1	$1-\rho$	ρ	$85-113\rho$	$444-194\rho$	
$(16+9\rho)$	A	$-\rho$	$1+\rho$	0	ρ	0	
	B	$-1+\rho$	$-2+\rho$	0	$-4+4\rho$	$5-8\rho$	

Table (2.4.2): The Birch, Swinnerton-Dyer Numbers

a		$c_\mathfrak{p}$	$\mid T \mid$	$\prod c_\mathfrak{p} / \mid T \mid^2$	$\alpha(E)$
$(3+5\rho) = (2+\rho)^2$	A	1	7	1/49	
	B	1	7	1/49	
$(8+\rho)$	A	1	6	1/36	9.72700227
	B	3	6	1/12	3.24233409
	C	2	6	1/18	4.86350113
	D	6	6	1/6	1.62116704
$(5+5\rho) = (1+\rho)(-5+5\rho)$	A	2, 1	8	1/32	8.94280685
	B	2, 2	8	1/16	4.47140343
	C	2, 1	4	1/8	2.23570171
	D	2, 4	8	1/8	2.23570171
	E	2, 8	8	1/4	1.11785086
	F	2, 2	4	1/4	1.11785086
	G	2, 1	2	1/2	0.55892543
	H	2, 1	2	1/2	0.55892543
$(9) = (1+\rho)^4$	A	3	9	1/27	7.30165666
	B	1	3	1/9	2.70287609
(11)	A	1	5	1/25	9.25771812
	B	5	5	1/5	1.85154362
	C	1	1	1	0.37030872
$(10+2\rho) = (2)(5+\rho)$	A	1, 1	5	1/25	9.21079465
	B	1, 5	5	1/5	1.84215893
	C	1, 1	1	1	0.36843179
$(12) = (1+\rho)^2(2)^2$	A	4, 3	12	1/12	4.17915519
	B	2, 3	6	1/6	2.55405786

Table (2.4.2) (continued)

a		$c_\wp$	$\mid T \mid$	$\prod c_\wp / \mid T \mid^2$	$\alpha(E)$
$(7+7\rho) = (1+\rho)(2+\rho)(1+2\rho)$	A	$4,1,1$	8	1/16	6.89661544
	B	$8,2,2$	16	1/8	3.44830772
	C	$16,1,1$	8	1/4	1.72415386
	D	$16,1,1$	8	1/4	1.72415386
	E	$2,2,2$	4	1/2	0.86207693
	F	$2,2,2$	4	1/2	0.86207693
	G	$1,2,2$	2	1	0.43103846
	H	$1,2,2$	2	1	0.43103846
$(9+6\rho) = (1+\rho)^2(5-3\rho)$	A	$2,1$	6	1/18	8.13941258
	B	$2,3$	6	1/6	2.71313753
	C	$2,2$	6	1/9	4.06970629
	D	$2,6$	6	1/3	1.35656876
$(8+8\rho) = (1+\rho)(2)^3$	A	$2,2$	8	1/16	7.27069404
	B	$1,2$	4	1/8	3.63534702
	C	$1,2$	4	1/8	3.63534702
	D	$2,4$	8	1/8	3.63534702
	E	$2,2$	4	1/4	1.81767351
	F	$2,2$	4	1/4	1.81767351
	G	$2,1$	2	1/2	0.90883675
	H	$2,1$	2	1/2	0.90883675

Table (2.4.2) (continued)

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp} / \mid T \mid^2$	$\alpha(E)$
$(14) = (2)(2+\rho)(-3+\rho)$	A	$2,1,1$	6	1/18	7.87875422
	B	$6,3,3$	18	1/6	2.62625141
	C	$1,2,2$	6	1/9	3.93937711
	D	$2,9,1$	6	1/2	0.87541735
	E	$18,1,1$	6	1/2	0.87541735
	F	$2,1,9$	6	1/2	0.87541735
	G	$3,6,6$	18	1/3	1.31312570
	H	$9,2,2$	6	1	0.43770857
	I	$1,2,18$	6	1	0.43770857
	J	$1,18,2$	6	1	0.43770857
$(14+2\rho) = (1+\rho)(2)(-2-3\rho)$	A	$5,2,1$	10	1/10	5.48223192
	B	$1,2,1$	2	1/2	1.09644638
	C	$10,1,2$	10	1/5	2.74111596
	D	$2,1,2$	2	1	0.54822319
$(15+\rho)$	A	1	4	1/16	8.81919611
	B	2	4	1/8	4.40959805
	C	4	4	1/4	2.20479903
	D	1	2	1/4	2.20479903
$(16) = (2)^4$	A	1	4	1/16	7.44117514
	B	2	4	1/8	4.42375798
	C	2	4	1/8	4.42375798
	D	2	4	1/8	4.42375798

Table (2.4.2) (continued)

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp} / \mid T \mid^2$	$\alpha(E)$
$(16+\rho) = (1+\rho)(2+\rho)(3+\rho)$	A	1,1,1	4	1/16	8.59663952
	B	2,2,2	8	1/8	4.29831976
	C*	1,1,1	4	1/16	2.14915988
	D	2,2,4	8	1/4	2.14915988
	E	2,2,8	8	1/2	1.07457994
	F	2,2,2	4	1/2	1.07457994
	G	2,2,1	2	1	0.53728997
	H	2,2,1	2	1	0.53728997
$(11+8\rho) = (1+\rho)(1+2\rho)(3+\rho)$	A	3,1,1	6	1/12	7.09684827
	B	1,3,3	6	1/4	2.36561609
	C	6,2,2	12	1/6	3.54842414
	D	2,6,6	12	1/2	1.18280805
	E	3,1,4	6	1/3	1.77421207
	F	12,4,1	12	1/3	1.77421207
	G	4,12,3	12	1	0.59140402
	H	1,3,12	6	1	0.59140402
$(13+6\rho)$	A	1	1	1	8.74990269
(17)	A	1	4	1/16	8.49575480
	B	2	4	1/8	4.24787740
	C	4	4	1/4	2.12393870
	D	1	2	1/4	2.12393870

Table (2.4.2) (continued)

a		$c_\wp$	$\mid T \mid$	$\prod c_\wp / \mid T \mid^2$	$\alpha(E)$
$(10+10\rho)=(1+\rho)(2)(5)$	A	$6,4,1$	12	1/6	3.88287042
	B	$2,12,3$	12	1/2	1.29429014
	C	$12,2,2$	12	1/3	1.94143521
	D	$4,6,6$	12	1	0.64714507
	E	$6,1,4$	6	2/3	0.97071761
	F	$24,1,1$	6	2/3	0.97071761
	G	$2,3,12$	6	2	0.32357254
	H	$8,3,3$	6	2	0.32357254
$(18)=(1+\rho)^4(2)$	A	$3,3$	9	1/9	5.63513523
	B	$3,1$	3	1/3	1.87837841
	C	$1,1$	3	1/9	5.63513523
	D	$3,9$	9	1/3	1.87837841
	E	$3,1$	3	1/3	1.87837841
$(14+7\rho)=(2+\rho)^2(2-3\rho)$	A	$4,1$	4	1/4	2.56241955
	B	$4,2$	4	1/2	1.28120977
	C	$2,2$	2	1	0.64060489
	D	$4,1$	2	1	0.64060489
$(19)=(3+2\rho)(-5+2\rho)$	A	$1,1$	3	1/9	8.41778108
	B	$3,3$	9	1/9	2.80592703
	C	$1,1$	3	1/9	0.93530901
	D	$9,1$	3	1	0.93530901
	E	$1,9$	3	1	0.93530901
$(11+11\rho)=(1+\rho)(11)$	A	$2,2$	4	1/4	2.05117195
	B	$2,1$	2	1/2	1.02558598
	C	$2,4$	4	1/2	1.02558598
	D	$2,1$	4	1/8	4.10234391
$(15+7\rho)$	A	1	1	1	8.41997711

Table (2.4.2) (concluded)

a		$c_\wp$	$\mid T \mid$	$\prod c_\wp / \mid T \mid^2$	$\alpha(E)$
$(17+5\rho)=(1+\rho)(2+\rho)(3+2\rho)$	A	$1,2,1$	4	$1/8$	4.97121381
	B	$2,4,2$	8	$1/4$	2.48560690
	C	$2,2,2$	4	$1/2$	1.24280345
	D	$1,8,1$	4	$1/2$	1.24280345
	E	$2,1,2$	2	1	0.62140173
	F	$2,1,2$	2	1	0.62140173
$(20)=(2)^2(5)$	A	$3,1$	6	$1/12$	6.42309566
	B	$3,2$	6	$1/6$	3.21154783
	C	$3,3$	6	$1/4$	2.14103189
	D	$3,6$	6	$1/2$	1.07051594
$(18+4\rho)=(2)(-11+9\rho)$	A	$2,1$	1	2	7.48003560
$(16+7\rho)=(1+\rho)(13-3\rho)$	A	$7,1$	7	$1/7$	4.75769923
	B	$1,1$	1	1	0.67967132
$(21)=(1+\rho)^2(2+\rho)(2-3\rho)$	A	$2,3,1$	6	$1/6$	3.91416051
	B	$2,1,3$	6	$1/6$	3.91416051
	C	$4,6,2$	12	$1/3$	1.95708026
	D	$4,2,6$	12	$1/3$	1.95708026
	E	$2,12,1$	6	$2/3$	0.97854013
	F	$2,3,4$	6	$2/3$	0.97854013
	G	$2,1,12$	6	$2/3$	0.97854013
	H	$2,4,3$	6	$2/3$	0.97854013
$(15+10\rho)=(3+2\rho)(-5\rho)$	A	$2,1$	4	$1/8$	5.79972220
	B	$2,2$	4	$1/4$	2.89986110
	C	$2,1$	2	$1/2$	1.44993055
	D	$2,4$	4	$1/2$	1.44993055
$(16+9\rho)=(1+3\rho)(4-7\rho)$	A	$1,1$	2	$1/4$	8.33178666
	B	$2,2$	2	1	4.16589333

