Essays on redistributive policies and household finance with heterogeneous agents

submitted by

Sylwia Patrycja Hubar

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Signature: Hubar

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Abstract

The overall objective of the thesis is to investigate needs and incentives of all income/wealth groups in order to explore ways and means to remedy the excessive economic inequality. A closer examination of individual decisions across richer and poorer households allows us to recognize conflicts of wants, needs and values and subsequently to draw recommendations for future policies.

The first chapter examines households’ preferences over the redistribution of wealth resources. The preferences of voting households are restricted by agents’ present and future resource constraints. The wealth resources vary over the business cycle, which affects the grounds for speculations of voting households. We augment the standard Real-Business-Cycle (RBC) model by the majority voting on lump-sum redistribution employing a balanced government budget. Our findings indicate that for the usual elasticity of labor supply both transfers’ level and share of output are procyclical, with the procyclicality increasing in the discrepancy between richer and poorer households.

In the second chapter we analytically demonstrate that all economic agents face subsistence costs that hinder economic and financial decisions of the poor. We find that the standard two-asset portfolio-selection model with a time-invariant subsistence component in the common-across agents Stone-Geary utility function is capable of explaining qualitatively and quantitatively three empirical regularities: (i) increasing saving rates in wealth, (ii) rising risky portfolio shares with wealth, (iii) more volatile consumption growth of the richer. On the contrary, “keeping-up-with-the-Joneses” utility with a time-varying weighted mean consumption produces identical saving rates and portfolio asset shares across richer and poorer agents, failing to match the micro data.
Finally, in the third chapter we use Epstein-Zin-Weil recursive preferences altered to include subsistence costs, as this form of utility function enables trade-off between stability and safety. We pursue an analytical investigation of a more complex multi-asset portfolio-choice model with perfectly insurable labor risk and no liquidity constraints and find further support of the data evidence. If households' total resources are anticipated to increase over time, poorer agents can afford to gradually escape subsistence concerns by choosing lower saving rates and accepting only minor portfolio risks as their consumption hovers close to the subsistence needs. The calibration part of the model economy shows that analytical results can quantitatively reconcile the data, too.
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My individual contributions to chapters

- Chapter 1: contributed to model description, data construction, wrote Gauss code and
  conducted simulations;

- Chapter 2: contributed to model description, data construction, calibration of parameter
  values, sensitivity analysis;

- Chapter 3: contributed to model description, calibration of parameter values, sensitivity
  analysis.
Definition of parameters

Chapter 1

\( t \) : time

\( Y \) : economy’s output

\( K \) : physical capital

\( H \) : capacity utilization

\( L \) : labor supply

\( \nu \) : intratemporal elasticity of substitution between capital and labor

\( I \) : investment

\( \delta \) : rate of depreciation

\( \delta_c \) : constant part of depreciation rate

\( b \) : slope parameter in the variant part of depreciation rate

\( \xi \) : elasticity of \( \delta' (H) \)

\( C \) : private consumption

\( G \) : public consumption

\( g \) : fraction of government consumption in total output

\( z_1 \) : total factor productivity shock

\( x_1 \equiv \ln (z_1) \)

\( z_2 \) : investment shock

\( x_2 \equiv \ln (z_2) \)

\( z_3 \) : government shock

\( x_3 \equiv \ln (z_3) \)

\( X \equiv [x_1 \ x_2 \ x_3]^T \) : vector comprising shocks

\( \alpha \) : share of capital in production function
\( \rho \) : persistence parameter
\( \varepsilon \) : random innovation
\( \sigma^2 \) : variance
\( \tau \) : marginal income tax rate
\( \tau' \) : next period marginal tax rate
\( T \) : lump-sum transfer payments
\( R \) : gross interest rate
\( w \) : wage rate
\( \mu \) : mass of household type
\( i \in \{r, m, p\} \) : determines household type
\( a \) : individual wealth endowments
\( \omega \) : individual labor productivity
\( l \) : individual labor supply
\( c \) : individual consumption
\( \theta \) : share of consumption in the utility function
\( \chi \) : intratemporal elasticity between labor and consumption
\( \eta \) : intertemporal elasticity of substitution
\( u \) : utility
\( r \) : net interest rate (after depreciation)
\( \bar{r} \) : after-tax interest rate
\( \bar{w} \) : after-tax wage rate
\( \Psi \) : policy rule
\( A \equiv [A_r \ A_m \ A_p]^T \) : vector comprising aggregate wealth holdings of each group
\( A \) : economic equilibrium law of motion
\( L \equiv [L_r \ L_m \ L_p]^T \): vector comprising aggregate levels of labor supply of each group
\( H \equiv [H_r \ H_m \ H_p]^T \): vector comprising aggregate capacity utilization of each group
\( V \): value function
\( h \): individual capacity utilization
\( \beta \): discount factor
\( A_{IE} \): intermediate equilibrium law of motion
\( \text{lss} \): labor supply at steady state
\( T^{ss} \): steady state transfers
\( \text{NetContrSS} \): net contribution to social system at steady state
\( \text{Contr} \): contribution to social system
\( \text{inv} \): individual investment

**Chapter 2**

\( t \): time
\( u \): utility function
\( c \): consumption
\( \chi \): subsistence parameter
\( \eta \): parameter directly connected to intertemporal elasticity of substitution
\( \bar{C} \): aggregate (average) consumption
\( \gamma \): weight assigned to average consumption
\( \rho \): time preference parameter
\( k \): total resources (wealth holdings and income)
\( y \): labor income
\( a \): wealth holdings
\( \phi \): share of risky assets
$R$ : rate of return on risky assets

$r_f$ : risk-free interest rate

$\sigma$ : standard deviation

$z(t)$ : standard Brownian process

$\varepsilon$ : disturbance

$J$ : value function

$\xi$ : slope coefficient in the decision rule for consumption

$\psi$ : constant parameter

$S$ : total savings

$\zeta$ : fraction of savings

$s$ : saving rate

$\bar{K}$ : average (aggregate) resources

$\rho_{a,y} \in (-1, 1)$ : coefficient of correlation between asset returns and income growth

$k_y$ : total lifetime resources of each income class

Chapter 3

$t$ : time

$u$ : utility function

$c$ : consumption

$\chi$ : subsistence parameter

$\eta$ : parameter directly connected to intertemporal elasticity of substitution

$\mu_y$ : average labor income growth

$\sigma_y$ : standard deviation of income growth

$I$ : total income (wealth holdings and income)
\( y \) : labor income
\( a \) : wealth holdings
\( \phi \) : share of risky assets
\( R \) : rate of return on risky assets
\( r_f \) : risk-free interest rate
\( \sigma \) : standard deviation
\( z(t) \) : standard Brownian process
\( \varepsilon \) : disturbance
\( \xi \) : slope coefficient in the decision rule for consumption
\( \psi \) : constant parameter
\( \Sigma \) : variance-covariance matrix
\( \sigma \) : \( N \times N \) matrix derived from decomposing \( \Sigma \)
\( p \) : asset price
\( e_i \) : \( 1 \times N \) vector, contains value 1 in the position of an asset \( i \in \{1, ..., N\} \) and zeros elsewhere
\( \rho_y \) : correlation between labor income and risky asset \( i \in \{1, ..., N\} \)
\( f \) : normalized “aggregator” function
\( J \) : continuation utility
\( \gamma \) : parameter that affects RRA coefficient
\( V \) : value function
\( \phi = [\phi_1 \ \cdots \ \phi_N] \) : row vector comprising portfolio shares allocated to risky assets
\( R = [R_1 \ \cdots \ R_N] \) : row vector comprising risky asset rates of return
\( \nu \) : squared reward-to-variability ratio
\( r_y \) : discount rate of labor income
1. Introduction

The widening discrepancy between rich and poor is a cause of political disputes, expanding divergence of opportunity and possibility. The economic inequality is especially evident in developing economies, in which dictatorial and authoritative leaders care more about their own interests than those of the society. However, the widening gap between rich and poor in industrialized countries is not unusual. In Tables I and II we report Gini coefficients for selected advanced and developing economies. The conspicuous high Gini coefficient in United States is comparable with the Gini values in some developing countries. Also, Portugal and the United Kingdom are featured by high measures of inequality.

In this thesis we show that the poor are rent-seekers during economic booms when fiscal redistribution of resources occurs. Also, we find that in the absence of government the poor’s economic and financial decisions are distorted. Agents confronted with the possibility to invest in financial markets and hence to earn high returns over short time period make more muted decisions if endowed with lower wealth holdings. This happens because all agents face subsistence costs that burden mainly those with scarce lifetime resources.

In the first article, we re-evaluate the implications of standard real-business-cycle (RBC) model on lump-sum transfer payments. We extend political-economy model of Krusell and Rios-Rull (1999) through RBC elements following King, Plosser and Rebelo (1988) and King and Rebelo (2000). We show that with widening gap between rich and poor voters become rent-seekers during economic upturns, benefiting from more leisure thanks to income effects through increased transfer receipts. In the second and third articles, we extend standard two-asset Merton (1969, 1971) model by considering subsistence needs. In this new environment, economic agents face great opportunities to increase their wealth endowments though risky asset investments. Nevertheless, the financial risks constrain decisions of the
poor whose wealth holdings just suffice to cover subsistence needs. We examine two concepts of subsistence consumption, namely constant and time varying bread-and-butter needs, and find that a constant part in determining subsistence level is crucial to match empirical patterns in the data. The Stone-Geary utility function with time-invariant subsistence cost generates saving rates, high-risk portfolio shares, consumption growth volatility, all positively related to initial and current financial wealth. The contrary is true for the time-variant subsistence consumption.

In summary, our results suggest that the poor need to be helped out otherwise they will remain poor. The procyclical demand for transfer payments indicates that the non-wealthy do not act at the expense of economy’s well-being, but only to support their exit from the poverty.

2. Background to the Study

In the first chapter, we introduce majority voting mechanism that enables us to examine political decisions of all income/wealth classes. Economic agents are heterogenous with respect to initial capital endowments and labor efficiency. The endogenous outcome of elections is the agents' political choice of optimal tax policy.

Agents pay income taxes and receive transfer payments. Government uses tax revenues to finance both transfers and government consumption, with the latter being “useless” to households. Government fiscal budget is balanced. There are two reasons why we do not allow for debt. First, our goal is to investigate incentives of agents with regard to the level of taxes and transfers today and whose incentives are not distorted by concerns over future fiscal finances. In light of Ricardian Equivalence principle the voters are forward-looking and internalize that today’s excessive government spending, if financed by debt, must be repaid
in the future. Second, incorporating another fiscal variable would increase the dimensions of “unknowns” to be voted on, technically challenging our analysis.

Households are classified into three groups as in Krusell and Rios-Rull (1999). We make use of the median voter theorem and set the optimal tax policy be equal to that of the median voter. In order to identify the median voter’s political preferences we choose the middle class to constitute 2% of the population.

In static models, the median voter theorem has been proved by means of single-peakedness or single-crossingness properties. Black (1948) gave proof of the median winner in the pairwise majority voting when preferences are single-peaked. In the static version of our model, we could compare pairwise tax rates across three groups and would find 49% supporters at each side of the median group. As a result, the rich (poor)’s optimal decision is to back the median voter so as to avoid higher (lower) than median tax rates.

