Modular Forms and Elliptic Curves over Imaginary Quadratic Number Fields

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I certify that all the material in this thesis which is not my own work has been identified and that no material is included for which a degree has previously been conferred upon me.

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ABSTRACT

The motivation for this thesis is two-fold.

First we investigate the correspondence between elliptic curves with conductor $\mathbf{a}$ and newforms of weight 2 for $\Gamma_0(\mathbf{a})$, where $\mathbf{a}$ is an ideal of ϑ_K and K is one of the 4 non-Euclidean imaginary quadratic number fields with class number 1. In Part I we develop an algorithm for finding rational newforms by calculating the action of the Hecke algebra on the first rational homology group of the hyperbolic upper half-space modulo $\Gamma_0(\mathbf{a})$. This work is an extension of Cremona's work [4] on modular forms over the 5 Euclidean fields.

We give tables of the results of implementing this algorithm on a computer. We list the dimensions of the +1 eigenspaces for the action of J on $H_1(\Gamma_0(\mathbf{a})\backslash\mathcal{H}_3^*, \mathbf{Q})$ along with the first few Hecke eigenvalues for each of the rational newforms. In addition we give tables of elliptic curves with small conductor, found via a systematic computer search using Tate's algorithm, and the trace of Frobenius at the first few primes. In all cases agreement was found in the Hecke eigenvalues and trace of Frobenius at the first 15 primes.

Secondly we provide extensive numerical evidence to support the Birch, Swinnerton-Dyer Conjecture. Part II is a description of joint work carried out with Cremona to calculate the quantities involved. We give tables of the results of these calculations over the 9 imaginary quadratic number fields with class number 1. We provide isogeny classes of curves of given conductor along with the order of the group of torsion points defined over K; the $c_\wp$ numbers; and the complex period of each curve. For each of the newforms corresponding to a class of elliptic curves without complex multiplication, we calculate the ratio $L(F,1)/\pi(F)$ where $L(F,1)$ is the value of the L-series of the newform, F, at $s = 1$ and $\pi(F)$ is the period. In the cases where $L(F,1)/\pi(F) \neq 0$ we list the values of $L(F,1)$ and $\pi(F)$. In the majority of cases we find agreement in the quantities predicted in the conjecture.