Table (2.4.3): Rational Newforms in $V^+(\mathrm{a})$; $\mathbf{K} = \mathbf{Q}(\sqrt{-3})$

a	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
$(8+\rho)$	1/6	0.27109401	1.62116704
$(5+5\rho)$	1/8	0.27946271	2.23570171
(11)	1/5	0.37030872	1.85154362
$(10+2\rho)$	1/5	0.36843179	1.84215893
$(7+7\rho)$	1/16	0.43103846	6.89661544
$(9+6\rho)$	1/6	0.45218958	2.71313753
$(8+8\rho)$	1/8	0.45441377	3.63534702
(14)	1/18	0.43770857	7.87875422
$(14+2\rho)$	1/10	0.56822319	5.48223192
$(15+\rho)$	1/4	0.55119976	2.20479903
$(16+\rho)$	1/8	0.53728997	4.29831976
$(11+8\rho)$	1/12	0.59140402	7.09684827
$(13+6\rho)$	0		
(17)	1/4	0.53098467	2.12393870
$(10+10\rho)$	1/6	0.64714507	3.88287042
(18)	1/9	0.62612614	5.63513523
$(14+7\rho)$	1/4	0.64060489	2.56241955
(19)	0		
$(11+11\rho)$	1/4	0.51279299	2.05117195
$(15+7\rho)$	0		
$(17+5\rho)$	1/8	0.62140173	4.97121381
(20)	1/6	0.53525797	3.21154783

Table (2.4.3) (concluded)

a	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
$(18+4\rho)$	0		
$(16+7\rho)$	1/7	0.67967132	4.75769923
(21)	1/6	0.65236009	3.91416051
$(15+10\rho)$	1/4	0.72496528	2.89986110
$(16+9\rho)$	0		

§2.5 The Results for $\mathbf{Q}(\sqrt{-7})$

We give the following tables:

Table (2.5.1): Isogeny classes of elliptic curves with conductor $\mathbf{a}$, $N(\mathbf{a}) \leq 300$.

Table (2.5.2): The Birch, Swinnerton-Dyer Numbers.

Table (2.5.3): Newforms in $V^{+}(\mathbf{a})$, $N(\mathbf{a}) \leq 300$.

Table (2.5.1): Elliptic Curves with Small Conductor over $Q(\sqrt{-7})$

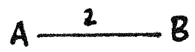
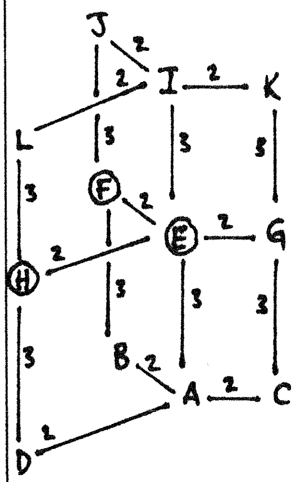
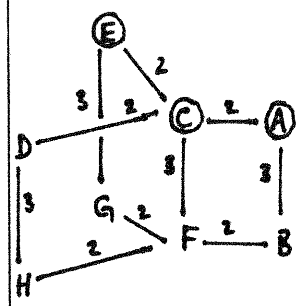
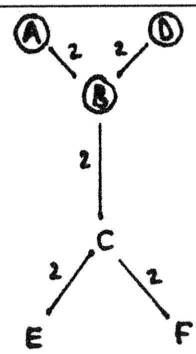
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(2-3\alpha)$	A	α	$2-\alpha$	α	$2-2\alpha$	$1-\alpha$	
	B	α	$-1-\alpha$	0	$11-15\alpha$	$26-7\alpha$	
$(2-4\alpha)$	A	1	0	1	-1	0	
	B	1	α	$-1+\alpha$	1	α	
	C	1	0	1	-11	12	
	D	1	$1-\alpha$	α	$1-\alpha$	$1-\alpha$	
	E	1	0	1	4	-6	
	F	1	α	$-1+\alpha$	$16-10\alpha$	$-16-4\alpha$	
	G	1	0	1	-36	-70	
	H	1	$1-\alpha$	α	$6+9\alpha$	$-20+4\alpha$	
	I	1	0	1	-171	-874	
	J	1	α	$-1+\alpha$	$-39+30\alpha$	$-154-29\alpha$	
	K	-1	0	-1	-2731	-55146	
	L	1	$1-\alpha$	α	$-9-31\alpha$	$-183+29\alpha$	
$(2+4\alpha)$	A	1	$1-\alpha$	0	1	1	
	B	1	$1-\alpha$	0	$36+55\alpha$	$446-87\alpha$	
	C	1	1	α	$2-2\alpha$	$2-\alpha$	
	D	1	1	α	$22-22\alpha$	$78-21\alpha$	
	E	1	α	1	$2+\alpha$	$-2+3\alpha$	
	F	1	1	α	$-23+13\alpha$	$19-28\alpha$	
	G	1	α	1	$-3-14\alpha$	$-14+19\alpha$	
	H	1	1	α	$37+3\alpha$	$219-136\alpha$	
$(1-5\alpha)$	A	1	$-\alpha$	$-\alpha$	$-2+\alpha$	1	
	B	1	$-1+\alpha$	$-1+\alpha$	0	0	
	C	1	$-1+\alpha$	$-1+\alpha$	$5+5\alpha$	$10-2\alpha$	
	D	1	$-2+\alpha$	α	$-3-\alpha$	$-\alpha$	
	E	1	$-1+\alpha$	$-1+\alpha$	10	$22+8\alpha$	
	F	1	$-1+\alpha$	$-1+\alpha$	$80+90\alpha$	$858-240\alpha$	

Table (2.5.1) (continued)

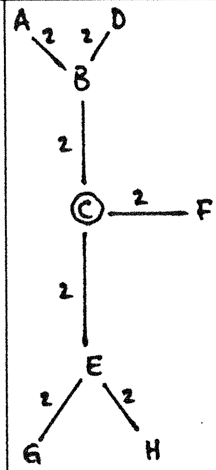
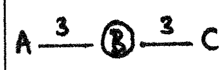
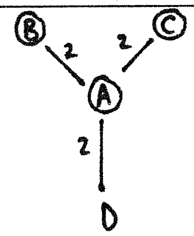
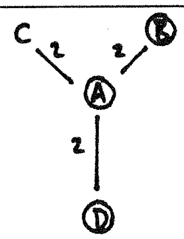
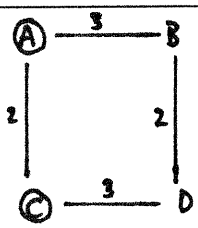
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
(7)	A	1	-1	0	-2	-1	
	B	1	-1	0	-37	-78	
$(3-6\alpha)$	A	1	$1-\alpha$	$-1+\alpha$	α	0	
	B	1	0	0	1	0	
	C	1	0	0	-4	-1	
	D	1	α	α	$1-2\alpha$	0	
	E	1	0	0	-49	-136	
	F	1	0	0	-39	90	
	G	1	0	0	-784	-8515	
	H	1	0	0	-34	-217	
$(4+5\alpha)$	A	$-1+\alpha$	$2-\alpha$	1	-1	0	
	B	$-1+\alpha$	$-1-\alpha$	α	$14-2\alpha$	$-8-14\alpha$	
	C	$-1+\alpha$	$-1-\alpha$	α	$-111+3\alpha$	$-349+108\alpha$	
$(3-7\alpha)$	A	1	$2-\alpha$	1	$-\alpha$	-1	
	B	1	$-\alpha$	1	$-1-2\alpha$	$-3+2\alpha$	
	C	1	$-2+\alpha$	1	$2-\alpha$	-1	
	D	1	$-1-\alpha$	0	$4-4\alpha$	$-11+3\alpha$	
$(7+3\alpha)$	A	$-1+\alpha$	α	$-1+\alpha$	α	1	
	B	$-1+\alpha$	0	$-1+\alpha$	$5-\alpha$	$1+2\alpha$	
	C	$-1+\alpha$	α	$-1+\alpha$	$-15+6\alpha$	$19-14\alpha$	
	D	$-1+\alpha$	$-1-\alpha$	0	$2+2\alpha$	$1-3\alpha$	
$(2+6\alpha)$	A	$1-\alpha$	0	$1-\alpha$	$-1+\alpha$	α	
	B	$-1+\alpha$	0	$-1+\alpha$	$14-4\alpha$	$2-21\alpha$	
	C	$-1+\alpha$	$-1+\alpha$	0	$-3+2\alpha$	$-1+\alpha$	
	D	$-1+\alpha$	$-1+\alpha$	0	$-28-3\alpha$	$72+2\alpha$	