Another study in support of the median voter result is that of Roberts (1977). Roberts (1977) introduced a principle of Hierarchical Adherence, which has assured the existence of majority voting equilibrium. Roberts (1977) proved that if the sequence of individual incomes does not hinge upon fiscal decisions, the preferences over tax rates are inversely related to incomes. Also, Meltzer and Richard (1981) followed Roberts’ (1977) condition by ordering preferred tax rates inversely to agent’s productivity. Gans and Smart (1996) show that Roberts’ (1977) Hierarchical Adherence implies a single-crossing of voters’ indifference curves. Voters’ marginal rates of substitution between two policy instruments, transfer payments $T$ and tax rate $\tau$, are upward sloping. If we fix a particular policy point $(\tau, T)$, the slopes of indifference curves can be ordered according to some parameter specific to each person, for example, according to individual’s initial wealth holdings. If the indifference curves are strictly increasing in that parameter, then single-crossingness in $(\tau, T)$ is guaranteed.
Yet, dynamic voting complicates the solution of the median-voter. Voting over next period’s tax rate requires a perfect foresight of possible outcomes. Voters need to be forward-looking about how their current political decisions will affect prices today and prices and policies in the future. For this reason we employ a structure-induced equilibrium of voting with rational and perfect-foresight voters as in Denzau and Mackay (1981). Voters learn the relationship between their decisions and ensuing outcomes over time and take both this relationship and their tastes into account. The institutional structure and procedure we employ is the majority voting rule. Nevertheless, our model’s setup, in contrast to Denzau and Mackay’s (1981) two-period model, has an infinite time horizon, which challenges our voting analysis, as we additionally need to ensure that the optimal government’s policy is consistent over time. For this reason we establish a politico-economic equilibrium as in Krusell et al. (1997) with government’s instantaneous pre-commitment as in Cohen and Michel (1988).

First, we define an economic equilibrium, in which current tax rate, $\tau$, is inherited from the previous period and future tax rates are given by a rule $\Psi$. The policy rule $\Psi (X, A, \tau)$ is an exogenously determined function of economic state and political variables. The value functions and aggregate laws of motion of state and choice variables depend on future political decisions and hence are indexed by the rule $\Psi (X, A, \tau)$. The decision rules are subsequently derived given the policy rule $\Psi$. Yet, in order to ensure that the government’s policy is consistent over time, namely given by the function $\Psi$ in every future period, we introduce an intermediate equilibrium, in which we deviate from $\Psi$ by setting next period tax rate equal to $\tau’$. The goal of this equilibrium is to investigate whether agents’ and government’s beliefs about future policies coincide with the political equilibrium outcomes. Here, agents

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1 The policy rule depends on the current state of the economy, namely shocks that hit economy (vector $X = [x_1, x_2, x_3]^T$), wealth holdings for each income group ($A = [A_r, A_m, A_p]^T$) and current tax rate ($\tau$).
express their voting preferences over next period tax rate, $\tau'$, taking as given the policy function $\Psi$ from period two onwards. Rational and fully forward-looking voters contemplating on the next period’s tax rates consider equilibrium responses of current and future prices and future policies. The political preferences of the pivotal group are aggregated into the function $\psi (X, A, A_m, \tau | \Psi)$. Evaluating the median voter’s political aggregator at economy-wide variables returns in equilibrium the policy function $\Psi (X, A, \tau)$. Government takes voters’ political preferences into account and pre-commits (honestly and accurately pre-announces) to the next period tax rate. The intermediate equilibrium can be defined as Feedback Stackelberg with government as a leader and private sector as a follower. Government is aware that its final policy decision directly affects the actions of private agents. Consequently, it does not deviate from its pre-commitment.

In support of the median voter result in a dynamic setting, we have found one study that proves the existence of majority equilibrium. Koulovatianos and Mirman (2010) manage to establish a result of the median winner in a sequential forward-looking voting for a certain model description, namely the AK model.

In summary, political and economic household decisions are examined in a dynamic and stochastic model economy. We introduce three shocks into the model. A positive productivity innovation increases the effectiveness of both factors, labor supply and utilized capital, in the production of output. Here, we follow Kydland and Prescott (1982), King, Plosser and Rebelo (1988), King and Rebelo (2000). The second shock is on investment as in Greenwood, Hercowitz and Huffman (1988) and Greenwood, Hercowitz and Krusell (2000). A positive technological change specific to investment lowers the price of capital in real terms raising the efficiency in producing capital goods. Also, this shock lowers the maintenance costs of capital, as the effective depreciation rate declines. Positive innovation
to government spending as in Chari et. al. (1994) acts as a distortion to the economy. The economic agents do not value government consumption goods but are burdened by higher taxation that finances these goods. Economic and fiscal policy decisions over the business cycle are valuable in recognizing households’ needs and incentives.

In the second research article, optimal decisions on portfolio allocation and consumption are explicitly derived in a continuous-time model. As in Merton (1969), agents have access to financial markets, allocating their wealth between risky and non-risky securities. Yet, our form of preferences is different from that in the standard two-asset Merton’s (1969) model. In particular, we modify agents’ values through incorporating a subsistence cost into agents' preferences. Since the subsistence consumption can take different forms, we confront two subsistence definitions in our investigation. The first subsistence form is defined as an invariable level of existential needs, which is given by a parameter $\chi$. The monthly amount of subsistence needs is determined based on the survey estimates of Koulovatianos (2007). The second subsistence cost is defined by the average consumption level, $\bar{C}(t)$, of the community, in which agents live, which is weighted by agents’ taste for the aggregate consumption with an equal (across agents) weight $\gamma$. We show that the usual Stone-Geary utility function with constant subsistence component is a better model variant than the “keeping-up-with-the-Joneses” preferences, as the first and not the latter can explain the micro data.

In the third research article, we enrich the model economy from the second chapter through a possibility of investment in a broad range of risky assets $N \geq 1$ and introduce an additional source of stochastic income, which comes from supplying labor. Here, we follow a multi-asset portfolio choice model with random labor income of Merton (1971). Both risky asset returns and labor earnings are uncertain. Random behaviors of asset prices and labor income evolve according to geometric Brownian motion process. In more detail, stochastic
returns of each asset $i \in \{1, \ldots, N\}$ are generated by geometric Brownian motion with a drift, as in Karatzas et al. (1986). The first term is a deterministic part representing an asset’s $i$ average return over unit of time. The second term is a stochastic part capturing asset $i$’s variability multiplied by the Brownian motion that captures random risk of an asset $i$. In a similar way, we model income fluctuations following Koo (1998). The deterministic term accounts for average income rate of growth over unit of time. The second stochastic term measures degree of income variation, which is multiplied by the Wiener process that accounts for random income fluctuations.\footnote{To model price and income dynamics, we add an average return per unit of time to the standard Wiener process. Moreover, since we are interested in a process that drives growth rates, namely asset and income returns, and additionally we need to assure that this process does not take on negative values, we model the stochastic processes as geometric Brownian motions.} The continuous time allows for analytical analysis of both economies. Yet, in the third chapter, we need to make use of one assumption in order to obtain analytical solutions: we presume that risky-asset trading perfectly offsets labor-income risk. The budget constraint in both models governs the evolution of wealth and is represented as a stochastic differential equation.\footnote{Thorough derivation of the stochastic differential budget equation from its discrete-time counterpart (stochastic difference equation) can be found in Merton (1969).} Finally, the optimization problem is formulated by means of Hamilton-Jacobi-Bellman (HJB) equation.

A huge novelty of our research are more realistic model economies, in which agents are exposed to business cycle shocks and individual specific existential, investment and labor-income risks. In all investigated economies the heterogeneity of wealth is considered. Subsequently, needs and incentives across all wealth groups can be recognized helping to draw implications for future policies.
Research Article 1

3. Cyclical Demand for Redistributive Policies

The chapter is based on the working paper “Cyclical Demand for Redistributive Policies”, co-authored with Carlos Cortinhas and Christos Koulovatianos.

3.1 Introduction

In light of fiscal issues that have arisen from the recent debt crisis, the public and private policy observers have understood that the interplay between the functionality of recommended policies and the variability of resource constraints is more important than during any other world-wide recessions. On the one hand voting households internalize negative effects of certain policies and want to retain the aggregate economic situation in good standing. On the other hand voters care about their own interests and as rent-seekers want to benefit from the resources of richer households or households that have a potential or incentives to produce larger income flows. Since the rent-seeking behavior is a root of suboptimality and inefficiency with real effects on the economy, it has attracted a huge attention of scholars who investigate structure-induced political equilibria. Yet, one issue remains uninvestigated, namely the cyclical behavior of the rent-seeking. This article fills this gap.

We re-evaluate the implications of a standard real-business-cycle (RBC) model on the lump-sum redistribution over the business cycle. We use the heterogeneous-agent model of Krusell and Rios-Rull (1999) and add RBC elements by the paradigm of King, Plosser and Rebelo (1988) and King and Rebelo (2000). As in Krusell and Rios-Rull (1999) we make assumption of a balanced fiscal budget. It is a crucial and necessary point for our

investigation as we use the concept of politico-economic equilibrium with a government that
pre-commits to its policy\textsuperscript{5} and the failure of commitment would make the government
default on its debt.\textsuperscript{6}

We explore that for standard calibration of the RBC model the level and the output frac-
tion of transfers are procyclical. This happens although we have assumed exogenous positive
innovations to government consumption (which is not valued by voters): as the government
budget is balanced, lump-sum redistribution acts, partly, as a shock-neutralizer to the higher
exogenous fiscal spending, which is financed by temporary increased tax burdens; this role
of fiscal redistribution is obvious when no inequality exists (i.e., there are no disputes over
transfers, shifting the political focus on the allocation of tax burdens over time); yet, as the
discrepancy between rich and poor expands, voters become rent-seekers during economic
upturns, benefiting from more leisure thanks to income effects through higher lump-sum
transfer payments. Yet, such a cyclical movement of transfers is not predominant in the
data. We observe procyclical transfers only in developing countries that are associated with
high wealth/income inequality\textsuperscript{7}. Consequently, the parameters used to calibrate the model
with politico-economic equilibrium for fiscal transfers must be re-investigated in future work.
One idea is to re-examine parameters in the elasticity of labor supply. Also, one can extend
the model by introducing incomplete markets to the model with voting.