Table (2.5.1) (continued)

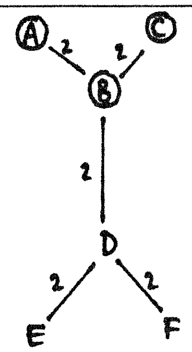
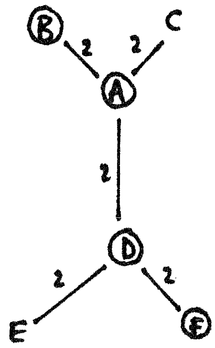
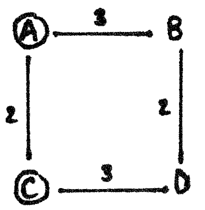
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(2-7\alpha)$	A	$-\alpha$	$-1+\alpha$	0	1	0	
	B	α	α	α	$-\alpha$	$1-\alpha$	
	C	α	-1	α	$6-3\alpha$	-4	
	D	α	α	α	$10-6\alpha$	$-7-\alpha$	
	E	α	α	α	$20-\alpha$	$-35+33\alpha$	
	F	α	α	α	$160-91\alpha$	$-611-175\alpha$	
(11)	A	0	-1	1	0	0	A —5— B —5— C
	B	0	-1	1	-10	-20	
	C	0	-1	1	-7820	-26358	
(8α)	A	0	$1-\alpha$	0	$-2+\alpha$	0	
	B	0	α	0	$-2-2\alpha$	$1+\alpha$	
	C	0	$1-\alpha$	0	$-22+21\alpha$	$16-40\alpha$	
	D	0	$-\alpha$	0	$-1-\alpha$	0	
	E	0	$-\alpha$	0	$-1-21\alpha$	$24-40\alpha$	
	F	0	$-1+\alpha$	0	$-4+2\alpha$	$-2+\alpha$	
$(12+\alpha)$	A	$1-\alpha$	$-\alpha$	1	$-1+\alpha$	0	A
$(3+8\alpha)$	A	0	$2-\alpha$	1	$-\alpha$	-1	A —3— (B) —3— C
	B	0	$-1-\alpha$	1	$9-9\alpha$	$-31+8\alpha$	
	C	0	$-1-\alpha$	1	$69-59\alpha$	$254-2\alpha$	
$(5-10\alpha)$	A	0	1	1	-1	0	A —3— B —3— C
	B	0	1	1	9	1	
	C	0	1	1	-131	-650	
$(2-10\alpha)$	A	$1-\alpha$	$2-\alpha$	$1-\alpha$	$2-\alpha$	$-\alpha$	
	B	$-1+\alpha$	$-1-\alpha$	0	$-4-24\alpha$	$24-44\alpha$	
	C	$-1+\alpha$	α	$-1+\alpha$	2	0	
	D	$-1+\alpha$	α	$-1+\alpha$	$-13+5\alpha$	4	

Table (2.5.1) (concluded)

a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
(14)	A	-1	$-2+\alpha$	-1	$-10+\alpha$	$-8-\alpha$	(A) —2— (B)
	B	1	$-1-\alpha$	$-1+\alpha$	$-8-2\alpha$	$-9+\alpha$	
$(10+5\alpha)$	A	0	$1-\alpha$	$-\alpha$	-1	0	A

Table (2.5.2): The Birch, Swinnerton-Dyer Numbers

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp} / \mid T \mid^2$	$\alpha(E)$
$(2-3\alpha) = (\alpha)^4$	A	4	8	1/16	
	B	2	4	1/8	
$(2-4\alpha) = (\alpha)(1-\alpha)(1-2\alpha)$	A	2, 2, 2	12	1/18	7.87875422
	B	1, 2, 1	6	1/18	7.87875422
	C	1, 1, 4	6	1/9	3.93937711
	D	2, 1, 1	6	1/18	7.87875422
	E	2, 2, 6	12	1/6	2.62625141
	F	1, 2, 3	6	1/6	2.62625141
	G	1, 1, 12	6	1/3	1.31312570
	H	2, 1, 3	6	1/6	2.62625141
	I	2, 2, 2	4	1/2	0.87541714
	J	1, 2, 1	2	1/2	0.87541714
	K	1, 1, 4	2	1	0.43770857
	L	2, 1, 1	2	1/2	0.87541714
$(2+4\alpha) = (\alpha)(1-\alpha)(1+2\alpha)$	A	2, 12, 1	12	1/6	5.04266466
	B	2, 4, 1	4	1/2	1.68088823
	C	2, 6, 2	12	1/6	5.04266466
	D	2, 3, 2	6	1/3	2.52133234
	E	2, 3, 1	6	1/6	5.04266466
	F	2, 2, 2	4	1/2	1.68088823
	G	2, 1, 1	2	1/2	1.68088823
	H	2, 1, 2	2	1	0.84044411
$(1-5\alpha) = (1-\alpha)(-5+2\alpha)$	A	8, 1	8	1/8	6.50007553
	B	4, 2	8	1/8	6.50007553
	C	2, 2	4	1/4	3.25003777
	D	2, 1	4	1/8	6.50007553
	E	1, 2	2	1/2	1.62501888
	F	1, 2	2	1/2	1.62501888

Table (2.5.2) (continued)

$\mathbf{a}$		$c_\wp$	$\mid T \mid$	$\prod c_\wp / \mid T \mid^2$	$\alpha(E)$
$(7) = (1-2\alpha)^2$	A	4	4	1/4	4.94450460
	B	2	2	1/2	2.47225230
$(3-6\alpha) = (1-2\alpha)(3)$	A	1, 1	4	1/16	6.89661544
	B	2, 2	8	1/16	6.89661544
	C	2, 4	8	1/8	3.44830772
	D	1, 1	4	1/16	6.89661544
	E	2, 2	4	1/4	1.72415386
	F	2, 8	8	1/4	1.72415386
	G	2, 1	2	1/2	0.86207693
	H	2, 1	2	1/2	0.86207693
$(4+5\alpha) = (\alpha)(-7+2\alpha)$	A	1, 1	3	1/9	6.44434572
	B	1, 3	3	1/3	2.14811524
	C	1, 1	1	1	0.71603841
$(3-7\alpha) = (1-\alpha)(7-2\alpha)$	A	2, 2	4	1/4	6.63209304
	B	1, 1	2	1/4	6.63209304
	C	4, 1	4	1/4	6.63209304
	D	1, 2	2	1/2	3.31604652
$(7+3\alpha) = (1-\alpha)^3(1+2\alpha)$	A	2, 2	4	1/4	5.26541403
	B	4, 1	4	1/4	5.26541403
	C	1, 2	2	1/2	2.63270701
	D	1, 1	2	1/4	5.26541403
$(2+6\alpha) = (\alpha)(1-\alpha)^2(-3+2\alpha)$	A	2, 3, 1	6	1/6	5.02948915
	B	2, 1, 1	2	1/2	1.67649638
	C	1, 3, 2	6	1/6	5.02948915
	D	1, 1, 2	2	1/2	1.67649638

Table (2.5.2) (continued)

a		$c_\wp$	$\mid T \mid$	$\prod c_\wp / \mid T \mid^2$	$\alpha(E)$
$(2-7\alpha) = (\alpha)^3(1+2\alpha)$	A	2, 1	4	1/8	5.84095523
	B	4, 2	8	1/8	5.84095523
	C	2, 1	4	1/8	5.84095523
	D	2, 2	4	1/4	2.92047762
	E	1, 2	2	1/2	1.46023881
	F	1, 2	2	1/2	1.46023881
$(11) = (1+2\alpha)(-3+2\alpha)$	A	1, 1	5	1/25	9.25771812
	B	5, 5	5	1	1.85154362
	C	1, 1	1	1	0.37030872
$(8\alpha) = (\alpha)^4(1-\alpha)^3$	A	2, 2	4	1/4	4.80100688
	B	1, 1	2	1/4	4.80100688
	C	2, 1	2	1/2	2.40050344
	D	4, 4	8	1/4	4.80100688
	E	4, 2	4	1/2	2.40050344
	F	2, 2	4	1/4	4.80100688
$(12+\alpha) = (\alpha)(7-6\alpha)$	A	2, 1	1	2	7.61021593
$(3+8\alpha) = (1-2\alpha)(5-2\alpha)$	A	1, 1	3	1/9	8.67458237
	B	3, 1	3	1/3	2.89152746
	C	1, 1	1	1	0.96384249
$(5-10\alpha) = (1-2\alpha)(5)$	A	2, 1	3	2/9	6.97477682
	B	6, 3	3	2	2.32492561
	C	2, 9	1	18	0.77497520
$(2-10\alpha) = (\alpha)(1-\alpha)^2(5-2\alpha)$	A	2, 3, 1	6	1/6	5.74460062
	B	2, 1, 1	2	1/2	1.91486687
	C	1, 3, 2	6	1/6	5.74460062
	D	1, 1, 2	2	1/2	1.91486687