3.2 The Model

In this section, we develop a dynamic stochastic general equilibrium model with voting
over fiscal transfer payments.\textsuperscript{8} The economy’s output is produced by employing two input

\textsuperscript{5} See Cohen and Michel (1988).
\textsuperscript{6} For more about this point see the relevant discussion in Klein et al. (2008, p. 804).
\textsuperscript{7} The U.S. data statistics for inequality that we use in our investigation coincide well with the inequality
levels in developing countries, which may explain procyclical fiscal redistribution in our model.
\textsuperscript{8} All steps and details on derivation of the model are provided in Appendix A.
factors, labor supply and utilized capital. There is a Hicks-neutral technology shock to the production function that augments total factor productivity. The production function is given by,

$$Y_t = z_{1,t} \left[ \alpha (K_t H_t)^{1-\frac{1}{v}} + (1 - \alpha) L_t^{1-\frac{1}{v}} \right]^\frac{v}{v-1},$$  \hspace{1cm} (1)

in which the productivity shock, $z_{1,t}$, follows a first-order autoregressive process,

$$\ln (z_{1,t+1}) = \rho_1 \ln (z_{1,t}) + \varepsilon_{1,t+1},$$  \hspace{1cm} (2)

where $\rho_1$ is a persistence parameter and $\varepsilon_{1,t+1}$ is a random innovation with zero mean and variance $\sigma_{1,t}^2$, identically and independently distributed over time, namely it is $\varepsilon_{1,t+1} \sim N(0, \sigma_{1,t}^2)$. We follow Greenwood et al. (1988) that capital, $K_t$, is utilized at a variable rate, $H_t$, over time. The capacity utilization, $H_t$, measures how intensively economy’s capital is used in the production of goods. Our motivation to introduce capacity utilization, $H_t$, comes from the fact that capital goods (machinery, factories, equipment, tools) are used less intensively during recessions and more intensively during expansions, which is especially important to consider in the cyclical investigation of redistributive policies that we perform here.\(^9\)

A large number of firms aim to maximize their profits by employing capital and labor in the production process. The intensive use of capital stimulates firms’ profits. Households supply firms with labor supply and capital goods, while firms supply households with the final consumption goods. Utility-maximizing households value consumption of goods and leisure time. Each consumer has one unit of time, which he allocates between work and leisure. Accordingly, consumers choose the amounts of labor and capital they provide to firms. Also, households propose the rate of capital utilization, $H$, to firms, controlling the intensity of capital use. Firms that demand capital for production need to agree to the

\(^9\) See King and Rebelo (2000) for more details.
utilization rates. The reason why households want to control capacity utilization is that the amount of capital maintained is influenced by the rate of intensity (utilization) the capital is being employed. Households, as capital owners, bear costs of intensive use and thus wear-out of capital, as shown by the capital’s law of motion,

\[ K_{t+1} = z_{2,t} \cdot I_t + (1 - \delta(H_t)) K_t , \]  

(3)

in which \( z_{2,t} \) is an autocorrelated investment-specific shock that evolves according to the process,

\[ \ln (z_{2,t+1}) = \rho_2 \ln (z_{2,t}) + \varepsilon_{2,t+1} , \]  

(4)

where \( \rho_2 \) is an autocorrelation parameter, while \( \varepsilon_{2,t+1} \) is a random disturbance with zero mean and variance, \( \sigma^2_{2,t} \), identically and independently distributed over time, hence \( \varepsilon_{2,t+1} \sim N (0, \sigma^2_{2,t} ) \). The innovation \( z_2 \) boosts aggregate investment \( I_t \) increasing production efficiency of new capital goods, \( K_{t+1} \). Greenwood et al. (2000, Figures 1 and 2) point to the importance of this type of shock by documenting a pattern of declining relative prices of equipment and at the same time a pattern of increasing demand for the equipment goods in the US postwar period. Scholars argue that the standard sector-neutral productivity shock does not explain these changes in relative prices. Hence, the investment-specific technological progress accounts for the increasing efficiency of new equipment over time that enables a lower-cost access to the new technology (equipment) over time (e.g. more powerful and faster computers).

The investment-related innovation makes (directly) the investment in new capital goods more efficient and (indirectly) the replacement of existing capital goods less costly, both in terms of consumption units. We can see this effect more clearly when we derive for investment, \( I_t \), in equation (3),
\[ I_t = e^{-x_{2,t}} \left[ K_{t+1} - (1 - \delta(H_t)) K_t \right], \] (5)

in which,

\[ x_{2,t} \equiv \ln(z_{2,t}). \]

Notice that the investment shock also lowers the utilization costs of existing capital, as the effective depreciation rate becomes \( e^{-x_{2,t}} \delta(H_t) \). In summary, new technology can be developed more efficiently due to the investment-related innovation (new better equipment becomes available), so higher utilization of old existing capital (since less costly) is optimal even though it implies an accelerated capital depreciation\(^{10}\).

Following Greenwood et al. (1988) and Greenwood et al. (2000), the rate of depreciation, \( \delta(H_t) \), is an increasing convex function of capacity utilization rate (\( \delta'(H_t) > 0 \) and \( \delta''(H_t) > 0 \)). The endogenous rate of capital depreciation is given by,

\[ \delta(H_t) = \delta_c + \frac{b_\delta}{1 + \xi} H_t^{1+\xi}; \quad \delta_c, b_\delta, \xi > 0. \] (6)

So, we assume that the depreciation rate consists of a constant part, \( \delta_c \), and of a variable part that varies together with the utilization rate\(^{11}\).

The economy’s aggregate resource constraint is given by,

\[ Y_t = C_t + I_t + G_t, \] (7)

in which total production output, \( Y_t \), is partly consumed by households, \( C_t \), partly invested in new capital, \( I_t \), and partly consumed by government, \( G_t \). Inserting the equation for investment (5) into the resource constraint gives,

\[ e^{-x_{2,t}} K_{t+1} = Y_t + e^{-x_{2,t}} (1 - \delta(H_t)) K_t - C_t - G_t. \] (8)

\(^{10}\)See Greenwood et al. (1988) and Greenwood et al. (2000) for more details.

\(^{11}\)Notice that parameter \( \xi \) measures the elasticity of \( \delta'(H) \).
We can see in equation (8) that a positive innovation to investment reduces the replacement value of current and future capital (the price of capital falls in real terms in response to an increase in \( x_2 \)). Also, intensity of capital use becomes less costly due to \( e^{-\delta_2 \delta H_t} \) (notice we have \( \delta(H_t) \) in the deterministic economy).

The government consumes fraction \( g \) of total production output, \( Y_t \). The amount of government consumption is given by,

\[
G_t = ge^{x_{3,t}}Y_t,
\]

in which \( g \) is exogenously defined and \( x_{3,t} \) is a positive government spending shock, as in Chari et. al. (1994). Notice that,

\[
x_{3,t} \equiv \ln(z_{3,t}).
\]

The government spending innovation, \( z_{3,t} \), follows an autoregressive process in logarithms,

\[
\ln(z_{3,t+1}) = \rho_3 \ln(z_{3,t}) + \varepsilon_{3,t+1},
\]

in which \( \rho_3 \) is a persistence parameter and \( \varepsilon_{3,t+1} \) is a random disturbance, identically and independently distributed, with zero mean and variance \( \sigma_{3,t}^2 \), namely \( \varepsilon_{3,t+1} \sim N(0, \sigma_{3,t}^2) \).

A positive fiscal policy shock boosts government consumption. In fact, fiscal innovation, \( z_{3,t} \), increases efficiency in provision of government goods. The government spending shock implies a higher fraction of production output allocated to government, which subsequently affects economy’s resources that are consumed and invested by individual households. Such a fiscal advancement may be deliberate, for example as a measure to stimulate economic activity. Nonetheless, in our closed economy government expenditures do not act as public goods that enter households’ preferences. Consequently, since agents do not derive satisfaction from consuming government goods, the fiscal innovation that drains economic resources
available to households negatively influences households’ decisions. The model’s simulation results show that the government spending shock has indeed a dampening effect on economic activity.

Government consumption, $G$, together with fiscal transfer payments, $T$, constitute total government expenditures. The amount of lump-sum fiscal transfers, $T_t$, is given by,

$$T_t = (\tau_t - ge^{z_{x,t}}) Y_t - \tau_t \delta(H_t)e^{-z_{e,t}} K_t,$$

(11)

where $\tau_t$ is a marginal income tax rate in period $t$. Multiplying tax rate, $\tau_t$, with total economy’s output, $Y_t$, gives the gross amount of government revenues in period $t$. And since the amount of depreciated capital is tax-exempt, the net amount of government tax revenues is given by $\tau_t \cdot (Y_t - \delta(H_t)e^{-z_{e,t}} K_t)$. As we can see from the budget constraint (11) government is bound to a balanced fiscal budget. So, it is not allowed to incur debt. The reason we restrict the government’s finances to be balanced is to eliminate difficulties related to the “curse of dimensionality”, which would occur in a multidimensional voting.

The linear-quadratic approximation of our model economy, which we employ here, cannot endogenously co-determine both the level of debt and the size of transfer payments in an environment with time-consistent public policy, such as in Krusell and Rios-Rull (1999) and Klein and Rios-Rull (2003).12

Inserting the expression for government spending from equation (9) into the resource constraint (8) yields,

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12See also the discussion in Klein and Rios-Rull (2003, p. 1218) about the technical requisites of introducing debt to such a model of time-consistent policy setting. Stockman (2001, 2004) shows that the policy-setting difference between balanced budgets and limits to run deficits (similar to those in the European Monetary Union) is not substantial under commitment, but it remains an open question whether time-consistent policy rules have similar properties.
\[ e^{-x_{2,t}} K_{t+1} = (1 - ge^{x_{2,t}}) Y_t + (1 - \delta(H_t)) e^{-x_{2,t}} K_t - C_t . \]  

(12)

In our economy with price-taking, profit-maximizing firms, perfectly competitive market implies that producers’ profits are driven down to zero. The economy’s production exhibits constant returns to scale,

\[ Y_t = R_t K_t H_t + w_t L_t . \]  

(13)

Plugging equation (13) into the resource constraint (12) gives,

\[ e^{-x_{2,t}} K_{t+1} = [1 + (1 - ge^{x_{2,t}}) e^{x_{2,t}} R_t H_t - \delta(H_t)] e^{-x_{2,t}} K_t + (1 - ge^{x_{2,t}}) w_t L_t - C_t . \]  

(14)

Price-taking representative producer employs labor supply and physical capital so as to maximize his profits.\(^{13}\) The cost of renting one unit of utilized capital, \( R_t \), is equal to the factor’s marginal physical product,

\[ R_t = \alpha z_{1,t}^{1 - \frac{1}{\nu}} \left( \frac{Y_t}{K_t H_t} \right)^{\frac{1}{\nu}} , \]  

(15)

and the marginal cost of hiring labor, \( w_t \), is equal to the marginal product of labor,

\[ w_t = (1 - \alpha) z_{1,t}^{1 - \frac{1}{\nu}} \left( \frac{Y_t}{L_t} \right)^{\frac{1}{\nu}} . \]  

(16)

Our model economy is inhabited by a continuum of infinitely-lived households with a total measure normalized to 1. Households are divided into three types, rich, median and poor (subscripts “r”, “m”, “p” are given respectively for each type). Each household type has a mass \( \mu_i \), \( i \in \{r, m, p\} \) that is normalized, \( \sum_i \mu_i = 1 \). Household heterogeneity stems from differences in initial wealth endowments, \( a_{i,0} > 0 \), and differences in individual labor productivity, \( \omega_i \), which is time-invariant and normalized so that \( \sum_i \mu_i \omega_i = 1 \), \( i \in \{r, m, p\} \).

\(^{13}\) For the statement of maximization problem see Appendix A.
Moreover, there is a very large number of identical households within each type class. Consequently, each individual household is like a “drop in the ocean” within own class. In every period a household $i$ decides on the allocation of one unit of time between work, $l_{i,t}$, and leisure, $(1-l_{i,t})$, and on the level of his own private consumption, $c_{i,t}$.

The expected lifetime utility function of a household $i$ is given by,

$$u(c_{i,t}, 1 - l_{i,t}) = \frac{\theta c_{i,t}^{1-\frac{1}{\gamma}} + (1 - \theta)(1 - l_{i,t})^{1-\frac{1}{\gamma}}}{1 - \frac{1}{\gamma}} - 1.$$  

(17)

The individual $i$’s budget constraint is given by,

$$e^{-z_{i,t}}a_{i,t+1} = (e^{-z_{i,t}} + \bar{r}_t) a_{i,t} + \bar{w}_t \omega d_{i,t} + T_t - c_{i,t},$$  

(18)

in which $\bar{r}_t$ is after-tax rental rate of capital,

$$\bar{r}_t \equiv (1 - \tau_t) \left( R_t H_{i,t} - e^{-z_{i,t}} \delta(H_{i,t}) \right), \quad t = 0, 1, ...,$$

(19)

and $\bar{w}_t$ is after-tax wage rate,

$$\bar{w}_t \equiv (1 - \tau_t) w_t, \quad t = 0, 1, ...$$  

(20)

The household maximization problem, derivation of first-order conditions and steady state calculations are provided in Appendix A.

### 3.3 Dynamic voting

The definition of politico-economic equilibrium in a dynamic infinite-horizon setting is a tedious task. The difficulty lies in assuring that optimal policy plans are time-consistent.

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With an exception to prices lower-case letters denote individual variables. Accordingly, the upper-case letters give aggregate variables.
over time. An optimal policy today, re-optimized at a later date, may no longer be optimal to implement. An example of such a policy is an altered open-loop Stackelberg equilibrium in Cohen and Michel (1988), in which the policy maker implements his policy once and for all time periods (having knowledge of the private sector’s best response) while the private sector plays every period. Since a time-inconsistent policy fails to comply with Bellman’s principle of optimality, our aim is to ensure that the policy maker’s initial decision is time-consistent.

Furthermore, we also need to make sure that time-consistent policy is indeed optimal. As shown in Cohen and Michel (1988) the optimality of time-consistent policy is only guaranteed in an economy with forward-looking agents. Scholars show that subgame perfect policies may be sub-optimal due to myopic policy maker’s actions. More precisely, a feedback Nash equilibrium, although time-consistent, is only sub-optimal as a myopic government, expected to follow a particular policy, chooses, based on the best expected response of the private sector, to follow another policy. A myopic government does not attempt to understand whether and how his new policy decision that deviates from its initial plan may affect the private sector.

The case in which the private sector needs to foresee government’s policy is contrasted with an event in which private agents monitor the implemented policy. Cohen and Michel (1988) show that only when government can really and truly announce its policy in front of the private sector’s actions, the subgame perfect equilibrium is optimal. Hence, time-consistent optimal policy is the feedback Stackelberg equilibrium policy. The government moves first knowing about its instantaneous impact on the private sector’s actions. The private sector observes the government’s policy decision (anticipating that the chosen policy will be implemented in each of the future periods) and responds to it. As a result, the policy maker needs to be forward-looking, namely be able to assess how his current policy decision
affects the private sector’s present and future actions.

Accordingly, in our analysis the policy decisions are made ahead of economic decisions and these are the voting households who need to perfectly and rationally foresee the impact of their policy decisions, namely the current level of income tax, on events today and in the future (prices today and prices and policies in the future). Since households are heterogeneous, different levels of income tax are favored by various wealth/income classes. Forward-looking voters internalize the distortionary effect of high taxes on the rich’s decisions but at the same time the alleviating effect on the poor’s decisions.

Furthermore, the concept of perfect-foresight and its importance in defining structure-induced equilibria were pointed out by Denzau and Mackay (1981). Scholars show that a structure-induced equilibrium with perfect-foresight does not coincide with the corresponding equilibrium in which agents have myopic predictions. In summary, scholars emphasize the importance of not only preferences and institutional procedures/structures but also voters’ forward-lookingness in determining equilibria. Yet, they point to some limitations associated with voters’ preferences under both assumptions of perfect and myopic expectations, arguing that preferences may be multi-peaked, resulting in no equilibrium. Scholars conclude that the institution within the model’s framework requires a more restrictive structure in order to resolve the issue of multi-peaked preferences. In our analysis, we adopt majority voting rule for the institutional structure of our model economy and argue that the median voter is decisive in every elections. Since it is extremely cumbersome to prove the median voter result in a dynamic model setting, we follow Krusell and Rios-Rull (1999) and perform a numerical cross-check, which verifies that the political preferences around the implied steady-state equilibrium are single-peaked in tax rate, $\tau_{t+1}$. To our knowledge, the only study that proves the existence of the decisive median voter in a dynamic setting is that of Koulovatianos and
Mirman (2010). Koulovatianos and Mirman (2010) manage to obtain the result of single-crossing preferences in a dynamic setting for a special case of the parametric model they use. Since the parametric variant of our model is similar to that of Koulovatianos and Mirman (2010), we feel confident in assuming the median voter result.\footnote{Notice, as outlined in the "Background to the Study", the median voter result was proved in static models by use of single-peakedness and single-crossingness criterions.}

To overcome the issue of time consistency, the median voter’s beliefs about future policies need to coincide with the equilibrium outcomes. Hence, the median voter has to take into account the decision and policy rules of his successors when contemplating on today’s policy. Otherwise, he does not have a well-defined problem. The fixed-point problem guarantees that the anticipated future political decisions coincide with the optimal future median voters’ behaviors. As in Krusell et al. (1997) and Krusell and Rios-Rull (1999) the politico-economic equilibrium guarantees time-consistency and makes the political views over $\tau_{t+1}$ be induced by the state of the economy. More precisely, the preferences over $\tau_{t+1}$ evaluated at the economy-wide values are in equilibrium consistent with the policy rule, $\Psi$, which is a function of the current state of the economy and which represents political decisions from period $t+2$ onwards.

### 3.3.1 Politico-economic equilibrium

In this section, we present the concept of politico-economic equilibrium with majority voting. Government balances its fiscal budget in every period. At the beginning of each period, households vote on the level of income tax rate endogenously determining the amount of resources redistributed. Our aim is to mimic the timing of presidential and parliamentary elections in democratic countries, in which governments pre-announce and commit to some level of fiscal budget. There exists two potential connections to the real world. Either, there
are elections every four years and a chosen government commits to its pre-announced fiscal policy for the period of four years (so, one period in the model encompasses four years), or political preferences of the voting agents are reflected in the fiscal budget at the beginning of every fiscal year irrespective of whether elections occur or not.

**Economic equilibrium with exogenous policy** The model assumes that elections take place every period. First, we define economic equilibrium\(^{16}\) assuming that next period tax rates are constant and follow policy rule, \(\Psi\),

\[
\tau' = \Psi (X, A, \tau);
\]

in which the vector \(A \equiv [A_r \ A_m \ A_p]^T\) contains the aggregate wealth holdings, \(A_i\), of each group \(i \in \{r, m, p\}\) and vector \(X \equiv [x_1 \ x_2 \ x_3]^T\) includes shocks to total factor productivity, \(x_1\), investment, \(x_2\), and government consumption, \(x_3\). The aggregate control and state variables of all three groups advance according to the law of motion \(A (X, A, \tau | \Psi)\) that is conditioned upon policy rule \(\Psi\):

\[
\begin{bmatrix}
L \\
A' \\
H
\end{bmatrix} = A (X, A, \tau | \Psi),
\]

in which vectors \(L \equiv [L_r \ L_m \ L_p]^T\) and \(H \equiv [H_r \ H_m \ H_p]^T\) contain respectively the aggregate levels of labor supply, \(L_i\), and capacity utilization, \(H_i\), for each group \(i \in \{r, m, p\}\).

The Bellman equation for each individual household’s problem is given by,

\[
V^i (X, A, a_i, \tau | \Psi) =
\]

\[
= \max_{a'_i \in [a_i, 1], l_i \in [0,1], h_i \geq 0} E \left\{ u \left( e^{-x_2} + \bar{r} \right) a_i + \bar{\omega} \omega_i l_i + T - e^{-x_2} a'_i, 1 - l_i \right\} +
\]

\(^{16}\)Detailed derivation of economic equilibrium is given in Appendix A, pages 134-140.
\[ + \beta V^i (X', A', a_i', \tau' \mid \Psi) \] (21)

subject to the fiscal budget constraint,

\[ T = (\tau - g e^{-\alpha}) Y - \tau \delta(H) e^{-x} K \] (22)

the aggregate law of motion,

\[
\begin{bmatrix}
L \\
A' \\
H
\end{bmatrix} = \mathcal{A} (X, A, \tau \mid \Psi),
\] (23)

and the assumption that future tax rates follow policy rule \( \Psi \),

\[ \tau' = \Psi (X, A, \tau). \] (24)

Through iterations on the value function \( V^i (X, A, a_i, \tau \mid \Psi) \) we obtain a fixed point for \( V^i (X, A, a_i, \tau \mid \Psi) \) and the corresponding individual economic decision rules. Furthermore, the iterations on \( \mathcal{A} (X, A, \tau \mid \Psi) \) lead to the fixed point for the aggregate law of motion. After having calculated the economic decision rules given policy \( \Psi \) we turn to specifying the “intermediate equilibrium” rule \( \mathcal{A}_{IE} (X, A, \tau, \tau' \mid \Psi) \) that takes into account a one-period deviation in tax rate.

**Political equilibrium** The definition of political equilibrium necessitates to consider economic equilibria in which tax rates deviate from the policy rule \( \Psi \). The voting households need to take into account such equilibria when contemplating on their current policy decision. For this reason, the next period policy tax is set to an arbitrary rate \( \tau' \) and only
tax rates from period \( t + 2 \) are given by the policy \( \Psi \). For the one-period tax deviation we determine the decision rule \( A_{IE} (X, A, \tau, \tau' \mid \Psi) \).\(^{17}\)

The individual two-period maximization problem, in which the next period tax rate is set arbitrarily, is given by a non-Bellman equation\(^{18}\),

\[
\hat{V}^i (X, A, \alpha_i, \tau, \tau' \mid \Psi) = \\
= \max_{\alpha'_i \in [\alpha_i, \alpha_i], \lambda_i \in [0, \lambda_i], \lambda_i \geq 0} E_0 \left\{ u \left( (e^{-s_2} + \bar{r}) \alpha_i + \bar{\omega} \lambda_i \lambda_i + T - e^{-s_2} \alpha'_i, 1 - \lambda_i \right) + \right. \\
\left. + \beta V^i (X', A', \alpha'_i, \tau' \mid \Psi) \right\} \\
\text{subject to the “intermediate” aggregate law of motion for labor, wealth holdings and capacity utilization,} \\
\begin{bmatrix}
    L \\
    A' \\
    H
\end{bmatrix} = A_{IE} (X, A, \tau, \tau' \mid \Psi). \quad (26)
\]

Notice that the individual value function, \( V^i (X', A', \alpha'_i, \tau' \mid \Psi) \), is only on the right hand side of equation (25). The iterations are on the “intermediate” law of motion \( A_{IE} (X, A, \tau, \tau' \mid \Psi) \) so that the individual economic decision rules computed by the non-Bellman equation (25) coincide with the aggregate decision rules \( A_{IE} (X, A, \tau, \tau' \mid \Psi) \). The value function for a one-period tax deviation, \( \hat{V}^i (X, A, \alpha_i, \tau, \tau' \mid \Psi) \), is a key by-product of the non-Bellman equation (25).