Table (2.5.2) (concluded)

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp} / \mid T \mid^2$	$\alpha(E)$
$(14) = (\alpha)(1-\alpha)(1-2\alpha)^2$	A	$1,2,2$	2	1	2.50944729
	B	$2,1,2$	2	1	2.50944729
$(10+5\alpha) = (\alpha)^3(5)$	A	$4,1$	1	4	6.68923046

Table (2.5.3): Rational Newforms in $V^+(\mathrm{a})$; K = $\mathbf{Q}(\sqrt{-7})$

a	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
$(2-4\alpha)$	1/6	0.43770857	2.62625141
$(2+4\alpha)$	1/6	0.84044411	5.04266466
$(1-5\alpha)$	1/8	0.81250944	6.50007553
$(3-6\alpha)$	1/8	0.43103846	3.44830772
$(4+5\alpha)$	1/3	0.71603841	2.14811524
$(3-7\alpha)$	1/4	1.65802326	6.63209304
$(7+3\alpha)$	1/4	1.31635306	5.26541403
$(2+6\alpha)$	1/6	0.83824819	5.02948915
$(2-7\alpha)$	1/8	0.73011940	5.84095523
(11)	0		
(8α)	1/4	1.20025172	4.80100688
$(12+\alpha)$	0		
$(3+8\alpha)$	1/3	0.96384249	2.89152746
$(5-10\alpha)$	0		
$(2-10\alpha)$	1/6	0.95743344	5.74460062
(14)	1	2.50944729	2.50944729
$(10+5\alpha)$	0		

§2.6 The Results for $\mathbf{Q}(\sqrt{-11})$

We give the following tables:

Table (2.6.1): Isogeny classes of elliptic curves with conductor $\mathbf{a}$, $N(\mathbf{a}) < 200$.

Table (2.6.2): The Birch, Swinnerton-Dyer Numbers.

Table (2.6.3): Newforms in $V^{+}(\mathbf{a})$, $N(\mathbf{a}) < 200$.

Table (2.6.1): Elliptic Curves with Small Conductor over $\mathbb{Q}(\sqrt{-11})$

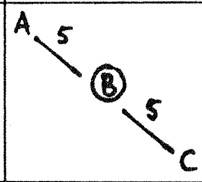
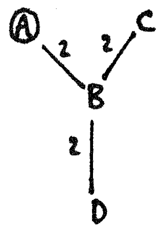
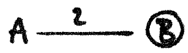
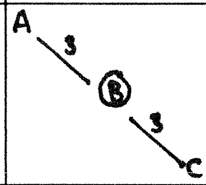
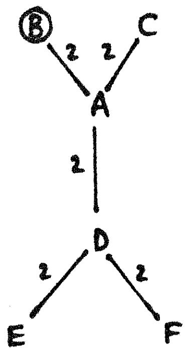
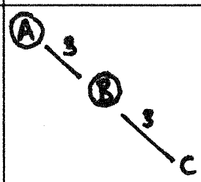
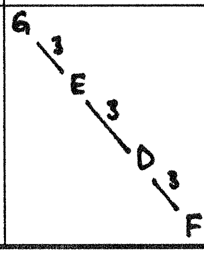
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(1-2\alpha)$	A	0	-1	1	0	0	
	B	0	-1	1	-10	-20	
	C	0	-1	1	-7820	-26358	
(3α)	A	$-\alpha$	α	$1-\alpha$	α	$2-\alpha$	
	B	α	α	$-1+\alpha$	$15-6\alpha$	$-13-2\alpha$	
	C	α	α	$-1+\alpha$	$30+4\alpha$	$-94+52\alpha$	
	D	α	α	$-1+\alpha$	$240-96\alpha$	$-1012-380\alpha$	
$(5+2\alpha)$	A	0	$1-\alpha$	1	-1	0	A
$(2+5\alpha)$	A	$-\alpha$	α	0	-1	0	
	B	α	α	0	4	$-3+5\alpha$	
$(8+2\alpha)$	A	$1-\alpha$	α	1	-1	0	
	B	$-1+\alpha$	α	1	$5+4\alpha$	$-18+4\alpha$	
	C	$-1+\alpha$	α	1	$-95+24\alpha$	$-322+116\alpha$	
$(3+5\alpha)$	A	1	$2-\alpha$	0	-2α	-1	
	B	1	$-1-\alpha$	1	$-17-10\alpha$	$6+41\alpha$	
$(3-6\alpha)$	A	1	1	0	-11	0	
	B	1	1	0	-6	-9	
	C	1	1	0	-146	621	
	D	1	1	0	44	55	
	E	1	1	0	$509-50\alpha$	$58+2414\alpha$	
	F	-1	1	0	$459+50\alpha$	$2472-2414\alpha$	
$(6+4\alpha)$	A	$-\alpha$	0	1	α	0	
	B	1	$-1+\alpha$	α	$-\alpha$	1	
	C	1	$-1+\alpha$	α	$30-21\alpha$	$-81+6\alpha$	
	D	$1-\alpha$	$2-\alpha$	1	1	0	
	E	$-1+\alpha$	$-1-\alpha$	α	$-12+7\alpha$	$5+10\alpha$	
	F	1	$-1+\alpha$	$-1+\alpha$	$2-2\alpha$	$3+\alpha$	
	G	$1-\alpha$	$-1-\alpha$	$-\alpha$	$-972+447\alpha$	$-7771+728\alpha$	

Table (2.6.1) (concluded)

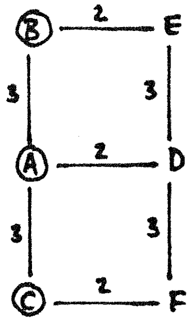
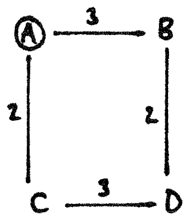
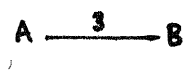
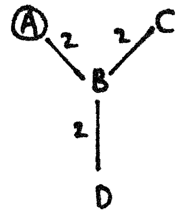
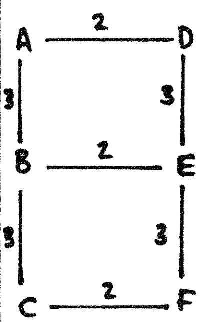
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
(6α)	A	$1-\alpha$	$-1-\alpha$	$-\alpha$	$-9+5\alpha$	$15-2\alpha$	
	B	$-\alpha$	0	-1	$12+2\alpha$	$12-10\alpha$	
	C	1	$-1+\alpha$	$-1+\alpha$	$5-9\alpha$	$27-7\alpha$	
	D	$-1+\alpha$	$-1-\alpha$	α	$-129+45\alpha$	$615-250\alpha$	
	E	α	0	1	$-18+12\alpha$	$24-62\alpha$	
	F	1	$-1+\alpha$	$-1+\alpha$	$-25+\alpha$	$105-15\alpha$	
(11)	A	1	1	0	-2	-7	Ⓐ
$(3+6\alpha)$	A	α	$1+\alpha$	α	α	0	
	B	α	$-2+\alpha$	0	$16-16\alpha$	$54+5\alpha$	
	C	α	$-2+\alpha$	0	$-4+4\alpha$	$-4-11\alpha$	
	D	α	$-2+\alpha$	0	$41-26\alpha$	-55	
$(2+7\alpha)$	A	$1+\alpha$	α	$1+\alpha$	$-4+\alpha$	-3	
	B	$-1+\alpha$	$-\alpha$	α	$3+8\alpha$	$-11-4\alpha$	
$(4-8\alpha)$	A	0	1	0	3	-1	
	B	0	1	0	-77	-289	
$(3+7\alpha)$	A	0	$1-\alpha$	α	α	0	A
$(6+6\alpha)$	A	α	$-1-\alpha$	0	4	0	
	B	α	$-1-\alpha$	0	-16	$28+12\alpha$	
	C	α	$-1-\alpha$	0	$-106+30\alpha$	$-296+174\alpha$	
	D	α	$-1-\alpha$	0	$-246-30\alpha$	$1584+378\alpha$	
(14)	A	1	0	1	-1	0	
	B	1	0	1	4	-6	
	C	1	0	1	-171	-874	
	D	1	0	1	-11	12	
	E	1	0	1	-36	-70	
	F	1	0	1	-2731	-55146	