The “intermediate equilibrium” rule \( A_{IE} (X, A, \tau, \tau' \mid \Psi) \) comprises how the aggregate economy responds to changes in the next period tax rate, namely to variations in \( \tau' \) only. The

\(^{17}\)For detailed derivation of intermediate equilibrium see Appendix A, pages 140-145.
\(^{18}\)This is a non-Bellman equation as the individual value function, \( V^i (X', A', \alpha'_i, \tau' \mid \Psi) \), is only on the right hand side of equation (25).
model’s one-period commitment to \( \tau' \) corresponds to the concept of instantaneous policy pre-commitment in Cohen and Michel (1988). The main by-product function \( \hat{V}^i (X, A, a_i, \tau, \tau' \mid \Psi) \) with an argument \( \tau' \) can reveal the individual representative household’s political preferences for each class \( i \in \{r, m, p\} \), given the economy’s current state \((X, A, a_i, \tau)\) and conditioned upon the rule \( \Psi \), which determines policies from period \( t + 2 \) onwards.

In particular, we are interested in the median voter’s political preferences, as voting households from both sides of the median class (poorer and richer agents) support the median tax rate. Consequently, we maximize the representative median agent’s non-Bellman equation (25) with respect to policy rate \( \tau' \) in order to assess political preferences of the median voter,

\[
\max_{\tau'} \hat{V}^m (X, A, a_m, \tau, \tau' \mid \Psi),
\]

which establishes whether the policy rule \( \Psi \) is, indeed, reproduced as an equilibrium political outcome. So, the pivotal preferences of the median group must coincide with the aggregate political decisions and thus the condition for the fixed point is,

\[
\Psi (X, A, \tau) = \psi (X, A, A_m, \tau \mid \Psi),
\]

in which,

\[
\psi (X, A, a_m, \tau \mid \Psi) \equiv \arg \max_{\tau'} \hat{V}^m (X, A, a_m, \tau, \tau' \mid \Psi).
\]

In summary, the fixed-point condition ensures that the representative agent’s policy rule returns the function \( \Psi \).

### 3.4 Calibration

We calibrate the model’s parameters according to standard values in macro literature, which are derived from long-run characteristics of the U.S. economy. In some cases the parameter-
ORIZATION is debatable and we choose such values that allow moments and cyclical patterns of the model to coincide with the ones of the US economy.

Moreover, we impose some targets to ensure that our model is well-behaved. The average consumption to output ratio should be around 60\%−65\%\textsuperscript{19}. The average labor supply should amount to one-third\textsuperscript{20}. Similar conditions were adopted by Krusell and Rios-Rull (1999). The endogenous depreciation rate of 9\% is chosen to comply with the average rates reported in the literature.\textsuperscript{21}

Cooley and Prescott (1995) point to the observations of constant leisure and steadily increasing real wages after year 1930 and suggest the elasticity of substitution between consumption and leisure to be close to one. We perform sensitivity analysis on parameter $\chi$. In particular, we want to know what happens when we vary the level of labor supply elasticity, which is given by $d\ln(l)/d\ln(w) = \chi \cdot (1 - lss)/lss$\textsuperscript{22}. Higher (lower) values of $\chi$ imply more (less) responsive labor supply to variations in wages. Our targets here are values chosen by Greenwood et al. (1988) who assign value of 1.7 to the intertemporal elasticity of substitution in labor supply, Aiyagari et al. (2002) who set the elasticity of labor supply to 2 and Prescott (2004) who estimates the elasticity of labor supply to changes in income tax rates to be nearly 3. Since our setup is a real business cycle model, we need to account for business cycle fluctuations for which a high value of labor supply elasticity is essential. The changes in parameter $\chi$ are accompanied by simultaneous changes in the share of consumption, namely the parameter $\theta$, so as to preserve the targeted level of labor

\textsuperscript{19}A similar assumption was imposed in Krusell & Rios-Rull (1999) and Nakajima (2005).
\textsuperscript{20}Similar values were used in Cooley & Prescott (1995), Krusell & Rios-Rull (1999) and Bachman & Bai (2010).
\textsuperscript{22}Definition of variables: percentage change in labor supply in response to a 1\% change in the wage rate: $d\ln(l)/d\ln(w)$, intratemporal elasticity of substitution between consumption and leisure: $\chi$, steady state labor supply: $lss$. 

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supply.

We assign a value of 0.5 to the intertemporal elasticity of substitution (IES), $\eta$, in order to obtain a risk aversion parameter of 2.\textsuperscript{23} In the microeconomic literature a common value chosen for the IES is 1 (see, for instance, Klein and Rios-Rull (2003)). Nonetheless, Krusell and Rios-Rull (1999) pick 0.25 in order to comply with the real business cycle literature. We perform comparative statics on the parameter $\eta$.

In our benchmark parameterization we calibrate the share of government consumption $g$ to 19%. Similar values are reported in Krusell and Rios-Rull (1999).\textsuperscript{24} Also, Barseghyan et al. (2010) choose based on U.S. data the government spending share of 18.4%, whereas Baxter and King (1993) calibrate it in accord with the postwar experience to 20%.

In the sensitivity analysis we decrease the share of government spending to 14%, which would be the case when schooling were not part of government consumption but instead of transfer payments (see, for instance, Krusell and Rios-Rull (1999)). Also, Xavier Mateos-Planas (2009) reports based on OECD Economic Outlook the government spending share of 15%.

We provide four sets of parameter values. Our benchmark is $\theta = 0.45$, $\chi = 1.2$, $g = 0.19$ and $\eta = 0.5$. For the sensitivity analysis, we increase the parameters $\chi$ and $\eta$ and decrease $g$. First, we study the effects of an increase in $\chi$ from 1.2 to 1.5. The elasticity of labor supply raises from 2.71 to 3.47. This value seems to be extremely high, but we show in the following sections that such values are crucial in the real business cycle environment. High levels of labor supply elasticity restore the observed procyclical pattern of labor and decrease the less evident procyclical behavior of transfer payments. In addition to changes in the parameter $\chi$

\textsuperscript{23}See, for instance, Itzetzki (2010) who sets the coefficient of relative risk aversion to 2.

\textsuperscript{24}Also, Klein & Rios-Rull (2003) assign values of 21.6% and 18.4% for the government spending process in their model, whereas Klein et al. (2008) report the output share consumed by government to be slightly under 20%.
we increase the value of the parameter $\theta$ from 0.45 to 0.475 in order to keep the steady state level of labor supply at about 1/3. Next, we run a simulation for a lower value of $g$ with benchmark parameters for $\theta$, $\chi$ and $\eta$. Finally, we increase the elasticity of intertemporal substitution $\eta$ from 0.5 to 0.8 in order to investigate a closer setup to the logarithmic one. The higher $\eta$ the greater is the agents’ willingness to substitute consumption and leisure over time.

The remaining parameters are well established in the literature and thus are not subject to change. The value of the discount rate $\beta$ is set to 0.96 (see, for example, Kydland and Prescott (1982) and Greenwood et al. (1988)). The technology parameter $\alpha$ is one-third. This is a generally agreed value for the capital share in the macro literature (see, for example, King and Rebelo (2000)). The parameter values $\delta_c$, $b_\delta$, $\xi$ in the formula for depreciation rate are assigned so that the steady state level of depreciation rate is about 0.09. The endogenous depreciation rate is increasing and convex in $H_t$ and thus the elasticity of depreciation $\xi = h \ast \delta''(h)/\delta'(h)$ is greater than 0. The first term $\delta_c$ represents a fixed part of the depreciation rate and is set to 0.03. The second term $b_\delta \frac{\delta_{1+\xi}}{1+\xi}$ depends on the utilization degree of capital. With the value $\xi$ approaching infinity, the importance of the second term diminishes and the depreciation rate approaches the constant component. On the contrary, when the value of the parameter $\xi$ nears zero ($\xi \rightarrow 0$), the marginal costs of depreciation of higher capital utilization decrease. Hence, the capital is more utilized. Also, a lower (higher) $\xi$ implies a less (more) responsive wage rate to changes in labor supply (see, for more details, King and Rebelo (2000)). Taking into account these offsetting effects and our depreciation target rate of 0.09 we assign a value of 1.42 to the parameter $\xi$.

The intratemporal elasticity of substitution between capital and labor $\nu$ is set to 0.99 in order to approximate the properties of the Cobb-Douglas production function. The persis-
tence and standard deviation parameters of the first-order autocorrelation processes driving the shocks are calibrated for the benchmark case as follows. We choose the autocorrelation coefficient of 0.9 (in line with King, Plosser and Rebelo (1988)) and the standard deviation of 0.0046 for the total factor productivity shock. The investment-specific technical change is assigned a persistence value of 0.64 as in Greenwood et al. (1998). In the case of government shock we follow Chari et al. (1994) and Stockman (2001) by setting the persistence parameter to 0.89. The benchmark case standard deviations $\sigma_2$ and $\sigma_3$ are set to 0.006 in order to comply with output fluctuations in the data. Accordingly, the values of standard deviations are adjusted for each subsequent simulation.

The preference and technology parameters are given in Table 1.1. Table 1.2 summarizes the parameters of the autocorrelation processes. In Table 1.3 we report distributional statistics of the U.S. economy, which we employ in our simulation exercises. The inequality distributions are crucial in the model as no redistribution of resources occurs in economies with equally endowed agents. Accordingly, the first group, 49 percent of the households, owns the lowest wealth, the next group of 2 percent constitutes the median wealth, and the last group of 49 percent owns the highest wealth. The middle group is chosen to be small in order to identify political preferences of the decisive voter.