Table (2.6.2): The Birch, Swinnerton-Dyer Numbers

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp}/\mid T \mid^2$	$\alpha(E)$
$(1-2\alpha)$	A	2	5	2/25	9.25771812
	B	10	5	2/5	1.85154362
	C	2	1	2	0.37030872
$(3\alpha)=(\alpha)^2(1-\alpha)$	A	4, 1	4	1/4	4.57926880
	B	4, 2	4	1/2	2.28963440
	C	2, 2	2	1	1.14481720
	D	4, 1	2	1	1.14481720
$(5+2\alpha)$	A	2	1	2	7.28081184
$(2+5\alpha)$	A	1	2	1/4	9.49556159
	B	2	2	1/2	4.74778080
$(8+2\alpha)=(2)(4+\alpha)$	A	2, 1	3	2/9	8.23872476
	B	6, 1	3	2/3	2.74624159
	C	2, 1	1	2	0.91541386
$(3+5\alpha)=(\alpha)^2(1-2\alpha)$	A	2, 1	2	1/2	4.90520376
	B	2, 2	2	1	2.45260188
$(3-6\alpha)=(\alpha)(1-\alpha)(1-2\alpha)$	A	2, 2, 4	8	1/4	2.05117195
	B	1, 1, 2	4	1/8	4.10234391
	C	1, 1, 8	4	1/2	1.02558598
	D	2, 2, 2	4	1/2	1.02558598
	E	2, 2, 1	2	1	0.51279299
	F	2, 2, 1	2	1	0.51279299
$(6+4\alpha)=(\alpha)^3(2)$	A	3, 1	3	1/3	7.35935349
	B	1, 3	3	1/3	7.35935349
	C	1, 1	1	1	2.45311783
	D	1, 3	3	1/3	6.13086731
	E	3, 9	3	3	2.04362244
	F	1, 1	3	1/9	6.13086731
	G	3, 1	3	1/3	0.18551127

Table (2.6.2) (concluded)

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp} / \mid T \mid^2$	$\alpha(E)$
$(6\alpha) = (\alpha)^2(1-\alpha)(2)$	A	$2,3,6$	6	1	2.26846155
	B	$2,1,2$	2	1	2.26846155
	C	$2,9,2$	6	1	2.26846155
	D	$4,6,3$	6	2	1.13423078
	E	$4,2,1$	2	2	1.13423078
	F	$4,18,1$	6	2	1.13423078
$(11) = (1-2\alpha)^2$	A	1	1	1	2.81770205
$(3+6\alpha) = (\alpha)(1-\alpha)^2(2-\alpha)$	A	$3,2,2$	6	1/3	5.96613233
	B	$1,2,2$	2	1	1.98871077
	C	$6,2,2$	6	2/3	2.98306616
	D	$2,2,2$	2	2	0.99435539
$(2+7\alpha) = (1-\alpha)(1+\alpha)(-1+2\alpha)$	A	$1,2,1$	2	1/2	4.20718792
	B	$2,1,2$	2	1	2.10359396
$(4-8\alpha) = (2)(1-2\alpha)$	A	$3,2$	3	2/3	3.68616879
	B	$3,2$	1	6	1.22872293
$(3+7\alpha) = (\alpha)(-8+\alpha)$	A	$9,1$	1	9	4.19537197
$(6+6\alpha) = (\alpha)(1-\alpha)(2)(1+\alpha)$	A	$1,1,4,2$	4	1/2	3.47974992
	B	$2,2,2,2$	4	1	1.73987496
	C*	$1,1,1,2$	2	1/2	0.86993748
	D	$2,2,1,2$	2	2	0.86993748
$(14) = (2)(7)$	A	$2,1$	6	1/18	7.87875422
	B	$6,3$	6	1/2	2.62625141
	C	$18,1$	2	9/2	0.87541735
	D	$1,2$	6	1/18	3.93937711
	E	$3,6$	6	1/2	1.31312570
	F	$9,2$	2	9/2	0.43770857

Table (2.6.3): Rational Newforms in $V^+(\mathrm{a})$; $\mathbf{K} = \mathbf{Q}(\sqrt{-11})$

a	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
$(1-2\alpha)$	2/5	0.74061745	1.85154362
(3α)	1/4	1.14481720	4.57926880
$(5+2\alpha)$	0		
$(2+5\alpha)$	1/2	2.37389040	4.74778080
$(8+2\alpha)$	2/3	1.83082772	2.74624159
$(3+5\alpha)$	1/2	2.45260188	4.90520376
$(3-6\alpha)$	1/8	0.51279299	4.10234391
$(6+4\alpha)a$	1/3	2.45311783	7.35935349
$(6+4\alpha)b$	0		
(6α)	1	2.26846155	2.26846155
(11)	1	2.81770205	2.81770205
$(3+6\alpha)$	1/3	1.98871077	5.96613233
$(2+7\alpha)$	1/2	2.10359396	4.20718792
$(4-8\alpha)$	0		
$(3+7\alpha)$	0		
$(6+6\alpha)$	1/2	1.73987496	3.47974992
(14)	0		

§2.7 The Results for $\mathbf{Q}(\sqrt{-19})$

We give the following tables:

Table (2.7.1): Isogeny classes of elliptic curves with conductor $\mathbf{a}$, $N(\mathbf{a}) < 500$.

Table (2.7.2): The Birch, Swinnerton-Dyer Numbers.

Table (2.7.3): Newforms in $V^+(\mathbf{a})$, $N(\mathbf{a}) < 500$.

Table (2.7.1): **Elliptic Curves with Small Conductor over Q($\sqrt{-19}$)**

a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(1-2\omega)$	A	0	1	1	1	0	A –3– (B) –3– C
	B	0	1	1	-9	-15	
	C	0	1	1	-769	-8470	
(2ω)	A	ω	$1-\omega$	1	-1	0	A –3– (B) –3– C
	B	ω	$1-\omega$	1	$9-5\omega$	$-24-2\omega$	
	C	ω	$1-\omega$	1	$-31-15\omega$	$-60-54\omega$	
$(4+2\omega)$	A	1	0	$1+\omega$	ω	0	A
$(2-4\omega)$	A	1	0	1	-16	22	A –3– B –3– C
	B	1	0	1	9	90	
	C	1	0	1	-86	-2456	
	D	1	1	1	0	1	D –5– E
	E	1	1	1	-70	-279	
$(5+3\omega)$	A	1	$-\omega$	0	-1	0	(A) –2– B
	B	1	$-\omega$	0	4	$1-4\omega$	
$(8+2\omega)$	A	1	$-\omega$	0	$-3+\omega$	ω	A
(11)	A	0	-1	1	0	0	A –5– B –5– C
	B	0	-1	1	-10	-20	
	C	0	-1	1	-7820	-263580	
$(3-6\omega)$	A	0	1	1	20	-32	A –5– B
	B	0	1	1	-4390	-113432	
	C	0	-1	1	-2	2	C
	D	1	0	1	-7	5	E –2– D, F –2– D, D –2– G
	E	1	0	1	-2	-1	
	F	1	0	1	-102	385	
	G	1	0	1	8	29	
$(1-6\omega)$	A	$-1+\omega$	$-2+\omega$	$-1+\omega$	$3+4\omega$	$-11-7\omega$	(A)
	B	ω	$-1-\omega$	1	0	0	B

Table (2.7.1) (continued)

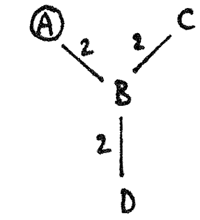
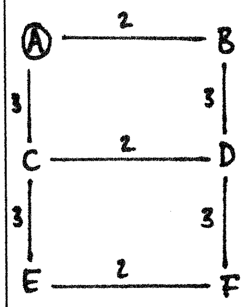
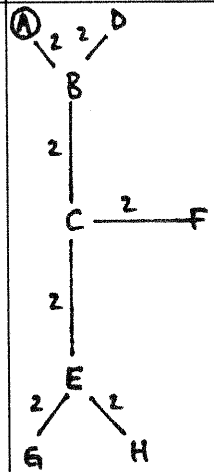
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(5+5\omega)$	A	ω	0	ω	$3-\omega$	1	
	B	ω	0	ω	$-2-\omega$	$-1-\omega$	
	C	ω	0	ω	$-77-6\omega$	$-236-33\omega$	
	D	ω	0	ω	$-7+4\omega$	$6-13\omega$	
(14)	A	1	0	1	-1	0	
	B	1	0	1	-11	12	
	C	1	0	1	4	-6	
	D	1	0	1	-36	-70	
	E	1	0	1	-171	-874	
	F	1	0	1	-2731	-55146	
$(2+6\omega)$	A	$-1+\omega$	$-2+\omega$	$-1+\omega$	$4-\omega$	2ω	Ⓐ
$(3+6\omega)$	A	0	ω	1	$-2+\omega$	-1	A
$(4+6\omega)$	A	1	$1-\omega$	0	-1	1	A
$(10+4\omega)$	A	$-1+\omega$	ω	ω	5	0	A
(15)	A	1	1	1	0	0	
	B	1	1	1	-5	2	
	C	1	1	1	-10	-10	
	D	1	1	1	-80	242	
	E	1	1	1	-135	-660	
	F	1	1	1	35	-28	
	G	1	1	1	-110	-880	
	H	1	1	1	-2160	-39540	
$(11+4\omega)$	A	0	$-1-\omega$	$-1+\omega$	$-6+\omega$	$8+2\omega$	A
(16)	A	0	0	0	1	$-2+4\omega$	Ⓐ
	B	0	0	0	1	$2-4\omega$	Ⓑ
$(3+7\omega)$	A	ω	$-2+\omega$	$-1+\omega$	$4-\omega$	ω	A
	B	1	$-2+\omega$	0	$-1-\omega$	1	B