3.5 Simulation results

3.5.1 First moments: benchmark and sensitivity analysis

First moments are reported in Table 1.4. The moments are given for wealth/income equality and inequality. The non-heterogeneous case is less important as equally endowed agents do not have incentives to become rent-seekers. Yet, the comparison of both simulation results is interesting. Our condition for a well-behaved model is maintained across different parameter values. For instance, reducing government spending from 19% to 14% makes
the ratio of consumption to output go up from 61% to 66% with both being close to our targeted measures. The rise in private consumption share is straightforward as government decreasing its consumption imposes lower taxes, boosting private demand. Our next target referring to labor supply is also achieved as shown in Table 1.4. The mean of labor supply varies between 30%–32% across all simulations. The benchmark ratio of transfers to output equals 14.6%. This ratio decreases to 12.7% in an environment with more elastic agents, as government can collect and redistribute less taxes. Accordingly, for a slightly lower labor supply elasticity (2.65 in simulation 3 versus the benchmark 2.71) the ratio of transfers to output is even higher. However, although the elasticity of labor supply reaches the lowest level of 2.59 in simulation 4, the share of transfers slightly decreases (from 15.8% in sim. 3 to 14.9% in sim. 4), which is in turn attributed to the higher value of $\eta = 0.8$. Our $T/Y$ values are slightly higher but close to those reported in the literature. Krusell and Rios-Rull (1999) estimate the U.S. transfers to be 10.6% of GDP when the means-tested transfers are incorporated. Furthermore, Chari et al. (1994) use balanced-growth path transfers as government obligations and report their approximated value to be 12% of GDP in 1985. In addition, we calculate the shares of redistribution as a percentage of GDP for selected developing and OECD countries. In order to construct the series of transfer payments we retrieve and sum up series for Social Benefits and Subsidies from the Government Finance Statistics database. The GDP series are retrieved from Penn World Tables 6.3. All details on the construction of the series are given in the Data Appendix A. The transfers to output shares and the correlations of transfers with output are reported in Table 1.7 for developing and Table 1.8 for OECD countries. Implied transfers to output ratios for developing countries are very low ranging from 1% to 8%. Yet, the OECD countries are featured with higher transfer payments between 8% and 23%. Averaging shares across OECD countries gives
14%, which is comparable with our simulated values. The ratio of investment to output is slightly above 19% in all simulations. Such an investment share is common in the literature. For example, Klein et al. (2008) calibrate it to a little over 20%, whereas Nakajima (2005) sets a target for investment share of 18%.

Based on the first results we conclude that the share of transfers and the mean of labor supply (slightly) increase with the declining elasticity of labor supply. Tax distortions are larger for less responsive agents to variations in wages ($\tau = 0.417$ versus $\tau = 0.395$) when $g$ is 19% and lower for agents more willing to substitute intertemporally ($\tau = 0.36$ versus $\tau = 0.37$) when $g$ is 14%. These findings indicate that governments can tax more heavily and thus redistribute a higher share of output in economies with more risk averse agents.

3.5.2 Second moments: benchmark and sensitivity analysis

Second moments of our model economy come close to business cycle statistics of the U.S. economy reported in King and Rebelo (2000). In Table 1.5 we report business cycle moments for our benchmark parameters, $\theta = 0.45$, $\chi = 1.2$, $g = 0.19$, $\eta = 0.5$, and for higher values of $\theta$ and $\chi$. In Table 1.6 we provide results for a lower share of government spending, $g = 14\%$, and investigate the effect of an increase in the intertemporal elasticity of substitution, $\eta$, from 0.5 to 0.8.\textsuperscript{25} Throughout, by high substitution economies we mean economies with a greater elasticity of labor supply (simulation 2 with $dln(l)/dln(w)$ of 3.47) and with a stronger intertemporal elasticity of substitution (simulation 4 with $\eta = 0.8$). The opposite is true for the remaining economies.

There is a positive and strong correlation of output with consumption, in line with the data. The procyclicality is slightly less pronounced for a model economy with $\eta = 0.8$, which

\textsuperscript{25}Accordingly, the parameter values are $\theta = 0.45$, $\chi = 1.2$, $g = 0.14$, $\eta = 0.5$ in the third and $\theta = 0.45$, $\chi = 1.2$, $g = 0.14$, $\eta = 0.8$ in the fourth simulation.
indicates a notable role of intertemporal elasticity of substitution in consumption decisions.

The correlation of output with investment is slightly above the correlation value in the data (benchmark: 0.85 versus data: 0.8). The procyclicality of investment is less pronounced in high substitution economies, which gives support to the choice of higher elasticities in order to come closer to reported business cycle moments.

The procyclical pattern of labor is, in particular, apparent in the economy with a high elasticity of labor supply (correlation is 0.79). Since agents react more elastically to changes in wages, their actions will be more pronounced in response to economic ups and downs. The cyclical patterns of labor are of lower magnitude in the remaining simulations varying from 0.52 to 0.59, which is notably below the evident correlation of 0.88. Hence, according to the data labor supply responds vividly to changes in wages.

Transfer payments comove positively with output fluctuations. This procyclical behavior of transfers is stronger for low substitution economies. Furthermore, the correlations of transfer-to-output ratio with output, of transfers with lagged output, of transfers with consumption and investment, all indicate a similar pattern of a positive but relatively lower comovement in high substitution environments. The more responsive agents are, the (relatively) less pronounced is the positive correlation of transfers and transfer-to-output ratio with $Y$. This happens because government cannot freely increase tax burdens in economies with elastic agents who are willing to postpone labor over time. The median voter internalizes these effects and votes for a relatively higher redistribution only when it is (relatively) less distortionary. Subsequently, the procyclical pattern of taxes is especially evident in low substitution economies. In addition, increases (decreases) in social care and medicare during booms (contractions) indicate that voters care about good standing of the economy. They vote for high redistribution only when the economy can afford it.
The procyclicality of redistribution observed in our simulated economy is less evident in the data. In particular, the OECD countries run countercyclical fiscal policies as reported in Table 1.8. Average correlation of cyclical components of transfer payments and GDP for selected OECD countries amounts to $-0.13^{26}$. Ilzetzki (2010) estimates this value for high-income countries, which is a much broader cross-section of countries, to be $-0.26$. Our lower countercyclical fiscal policy result based on OECD countries is brought about by highly procyclical transfers in Hungary and South Korea. Also, Belgium, Norway and Luxembourg are featured by a procyclical fiscal redistribution, which, however, increases by less than output as shown by a negative correlation of transfer-to-output ratio with output. The correlation of the output share of fiscal transfers $T/Y$ with GDP is only positive for Hungary and South Korea and highly negative for other OECD countries. The opposite is true for developing countries. Most developing countries run a procyclical fiscal policy as shown in Table 1.7. The average correlation between cyclical components of transfer payments and GDP comes to 0.17. The average correlation for Latin American countries in Ilzetzki (2010) is estimated to be 0.11. Consequently, high procyclical fiscal policy in our simulated economies resembles better the cyclical patterns in developing countries. We think that our results are driven by the large income/wealth discrepancy in our model, which reflects the distribution dynamics of the U.S. economy. In the U.S. the inequality is striking and comparable to the wealth/income distribution dynamics in some developing countries as measured by the Gini coefficient.²⁷

²⁶Missings in Government Finance Statistics dataset and thus lack of a comprehensive time series for Subsidies and/or Social Benefits for some OECD and developing countries has limited our results to “selected” countries.
²⁷See Tables 1 and 2 for Gini coefficients for some selected countries. The Gini coefficient for the United States amounts to 0.46. Source: UNU-WIDER (United Nations University World Institute for Development Economics Research, World Income Inequality Database).
so that the output variations in our artificial economy coincide with output fluctuations in
the data. The consumption fluctuates slightly more in the simulated economy (benchmark
1.47 versus 1.35 in the data). The variations are larger in high substitution environments. In
contrast, the fluctuations in investment are slightly subdued in the model (benchmark 4.34
versus 5.30 in the data). Higher intertemporal elasticity of substitution, $\eta$, implies larger
investment fluctuations of 4.77 matching better the data. The standard deviation of labor
is considerably lower than the one evident in the data (benchmark 0.64 versus 1.79 in the
data). Labor supply is most volatile in high substitution economies, as agents respond more
actively to economic events.

So far we have compared high substitution against low substitution economies. Yet, no
remarkable differences are apparent between economies that are distinguished by the share of
government consumption (19% versus 14% respectively).\textsuperscript{28} Minor differences result rather
from slightly different elasticities of labor supply (2.71 versus 2.65 respectively) than from
different levels of government consumption. We find that, in the benchmark economy, the
procyclical behavior of transfers, transfer-to-output ratio and taxes is less distinct and that
of labor more pronounced.

High substitution economies match better the data. In particular, high elasticity of labor
supply is important for the real business cycle environment. Economies with higher labor
supply and intertemporal elasticity are more responsive to shocks. Agents are more flexible
and willing to substitute between consumption and leisure (higher parameter $\chi$), to postpone
labor over time (higher $\chi \cdot (1 - \text{loss})/\text{loss}$) and to substitute intertemporally (higher parameter
$\eta$). Hence, there are stronger comovements of labor with output and weaker comovements of
transfers with output, resulting in a better match to the data. Governments cannot freely

\textsuperscript{28}See Table 7 for the benchmark and Table 8 for the corresponding economy with $\eta = 0.5$, $\theta = 0.45$, $\chi = 1.2$, but a lower value of $g = 0.14$. 

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decide on the level of taxes in an economy with elastic agents, as they are more willing to refrain from work and investment.

The redistribution of resources occurs only in economies with wealth/income discrepancies. This result is present in our numerical investigation. We are only able to observe increases in transfer payments as a share of GDP when we widen the income/wealth gap. Also, the procyclical pattern of fiscal policy instruments, transfers and taxes, becomes most pronounced for the last level of inequality that reflects the distributional statistics of the U.S. economy. Moreover, the positive correlations of transfers with consumption, with investment and with lagged output show similar gradual increases in wealth/income gap. Finally, the correlation of output with transfers and taxes increases with inequality, whereas the correlation of output with labor decreases with inequality. So, the greater the need for medicare and social security systems resulting from the larger discrepancy between the rich and the poor, the (relatively) lower are the procyclical properties of labor. The poor and the median want to gain at the expense of the rich. The rich, on the other hand, internalize their role of net-contributors. Consequently, all income groups increase labor supply by less than they would do if they faced equal wealth endowments.

### 3.6 Impulse responses

In this section, we analyze impulse responses. In Figures 1.1 to 1.3 the responses to 1% total factor productivity (TFP) shock, to 1% investment-specific shock and 1% government spending shock are plotted.

Figure 1.1 shows the evolution of shocks and the responses of output, consumption and investment. The response of output is positive to all three shocks, but of significantly lower magnitude following a government spending shock. The deviation from steady state output is larger and more persistent after the TFP than the investment shock. Moreover, the output
deviation after investment shock becomes slightly negative from year 20 onwards. Output increases only a little in response to a government shock and abruptly declines below the steady state level already in period one. This happens due to a prompt increase in taxation levied to finance this sudden boost in government spending.

Consumption reacts strongly positively to the productivity shock. Higher productivity increases output and thus resources that can be consumed and/or invested. Consumption drops slightly due to a positive response of tax rates in year one, then increases a little to decline steadily over time. The distortionary taxation lessens the hump-shaped pattern of consumption that is in turn evident in the event of an investment shock. Consumption reacts negatively to the investment shock, but increases over the first 6 years and then gradually converges back to its steady state level. It remains above the steady state level for over 20 years. There is a negative response of private consumption to the positive innovation in government spending. In addition, the consumption declines even more in period one due to increased tax payments. An increase in government spending is not beneficial to private agents in any way, as they do not value government consumption goods. Moreover, there is an adverse effect on the private sector’s decisions coming from government innovation as a positive shock to government spending is financed by distortionary taxes crowding out private consumption and investment. An economy hit by a government spending shock anticipates in period zero higher future tax burdens and experiences indeed higher taxes in the periods after the shock.

Investment responds positively to the productivity and investment innovations. However, the effect of the productivity shock is more persistent due to a higher autocorrelation coefficient. The level of investment above the steady state lasts for 22 years following the TFP shock and only 5 years after the investment shock. Investment does not respond to the
government spending boost in period zero, but already in period one it drops due to higher
tax burdens. The level of investment remains below the steady state over 14 years with
its level steadily increasing year after year. From period 15 onwards agents seem to invest
slightly above the steady state level, which signals agents’ desire for rebound and recovery.

The graphs of Figure 1.2 depict impulse responses of labor, transfers, taxes and transfers
to output ratio. Labor reacts positively to all three shocks, but abruptly declines due to
tax increases in period one. Labor remains above the steady state level for 8 years following
the TFP shock and for only 3 years after the investment shock, steadily decreasing to below
the steady state level. The positive response of labor supply to government shock drops
below the steady state level already in period one due to increase in taxes. The labor supply
remains below the steady state for over 15 years and then it increases slightly above the
steady state pointing to agents’ desire for a rebound and improvement in living standards.

Transfer payments respond positively to both TFP and investment shock. Yet, the
response to TFP shock is larger and more persistent than the reaction to investment shock.
In period one the deviations from the steady state transfers increase even more as the
government sees space for a rise in taxes and redistribution, improving its image among
poorer voters for the next elections. In the event of a government spending shock transfers
drop sharply below the steady state, as the government finances its higher consumption
at the cost of transfers. Although higher government revenues in period one rise slightly
transfers, the transfer payments remain below the steady state (with its level gradually
increasing) over ensuing years.

The impulse responses of transfer to output ratio \(T/Y\) exhibit similar patterns as
the ones of transfers, but are of lower magnitude. There is a positive response of \(T/Y\) in
period zero in the event of productivity and investment shocks. In period one the \(T/Y\)
ratio moves further away from the steady state following the TFP and investment shock, as the government is able to redistribute more to households. In later years the $T/Y$ ratio converges steadily to the steady state. The transfer-to-output ratio responds negatively to a positive government shock in year zero, but increases sharply already in year one due to higher tax revenues.

Although by construction the elections take place every period, the voting results over income taxes are implemented with a one period lag. A booming economy in year zero implies higher taxes in the next period. Governments can afford to raise taxes following positive innovations to productivity or investment as their distortionary impact will be offset by positive economic developments. On the contrary, the government spending shock drags the whole economy down as the resources available to the private sector are dwindling.

Figure 1.3 depicts the impulse responses of capacity utilization, capital, wages and interest rates. Since the capital depends on the previous state of the economy, it responds with a lag to all shocks. Capital responds strongly positively to TFP and investment shocks. The mechanics are as follows. Total factor productivity shock increases the marginal product of labor and capital. Hence, wages and interest rates soar, and thus labor supply and capital investment in period zero. The investment shock, on the other hand, lowers the price of capital in real terms raising efficiency in producing capital goods and brings down the maintenance costs of capital increasing the gross return on capital. As a result, there is an excess supply of capital in the economy following productivity and investment shocks. The effect is more persistent and of larger magnitude ensuing the TFP shock. Levels of capital above steady state will gradually go down as agents will start to enjoy more leisure over time, which results in a hump-shaped pattern of capital responses.

\footnote{Capital in the period after the shock depends on the steady state of the economy in the period prior to the shock.}
As already pointed out, a positive investment shock does not only enhance the production efficiency of today’s and tomorrow’s capital but it also lowers the maintenance cost of capital making the effective depreciation rate equal to $e^{-2\delta} (H_t)$. Consequently, this effect reduces the marginal cost of utilization. The productivity shock, in turn, affects positively the marginal product of capacity utilization. As a result, both shocks lead to a pick-up of capacity utilization in period zero. The excessive levels of utilization decline below the steady state quite rapidly ensuing both TFP (in year 6) and investment (in year 3) shocks. This happens because agents start to consume (hump-shaped pattern of consumption in Figure 1.1) and to enjoy more leisure (declining labor supply in Figure 1.2) by cutting down on investment (see Figure 1.1). Hence, the capital after having reached a peak is less utilized over time.

The wage rate following the investment shock is at its steady state level in period zero, but falls slightly below the steady state in year one due to higher taxes and then moves upwards, above the steady state, in the next period, which in turn is due to a sharp decline in labor supply. The wage rates remain above the steady state for 19 years. Also, wages jump above the steady state in response to the productivity shock, but decrease rapidly already in year one as a result of higher taxes and converge slowly back to the steady state over the ensuing years. Excess taxation used to finance innovation in valueless government consumption distorts wages considerably. Wages drop below the steady state in period one and then gradually increase as the economy recovers.

Interest rates respond positively to a government consumption shock but drop below the steady state in period one due to higher taxes. The interest rates soar already in period 5, which indicates that taxes on capital gains finance the excessive government consumption. Since the productivity and investment shocks increase the marginal product of capital and
gross interest rate, the interest rate surges in the event of these innovations, but declines quite quickly below the steady state. The reverse hump-shaped pattern of interest rates is a result of a hump-shaped capital.

The nature of a positive government spending shock that acts as a burden to the economy is emphasized once more. Since the government consumption is “useless” to agents, positive innovations to government that are absorbed by higher taxes significantly distort the economy, as agents do not benefit from the increased government spending but only experience a reduction in their disposable incomes. The responses to a positive government shock are rises in labor supply, capacity utilization, output, interest rates, and fall in consumption. The intuition is that households consume less and work more today (and hence utilize more capital today, which results in higher interest rates) as they anticipate higher future taxes to finance today’s boost in government expenditures. Once the taxes are increased in period one, output, consumption, investment, labor and capacity utilization together with prices deteriorate below the steady state levels. Furthermore, capital declines and its level remains below the steady state. Reduced levels of capital induce returns on capital to increase above the steady state. Also, the extremely low levels of capital are utilized at a relatively faster rate. Therefore, the utilization rates hover above the steady state level from year 3 onwards.

3.6.1 Impulse responses of each individual group

In this section we investigate impulse responses for each income group and analyze net contributions to the social system. We expect to find that the median voter who is poorer than the mean voter is a net-receiver. Consequently, he will vote for higher tax rates in order to augment his resources by larger transfer receipts. To monitor which class is indeed a net contributor and which is a net receiver we plot net contributions for each class $i = \{r, m, p\}$. The net contribution of class $i$ is defined as net income of this class divided by transfer
receipts. The net income of class $i$ is determined as a contribution to the social system of this class less transfer payments, i.e. class $i$’ tax payments used to finance transfers less transfer receipts. We depict net contributions as deviations from steady state transfers ($T^{ss}$),

$$NetContriSS^i_t = \frac{Contri^i_t - T^{ss}}{T^{ss}} ,$$ (27)

and net contributions as deviations from transfers in every period ($T_t$),

$$NetContri^i_t = \frac{Contri^i_t - T_t}{T_t} ,$$ (28)

for each class $i = \{r, m, p\}$. The contribution to the social system ($Contri^i_t$ in the above formulas) is calculated as follows:

$$Contri^i_t = (\tau_t - g) \cdot (r_t a_{i,t} + \omega_t w_{i,t}) = (\tau_t - g) \cdot ((R_t H_{i,t} - e^{-\tau_t} \delta(H_{i,t}))a_{i,t} + \omega_t w_{i,t}) .$$ (29)

Furthermore, we plot individual group’s impulse responses for income, consumption, investment and labor subject to the class $i$’s budget constraint. The budget constraint of class $i$, where $i \in \{r, m, p\}$, is as follows:

$$(1 - \tau_t)R_t H_{i,t}a_{i,t} + (1 - \tau_t)w_{i,t}a_{i,t} + T_t = e^{-\tau_t} a_{i,t+1} - e^{-\tau_t} (1 - (1 - \tau_t) \delta(H_{i,t}))a_{i,t} + c_{i,t} .$$ (30)

The income of class $i$ is,

$$y_{i,t} = (1 - \tau_t)(R_t H_{i,t} - e^{-\tau_t} \delta(H_{i,t}))a_{i,t} + (1 - \tau_t)w_{i,t}a_{i,t} ,$$ (31)

the class $i$’s investment decision is,

$$i_{i,t} = e^{-\tau_t} a_{i,t+1} - e^{-\tau_t} (1 - (1 - \tau_t) \delta(H_{i,t}))a_{i,t} ,$$ (32)

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and the consumption decision of group $i$ is defined as,

$$c_{i,t} = y_{i,t} + T_t - e^{-z_{x,t}}(a_{i,t+1} - a_{i,t}).$$

(33)

The individual labor decision is extracted from the law of motion $A(X, A, \tau \mid \Psi)$ of aggregate state and choice variables.

Figure 1.4 depicts impulse responses to total factor productivity shock for each income group. The poor profit mostly from the innovation in TFP. The positive deviations from the steady state levels of income, consumption and investment are the largest for the poor class even though the taxes in period one decrease notably the poor's income. In contrast, the positive deviation from the steady state labor supply is the largest for the rich, followed by the median class. Consequently, the rich work more than the median and poor, following the TFP shock. The deviations from the steady state labor supply become negative over time. Yet, the rich class keep working the most hours.

Figure 1.5 displays impulse responses to the investment-specific technological change for each income group. The income response is the largest for the rich. However, as agents face increased taxation in period one, the positive deviation from the steady state income of the poor class becomes larger than those of the median and rich. The income’s deviation of the median class overtakes the one of the poor in year 15. The median class’s income remains above the steady state over the entire time path. The deviation from steady state consumption (investment) is the highest (lowest) for the rich in the event of investment shock in period zero. The poor class in turn consumes less, but invest highly above the steady state level in year zero. The rationale is that the rich prefer to increase evenly both consumption and investment in a boom, whereas the poor aspire to exploit favorable investment opportunities by reducing consumption. The consumption of the poor increases rapidly above the steady state level already in year one, reaching levels higher than the
positive deviations of the median and the rich. The main reason behind this enormous rise in consumption is increased direct transfer payments to the poor and median in period 1. Labor responds positively to the investment boost, but declines abruptly in period one as a consequence of an increased tax burden. The evolution of each class’s labor supply indicates that the rich work the most hours. Furthermore, it is worth mentioning that the impulse responses of each class indicate that the median group is better-off in the long run.

The impulse responses to the government spending shock of each income group are plotted in Figure 1.6. The poor are the most negatively affected. The income of each class responds positively to the shock, but drops below the steady state in period one due to tax distortions. The consumption of each group responds adversely to the innovation in government spending. The households seem to anticipate higher future tax payments and thus consume less already in year zero. The investment responds slightly positively only for the rich class, but drops below the steady state next period due to tax burdens. The interest rates that raise above the steady state influence demand of the rich and median for capital investment. Consequently, their investment decisions increase from period 1 onwards reaching levels slightly above the steady state. The poor cannot afford to invest. Labor responds highly positively, but sharply declines due to tax burdens in period one. The rich and the median can afford to work less over the entire time path. The poor, however, rely heavily on transfer payments, which drop below the steady state. Consequently, the poor class has to work more, far above the steady state.

The last Figure 1.7 plots the net contributions of each income class when all three shocks hit randomly the economy. We can clearly monitor which classes are net receivers and which are net contributors. As expected the rich contribute more than they receive from the social system. The median and the poor gain at the expense of the rich class, as they both are net
receivers.