Table (2.7.1) (continued)

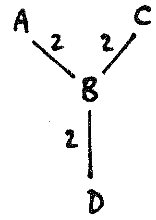
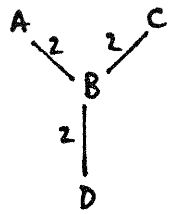
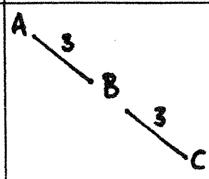
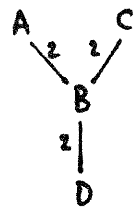
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(10+5\omega)$	A	1	$1-\omega$	1	$-5+2\omega$	$2-\omega$	
	B	1	$1-\omega$	1	$-10+2\omega$	$-10+3\omega$	
	C	1	$1-\omega$	1	$-135+22\omega$	$-580+191\omega$	
	D	1	$1-\omega$	1	$35-18\omega$	$-108-9\omega$	
(17)	A	1	-1	1	-1	0	
	B	1	-1	1	-6	-4	
	C	1	-1	1	-91	-310	
	D	1	-1	1	-1	-14	
$(4-8\omega)$	A	0	-1	0	-21	-31	A
$(6-8\omega)$	A	ω	$-1-\omega$	$-1+\omega$	ω	ω	
	B	ω	$-1-\omega$	$-1+\omega$	$140-34\omega$	$210+176\omega$	
$(16+2\omega)$	A	ω	0	1	0	0	
	B	ω	0	1	-5ω	$-2+4\omega$	
	C	ω	0	1	$-20-165\omega$	$1662-1132\omega$	
$(15+3\omega)$	A	ω	$-\omega$	ω	$3+\omega$	$1-\omega$	
	B	ω	$-\omega$	ω	$-2+\omega$	$-1+2\omega$	
	C	ω	$-\omega$	ω	$-77+16\omega$	$-196+86\omega$	
	D	ω	$-\omega$	ω	$-7-14\omega$	$-34+50\omega$	
(8ω)	A	0	$-1+\omega$	0	-1	0	
	B	0	$-1+\omega$	0	4	$-4+4\omega$	
$(7+7\omega)$	A	0	$-1+\omega$	$-1+\omega$	-1	0	A
$(4+8\omega)$	A	0	$1-\omega$	0	$-4-3\omega$	$-7+4\omega$	Ⓐ
$(18+2\omega)$	A	$-1+\omega$	ω	1	0	-1	
	B	$-1+\omega$	ω	1	50ω	$311+88\omega$	
$(5+8\omega)$	A	ω	$1-\omega$	$-1+\omega$	$5+\omega$	$4-2\omega$	A
$(19+\omega)$	A	ω	$-2+\omega$	0	$3-2\omega$	$-2+2\omega$	
	B	ω	$-2+\omega$	0	$8+18\omega$	$67+3\omega$	

Table (2.7.1) (continued)

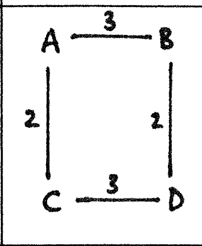
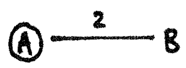
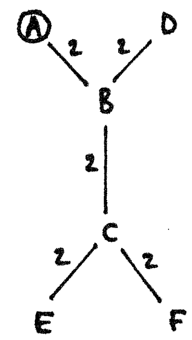
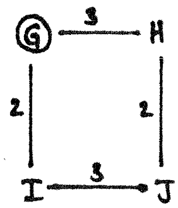
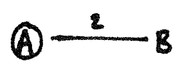
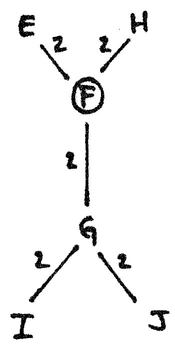
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
(20)	A	0	1	0	-1	0	
	B	0	1	0	4	4	
	C	0	1	0	-41	-116	
	D	0	1	0	-36	-140	
$(7+8\omega)$	A	ω	$-\omega$	ω	$1-\omega$	2	
	B	ω	$-\omega$	ω	$-19-6\omega$	$35+27\omega$	
$(15+5\omega)$	A	$-1+\omega$	1	$-1+\omega$	0	0	
	B	$-1+\omega$	1	$-1+\omega$	-5	$7+\omega$	
	C	$-1+\omega$	1	$-1+\omega$	$15+5\omega$	$40+3\omega$	
	D	$-1+\omega$	1	$-1+\omega$	$-105-5\omega$	$422+43\omega$	
	E	$-1+\omega$	1	$-1+\omega$	$210-15\omega$	$27+48\omega$	
	F	$-1+\omega$	1	$-1+\omega$	$140+105\omega$	$2065-252\omega$	
	G	$-1+\omega$	ω	$-1+\omega$	$1+3\omega$	$2+2\omega$	
	H	$-1+\omega$	ω	$-1+\omega$	$-19-2\omega$	$61-2\omega$	
	I	$-1+\omega$	ω	$-1+\omega$	$11-22\omega$	$41-36\omega$	
	J	$-1+\omega$	ω	$-1+\omega$	$16-302\omega$	$4832-2707\omega$	
(21)	A	ω	$-1+\omega$	ω	$2-2\omega$	3	
	B	ω	$-1+\omega$	ω	$12-7\omega$	$24+7\omega$	
	C	$-1+\omega$	$-\omega$	$-1+\omega$	2ω	3	
	D	$-1+\omega$	$-\omega$	$-1+\omega$	$5+7\omega$	$31-7\omega$	
	E	1	0	0	1	0	
	F	1	0	0	-4	-1	
	G	1	0	0	$-49i$	-136	
	H	1	0	0	-39	90	
	I	1	0	0	-784	-8515	
	J	1	0	0	-34	-217	
$(14+6\omega)$	A	$-1+\omega$	$-\omega$	$-1+\omega$	$2+3\omega$	0	A
	B	1	0	ω	1	2	B

Table (2.7.1) (concluded)

a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(5+9\omega)$	A	0	$0-\omega$	1	$-5+\omega$	$1+\omega$	A
$(18+4\omega)$	A	$-1+\omega$	0	1	0	0	
	B	$-1+\omega$	0	1	$5-5\omega$	$-16+4\omega$	A —3— B —3— C
	C	$-1+\omega$	0	1	$5-75\omega$	$516-304\omega$	
	D	1	1	1	$1+\omega$	2	D
$(2-10\omega)$	A	$-1+\omega$	$-2+\omega$	0	$2-\omega$	$-4-\omega$	A
(10ω)	A	ω	-1	1	2	$-2-\omega$	A

Table (2.5.2): The Birch, Swinnerton-Dyer Numbers

$\mathfrak{a}$		$c_{\mathfrak{p}}$	$\mid T \mid$	$\prod c_{\mathfrak{p}}/\mid T\mid^2$	$\alpha(E)$
$(1-2\omega)$	A	2	3	2/9	8.41778108
	B	6	3	2/3	2.80592703
	C	2	1	2	0.93530901
$(2\omega)=(2)(\omega)$	A	1,2	3	2/9	9.37616502
	B	3,2	3	2/3	3.12538834
	C	1,2	1	2	1.04179611
$(4+2\omega)=(2)(2+\omega)$	A	1,2	1	2	4.01644151
$(2-4\omega)=(2)(1-2\omega)$	A	3,2	3	2/3	3.41059020
	B	9,6	3	6	1.13686340
	C	27,2	1	54	0.37895447
	D	5,2	5	2/5	4.82527981
	E	1,2	1	2	0.96505596
$(5+3\omega)=(\omega)(4-\omega)$	A	2,1	2	1/2	8.23000912
	B	2,2	2	1	4.11500456
$(8+2\omega)=(2)(4+\omega)$	A	1,2	1	2	4.19314847
$(11)=(3-\omega)(2+\omega)$	A	1,1	5	1/25	9.25771812
	B	5,5	5	1	1.85154362
	C	1,1	1	1	0.37030872
$(3-6\omega)=(3)(1-2\omega)$	A	10,2	5	4/5	1.35882915
	B	2,2	1	4	0.27176583
	C	2,2	1	4	5.32864412
	D	2,2	4	1/4	3.26468900
	E	1,2	4	1/8	6.52937800
	F	4,2	4	1/2	1.63234450
	G	1,2	2	1/2	1.63234450
$(1-6\omega)=(1-\omega)^2(1+\omega)$	A	2,1	1	2	2.38585014
	B	2,3	1	6	5.33492311

Table (2.7.2) (continued)

a		$c_{\mathfrak{p}}$	$\mid T \mid$	$\prod c_{\mathfrak{p}} / \mid T \mid^2$	$\alpha(E)$
$(5+5\omega) = (\omega)(1-\omega)(1+\omega)$	A	$1,1,2$	2	1/2	7.51411790
	B	$2,2,4$	4	1	3.75705895
	C	$1,4,2$	2	2	1.87852948
	D	$4,1,8$	4	2	1.87852948
$(14) = (2)(1-\omega)(1+\omega)$	A†	$2,1,1$	6	1/18	7.87875422
	B†	$1,2,2$	6	1/9	3.93937711
	C	$6,3,3$	6	3/2	2.62625141
	D	$3,6,6$	6	3	1.31312570
	E	$18,1,1$	2	9/2	0.87541714
	F	$9,2,2$	2	9	0.43770857
$(2+6\omega)(2)(2-\omega)^2$	A	$1,1$	1	1	3.75121306
$(3+6\omega) = (3)(1+2\omega)$	A	$2,2$	1	4	5.60074752
$(4+6\omega) = (2)(1-\omega)(\omega-3)$	A	$2,1,1$	1	2	5.94753238
$(10+4\omega) = (2)(\omega)(3-\omega)$	A	$1,4,2$	1	8	3.79212444
$(15) = (\omega)(1-\omega)(3)$	A	$1,1,1$	4	1/4	8.94280685
	B	$2,2,2$	8	1/8	4.47140343
	C	$4,4,4$	8	1	2.23570171
	D‡	$1,1,1$	4	1/16	2.23570171
	E	$2,2,8$	4	2	1.11785086
	F	$8,8,2$	8	2	1.11785086
	G	$1,1,16$	2	4	0.55892543
	H	$1,1,4$	2	1	0.55892543
$(11+4\omega) = (1-\omega)(1+\omega)^2$	A	$2,1$	1	2	3.27119177
$(16) = (2)^4$	A	2	1	2	2.48119422
	B	2	1	2	2.48119422
$(3+7\omega) = (\omega-1)^2(2+\omega)$	A	$8,1$	1	8	3.83170094
	B	$1,1$	1	1	8.56794378

Table (2.7.2) (continued)

a		$c_{\wp}$	$\mid T \mid$	$\prod c_{\wp} / \mid T \mid^2$	$\alpha(E)$
$(10+5\omega)=(\omega)(1-\omega)(2+\omega)$	A	1,3,2	2	3/2	4.23512197
	B	2,6,4	4	3	2.11756098
	C	1,12,2	2	6	1.05878049
	D	4,3,8	4	6	1.05878049
$(17)=(4-\omega)(3+\omega)$	A	1,1	4	1/16	8.49575480
	B	2,2	4	4	4.24787740
	C	1,1	2	1/4	2.12393870
	D	4,4	4	1	2.12393870
$(4-8\omega)=(2)^2(1-2\omega)$	A	3,2	1	6	2.41539189
$(6-8\omega)=(-2)(1+\omega)(2+\omega)$	A	1,5,1	5	1/5	3.48548432
	B	1,1,5	1	5	6.97096863
$(16+2\omega)=(2)(1+\omega)(3-\omega)$	A	2,1,1	3	2/9	6.86135209
	B	6,3,1	3	2	2.28711736
	C	2,1,1	1	2	0.76237245
$(15+3\omega)=(\omega)(2-\omega)(3)$	A	2,2,1	4	1/4	5.80769969
	B	4,2,2	4	1	2.90384985
	C	2,2,1	2	1	1.45192492
	D	8,2,4	4	1/4	1.45192492
$(8\omega)=(2)^3(\omega)$	A	2,1	2	1/2	7.37780615
	B	2,2	2	1	3.68890308
$(7+7\omega)=(2-\omega)(1+\omega)^2$	A	1,4	1	4	5.25564883
$(4+8\omega)=(2)^2(1+2\omega)$	A	1,2	1	2	3.01595277
$(18+2\omega)=(2)(1-\omega)(2\omega-1)$	A	3,3,1	3	1	5.07836247
	B	1,1,3	1	3	1.69278749
$(5+8\omega)=(\omega)(2-\omega)(2+\omega)$	A	1,1,2	1	2	5.24553796
$(19+\omega)=(1-\omega)(2-\omega)(\omega-3)$	A	3,2,1	3	2/3	5.12176490
	B	1,6,1	1	6	1.70725497

Table (2.7.2) (continued)

a		$c_\wp$	$\mid T \mid$	$\prod c_\wp / \mid T \mid^2$	$\alpha(E)$
$(20) = (2)^2(\omega)(1-\omega)$	A	3,1,1	6	1/12	6.42309566
	B	3,2,2	6	1/3	3.21154783
	C	3,1,1	2	3/4	2.14103189
	D	3,2,2	2	3	1.07051594
$(7+8\omega) = (1-\omega)^2(-3-\omega)$	A	4,1	2	1	3.68057197
	B	4,2	2	2	1.84028598
$(15+5\omega) = (\omega)(1-\omega)(3+\omega)$	A	2,2,1	4	1	5.61588948
	B	4,4,2	8	1/2	2.80794474
	C	8,2,4	4	4	1.40397237
	D	2,8,1	4	1	1.40397237
	E	16,1,2	2	8	0.48336186
	F	4,1,8	2	8	0.70198619
	G	2,2,3	6	1/3	2.90141556
	H	2,2,6	6	2/3	1.45070778
	I	2,2,1	2	1	0.96713852
	J	2,2,2	2	2	0.48356926
$(21) = (2-\omega)(1+\omega)(3)$	A	2,2,1	2	1	6.08126033
	B	4,1,2	2	2	3.04063017
	C	2,2,1	2	1	6.08126033
	D	1,4,2	2	2	3.04063017
	E	1,1,2	4	1/8	6.89661544
	F	2,2,4	8	1/4	3.44830772
	G	2,2,2	4	1/2	1.72415386
	H	1,1,8	8	1/8	1.72415386
	I	2,2,1	2	1	0.86207693
	J	2,2,1	2	1	0.86207693
$(14+6\omega) = (2)(1-\omega)(2\omega-3)$	A	1,3,1	1	3	3.13190611
	B	1,1,1	1	1	4.83737244

Table (2.7.2) (concluded)

a		$c_{\mathfrak{p}}$	$\mid T \mid$	$\prod c_{\mathfrak{p}} / \mid T \mid^2$	$\alpha(E)$
$(5+9\omega)=(\omega)^2(1-2\omega)$	A	1, 2	1	2	3.76454614
$(18+4\omega)=(2)(1+\omega)(4-\omega)$	A	1, 1, 1	3	1/9	7.93953245
	B	3, 3, 1	3	1	2.64651082
	C	1, 1, 1	1	1	0.88217027
	D	1, 1, 1	1	1	4.21114018
$(2-10\omega)=(-2)(2+\omega)^2$	A	3, 2	1	6	2.99243282
$(10\omega)=(2)(\omega)^2(1-\omega)$	A	1, 2, 2	1	4	4.19314847

Table (2.7.3): Rational Newforms in $V^+(\mathrm{a})$; $\mathbf{K} = \mathbf{Q}(\sqrt{-19})$

a	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
$(1-2\omega)$	2/3	1.87061802	2.80597203
(2ω)	2/3	2.08359223	3.12538834
$(4+2\omega)$	0		
$(2-4\omega)a$	0		
$(2-4\omega)b$	4/5	1.93011193	2.41263991
$(5+3\omega)$	1/2	4.11500456	8.23000912
$(8+2\omega)$	0		
(11)	0		
$(3-6\omega)a$	64/5	4.34825328	0.33970729
$(3-6\omega)b$	0		
$(3-6\omega)c$	1	0.81617225	0.81617225
$(1-6\omega)a$	2	4.77170028	2.38585014
$(1-6\omega)b$	0		
$(5+5\omega)$	1/2	3.75705895	7.51411790
(14)	1/2	3.93937711	7.87875422
$(2+6\omega)$	1	3.75121306	3.75121306
$(3+6\omega)$	0		
$(4+6\omega)$	0		
$(10+4\omega)$	0		
(15)	1/4	2.23570171	8.94280685
$(11+4\omega)$	0		
$(16)a$	2	4.96238844	2.48119422
$(16)b$	2	4.96238844	2.48119422
$(3+7\omega)a$	0		
$(3+7\omega)b$	0		
$(10+5\omega)$	0		
(17)	0		
$(4+8\omega)$	0		

Table (2.7.3) (concluded)

a	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
$(6-8\omega)$	0		
$(16+2\omega)$	0		
$(15+3\omega)$	0		
(8ω)	1/2	3.68890308	7.37780615
$(7+7\omega)$	0		
$(4+8\omega)$	2	6.03190554	3.01595277
$(18+2\omega)$	12	5.07836247	0.42319687
$(5+8\omega)$	0		
$(19+\omega)$	0		
(20)	0		
$(7+8\omega)$	1	3.68057197	3.68057197
$(15+5\omega)a$	1	5.61588948	5.61588948
$(15+5\omega)b$	1/3	0.96713852	2.90141556
$(21)a$	1	6.08126033	6.08126033
$(21)b$	1	6.08126033	6.08126033
$(21)c$	1/4	0.86207693	3.44830772
$(14+6\omega)a$	0		
$(14+6\omega)b$	0		
$(5+9\omega)$	8	7.52909228	0.94113654
$(18+4\omega)a$	0		
$(18+4\omega)b$	0		
$(2-10\omega)$	0		
(10ω)	0		

§2.8 The Results for $\mathbf{Q}(\sqrt{-43})$

We give the following tables:

Table (2.8.1): Isogeny classes of elliptic curves with conductor $\mathbf{a}$, $N(\mathbf{a}) < 230$.

Table (2.8.2): The Birch, Swinnerton-Dyer Numbers.

Table (2.8.3): Newforms in $V^+(\mathbf{a})$, $N(\mathbf{a}) < 230$.

Table (2.8.1): Elliptic Curves with Small Conductor over $Q(\sqrt{-43})$

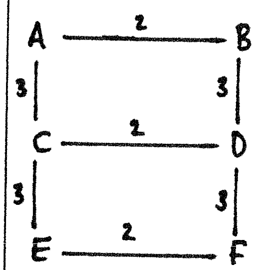
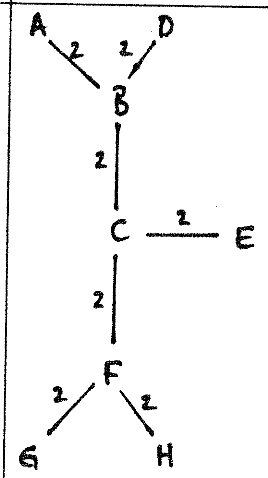
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(1-2\omega)$	A	0	1	1	0	0	A
(11)	A	0	-1	1	-10	-20	C —5— A —5— B
	B	0	-1	1	-7820	-263580	
	C	0	-1	1	0	0	
$(8+2\omega)$	A	$-1+\omega$	ω	1	$-\omega$	0	A —3— B —3— C
	B	$-1+\omega$	ω	1	$5+4\omega$	$-50-4\omega$	
	C	$-1+\omega$	ω	1	$105-16\omega$	$-285-84\omega$	
$(13+\omega)$	A	0	$-\omega$	$-1+\omega$	-2	1	A
(14)	A	1	0	1	-1	0	
	B	1	0	1	-11	12	
	C	1	0	1	4	-6	
	D	1	0	1	-36	-70	
	E	1	0	1	-171	-874	
	F	1	0	1	-2731	-55146	
(15)	A	1	1	1	0	0	
	B	1	1	1	-5	2	
	C	1	1	1	-10	-10	
	D	1	1	1	-80	242	
	E	1	1	1	-135	-660	
	F	1	1	1	35	-28	
	G	1	1	1	-110	-880	
	H	1	1	1	-2160	-39540	

Table (2.7.2): The Birch, Swinnerton-Dyer Numbers

a		$c_{\mathfrak{p}}$	$\mid T \mid$	$\prod c_{\mathfrak{p}} / \mid T \mid^2$	$\alpha(E)$
$(1+2\omega)$	A	2	1	2	7.45482142
$(11) = (1-\omega)(\omega)$	A	5, 5	4	1	1.85154362
	B	1, 1	1	1	0.37030872
	C	1, 1	5	1/25	9.25771812
$(8+2\omega) = (2)(4+\omega)$	A	2, 1	3	2/9	6.89959811
	B	6, 3	3	2	2.29986604
	C	2, 9	1	18	0.75476655
$(13+\omega)$	A	1	1	1	7.21827612
$(14) = (2)(7)$	A	2, 1	6	1/18	7.87875422
	B	1, 2	6	1/18	3.93937711
	C	6, 3	6	1/2	2.62625141
	D	3, 6	6	1/2	1.31312570
	E	18, 1	2	9/2	0.87541714
	F	9, 2	2	9/2	0.43770857
$(15) = (3)(5)$	A	1, 1	4	1/16	8.94280685
	B	2, 2	8	1/16	4.47140343
	C	4, 4	8	1/4	2.23570171
	D	1, 1	4	1/16	2.23570171
	E	8, 2	4	1	1.11785086
	F	2, 8	8	1/4	1.11785086
	G	16, 1	2	4	0.55892543
	H	4, 1	2	1	0.55892543

Table (2.8.3): Rational Newforms in $V^+(\mathfrak{a})$; $\mathbf{K} = \mathbf{Q}(\sqrt{-43})$

$\mathfrak{a}$	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
$(1-2\omega)$	0		
(11)	0		
$(8+2\omega)$	0		
$(13+\omega)$	0		
(14)	0		
(15)	0		

§2.9 The Results for $\mathbf{Q}(\sqrt{-67})$

We give the following tables:

Table (2.9.1): Isogeny classes of elliptic curves with conductor $\mathbf{a}$, $N(\mathbf{a}) < 265$.

Table (2.9.2): The Birch, Swinnerton-Dyer Numbers.

Table (2.9.3): Newforms in $V^+(\mathbf{a})$, $N(\mathbf{a}) < 265$.

Table (2.9.1): Elliptic Curves with Small Conductors over $Q(\sqrt{-67})$

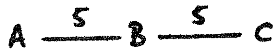
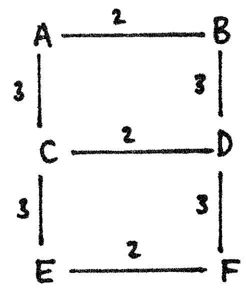
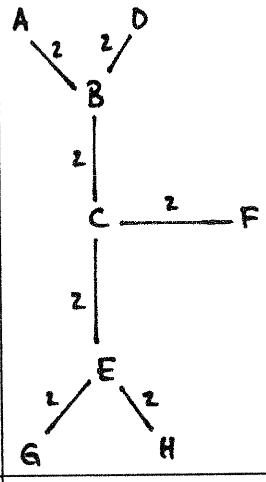
a		a_1	a_2	a_3	a_4	a_6	Isogeny Class
$(1-2\omega)$	A	0	1	1	-12	-21	A
(11)	A	0	-1	1	0	0	
	B	0	-1	1	-10	-20	
	C	0	-1	1	-7820	-263580	
(14)	A	1	0	1	-1	0	
	B	1	0	1	-11	12	
	C	1	0	1	4	-6	
	D	1	0	1	-36	-70	
	E	1	0	1	-171	-874	
	F	1	0	1	-2731	-55146	
(15)	A	1	1	1	0	0	
	B	1	1	1	-5	2	
	C	1	1	1	-10	-10	
	D	1	1	1	-80	242	
	E	1	1	1	-135	-660	
	F	1	1	1	35	-28	
	G	1	1	1	-110	-880	
	H	1	1	1	-2160	-39540	
$(12-3\omega)$	A	$-1+\omega$	$1-\omega$	ω	$3+3\omega$	$13-\omega$	A

Table (6.8.2): The Birch, Swinnerton-Dyer Numbers

a		$c_{\mathfrak{p}}$	$\mid T \mid$	$\prod c_{\mathfrak{p}} / \mid T \mid^2$	$\alpha(E)$
$(1-2\omega)$	A	2	1	2	3.85948296
(11)	A	1	5	1/25	9.25771812
	B	5	5	1/5	1.85154362
	C	1	5	1/25	0.37030872
$(14) = (2)(7)$	A	2, 1	6	1/18	7.87875422
	B	1, 2	6	1/18	3.93937711
	C	6, 3	6	1/2	2.62625141
	D	3, 6	6	1/2	1.31312570
	E	18, 1	2	9/2	0.87541714
	F	9, 2	2	9/2	0.43770857
$(15) = (3)(5)$	A	1, 1	4	1/16	8.94280685
	B	2, 2	8	1/16	4.47140343
	C	4, 4	8	1/4	2.23570171
	D	1, 1	4	1/16	2.23570171
	E	8, 2	4	1	1.11785086
	F	2, 8	8	1/4	1.11785086
	G	16, 1	2	4	0.55892543
	H	4, 1	2	1	0.55892543
$(12-3\omega) = (3)(4-\omega)$	A	1, 1	1	1	8.75824911

Table (2.9.3): Rational Newforms in $V^+(\mathrm{a})$; $\mathbf{K} = \mathbf{Q}(\sqrt{-67})$

a	$L(F,1)/\pi(F)$	$L(F,1)$	$\pi(F)$
$(1-2\omega)$	232/5	7.1896591	0.16635702
(11)	7/5	3.33277848	2.38055606
(14)	0		
(15)	1/2	4.77003032	9.54006064
$(12-3\omega)$	39/2	8.75824911	0.44914098

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