3.7 Conclusions

The task we have undertaken in this chapter was to re-evaluate what a standard real-business-cycle model has to say about redistribution through lump-sum taxes over the business cycle. We have employed a variant of the heterogeneous-agent model of Krusell and Rios-Rull (1999), adding RBC elements by the paradigm of King, Plosser and Rebelo (1988) and King and Rebelo (2000). We have found that for standard parameterizations of the RBC model both the level and the output share of fiscal transfers are procyclical. This happens despite the fact that we have assumed exogenous procyclical shocks to government consumption (which does not enter utility in our model): as the fiscal budget is balanced, lump-sum transfers act, partly, as a shock-absorber to the elevated exogenous demand for government spending and temporary tax increases; this role of transfers is obvious when we presume no inequality (i.e., no conflicts about transfers, shifting the focus of voting on the intertemporal allocation of tax burdens). Yet, once we incorporate income and wealth distribution statistics of the U.S., which are skewed to the right, the rent-seeking behavior becomes evident in the simulated economy. Poor and median voters see space for increasing tax burdens during booms, obtaining welfare from increases in leisure due to income effects through the increased lump-sum transfers. Furthermore, the procyclical pattern of transfers and transfers to output ratio indicate that rent-seeking voters care about the good standing of the economy, as they do not behave selfishly when the economy deteriorates.

The sensitivity analysis allows us to conclude that high levels of elasticity of labor supply are essential in the real business cycle environment. Furthermore, increasing the intertemporal elasticity of substitution improves implied cyclical patterns matching better the data. The procyclicality of transfers and transfer-to-output ratio in the model resembles the cycli-
cal patterns in developing countries. This outcome may result from the extremely large discrepancy between the rich and poor in the U.S.\textsuperscript{30} Nevertheless, such high cyclical features of redistributive transfers are not predominant in the data. Consequently, our results imply that the parameters used in the model with the politico-economic equilibrium for redistribution should be re-examined in future work. One venue is to re-examine parameters behind the elasticity of labor supply. Another plausible extension may be to introduce incomplete markets to a voting model. Yet, the complexity of studying voting with incomplete markets may be prohibitively high for the state-of-the-art knowledge on computing equilibrium in dynamic games, requiring first some stepping stones to be developed.

4. **Stylized Facts on Household Finance**

A number of stylized facts cannot be reconciled with standard portfolio-choice models. We argue that the failure of the standard finance literature to match the empirical evidence lies in the non-consideration of subsistence risks that notably affect savings and investment decisions of the poor. In the following two chapters we extend the standard Merton (1969, 1971) model to include subsistence costs. We find that introducing subsistence consumption into agents’ preferences has a huge potential to match stylized facts on portfolio selection, savings and consumption decisions. The first stylized fact we aim to match refers to larger saving rates among high-income households. Dynan et al. (2004) find a robust positive relationship between saving and current income (see Figure 1 in Dynan et al. (2004)). Also, scholars using different instruments for lifetime income (ranging from food consumption, consumption of vehicles, lagged and future earnings to education) document that saving rates rise with permanent income (see Figure 2 in Dynan et al. (2004)). Finally, scholars report a positive correlation between lifetime income and marginal savings propensity.

\textsuperscript{30}The Gini coefficient for U.S in 2004 was 46.4 which is very close to the value in developing countries.
A second empirical fact refers to riskier portfolios among wealthier households. Carroll (2002) using U.S. survey data provides evidence that the top richest 1% of U.S. households select riskier portfolio investments than the remaining 99% do. Fractions of net worth allocated to privately held businesses and of financial assets invested in stocks are larger for the richest 1% (respectively 38% and 35% against 15% for the rest 99%, see Table 3 in Carroll (2002)). Moreover, the richest 1% invested on average 63% of financial wealth in risky assets over the period 1962-1995 compared with 36% invested by the rest 99% (see Table 4 in Carroll (2002)). Also, Carroll (2002) conducts an international comparison and finds riskier portfolio structures among the wealthy in all investigated countries, namely the US, the Netherlands, Italy, Germany and the UK (see Table 7 in Carroll (2002)). In addition, Wachter and Yogo (2010) document risky portfolio shares rising in wealth. Firstly, scholars, using the Survey of Consumer Finances data, find a statistically robust positive relation between risky portfolio share (share of net worth allocated to risky assets) and net worth (see Table 4 in Wachter and Yogo (2010)). Secondly, their model specification with nonhomothetic preferences over basic and luxury goods implies that agents with larger permanent income are more risk tolerant and invest higher fractions of wealth in stocks. Finally, scholars provide data evidence for their model's prediction. To be exact, Wachter and Yogo (2010, Table 8) report median values of portfolio shares by wealth quartile from

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31 Carroll (2002) uses two data surveys, the Survey of Financial Characteristics of Consumers and the Survey of Consumer Finances, in order to provide a statistical summary of net worth composition.

32 Carroll (2002, Table 3 and Table 4) reports net worth composition for the extremely rich (top 1% of net worth) and the rest (0-99% of net worth) averaged over the 1962-1995 period.

33 In Carroll (2002, Table 7) the riskier portfolios of the top 5% are featured by larger shares of risky assets and private businesses compared with the equivalent investments of the bottom 95%. Risky assets as a ratio to Gross Financial Assets: 50.7% for top 5% and 30% for bottom 95% in the US (year 1995); 53% for top 5% and 14% for bottom 95% in the Netherlands (year 1995); 30.6% for top 5% and 9% for bottom 95% in Italy (year 1995); 19.7% for top 5% and 10% for bottom 95% in Germany (year 1993); 25% for top 5% and 15% for bottom 95% in the UK (years 1997/98). Net value of private business as a ratio to Net Worth: 32% for top 5% and 6.5% for bottom 95% in the US (year 1995); 19% for top 5% and 3% for bottom 95% in the Netherlands (year 1995); 37% for top 5% and 15.6% for bottom 95% in Italy (year 1995).
the Survey of Consumer Finances that clearly indicate higher risky portfolio investments among wealthier as opposed to poorer households.\textsuperscript{34}

The third empirical regularity relates to a more volatile consumption growth among richer households. Ogaki and Atkeson (1997) using Indian panel data estimate a simple statistical model that summarizes the extent of variation in the mean and volatility of consumption growth across wealth-heterogenous agents. Their empirical results conform with the view that the rich experience larger volatility of consumption growth (while the mean is the same for all agents). Furthermore, the authors provide an economic model and show that differences in the consumption growth volatility across rich and poor agents are attributed to the variations in the IES across agents. Malloy et al. (2009) is another relevant study corroborating the third stylized fact. Malloy et al. (2009) estimate consumption sensitivity of three groups (top wealthiest stockholders, remaining stockholders and non-stockholders) to the aggregate consumption shocks over different time horizons. Malloy et al. (2009) find that the consumption stream of the top wealthiest stockholders is more responsive to long-term aggregate shocks and is more correlated with asset portfolio returns than the consumption patterns of other (less wealthy) stockholders. In addition, long-term stockholders’ consumption series are more responsive to aggregate consumption shocks than the consumption paths of non-stockholders.\textsuperscript{35}

Finally, the fourth empirical perception is that wealthier households exhibit greater willingness to substitute intertemporally. The empirical findings of Ogaki and Atkeson (1997) comply with the view that the intertemporal elasticity of substitution (IES) increases with the level of wealth (in contrast to the rate of time preference that is shown to be constant

\textsuperscript{34}See Table 8 in Wachter and Yogo (2010) for empirical evidence (Panel A) and their model’s predicton (Panel B).

\textsuperscript{35}See Table 1 in Malloy et al. (2009) for the consumption sensitivity of three classes (top richest stockholders, remaining two thirds stockholders and non-stockholders) over different horizons of time.
across agents). Also, Blundell et al. (1994) show that IES positively varies with the level of consumption which acts as a proxy for lifetime assets. Their result estimates indicate up to three times larger IES for the top highest as compared with the lowest percentile of income distribution (see Table VI in Blundell et al. (1994)). Furthermore, Guvenen (2006) points to the importance of IES heterogeneity in aligning two different views, empirical studies and calibrated models, which suggest IES to be close to 0 and 1 respectively. He finds that two characteristics, IES increasing in wealth and limited participation in stock markets, are essential to produce enormous wealth inequality as evident between stockholders and non-stockholders in the data. Also, his model indicates that investment and output fluctuations are affected mainly by decisions of high-IES (wealthy) stockholders whereas consumption is influenced by the remaining majority of the population (i.e. low-IES, poor non-stockholders), reconciling two contradicting views about the IES. In addition, some literature points that the wealthy are more willing to substitute their consumption over time as they postpone mainly consumption of luxuries. On the contrary, the poor cannot afford large fluctuations in consumption as they barely meet their necessities. The studies of Wachter and Yogo (2010) and Browning and Crossley (2001), in which luxury consumption bundles are separated from basic goods, corroborate such a view.

The evidence on differing decisions and perceptions of risk across richer and poorer households motivates the introduction of subsistence consumption into agents’ preferences. The need to satisfy subsistence consumption may burden the poor, which may be behind their subdued decisions. Consequently, our objective is to examine whether the subsistence consumption in a single utility function is capable of simultaneously reconciling the four empirical regularities detailed above.

In our investigation, we do not assume exogenously different IES in the same way as we
do not presume exogenously various saving rates or portfolio shares across richer and poorer agents. By contrast, our research objective is to examine whether the model specification with subsistence consumption can produce endogenously heterogeneous IES, saving rates and risky asset shares. Some researchers, for instance, Guvenen (2006 and 2009) and De Graeve et al. (2009) assign higher IES values for stockholding and lower for non-stockholding households in their calibration parts. Yet, we argue the assumption of exogenously different IES may directly lead to larger proportions of wealth allocated to risky assets, more volatile consumption growth and greater saving rates for more risk tolerant agents. In a nutshell, the assumptions may generate the conclusions. Moreover, Carroll (2002) points to arguments that are against the theory of exogenously heterogeneous risk attitudes, which gives support to our view that assuming different IES up front is incorrect.36 Interestingly, his findings indicate that the Stone-Geary utility formulation with wealth that captures bequest or capital greed motives as in Carroll (2000) explains greater saving rates of the rich. The proposition that the rich unlike the poor can afford to be greedy, save for their descendants and consume luxury goods conforms with our view that the wealth holdings of the rich exceed considerably the subsistence needs. So, only the non-rich, whose wealth may be just enough to cover the subsistence costs, feel pressure of existential needs and cannot afford to be greedy or generous and consume as much as the rich do. In the following chapters, our focus is on the poor whose abilities to satisfy the subsistence needs may be at risk. We believe that the understanding of poorer agents’ problems and constraints is essential to draw implications for economic policies.

36Carroll (2002) argues the exogenous preference heterogeneity does not explain the patterns evident in the data: i) undiversified portfolios of the rich, ii) huge amount of household’s wealth invested in enterprises which are actively self-managed, iii) failure of the old rich to run down their wealth holdings.
5. Saving Rates and Portfolio Choice with Subsistence Consumption

This chapter is based on the paper “Saving rates and portfolio choice with subsistence consumption”, co-authored with Carolina Achury and Christos Koulovatianos. This paper was published in January 2012 in the Review of Economic Dynamics (Volume 15, Issue 1, Pages 108–126).

5.1 Introduction

We use the standard two-asset Merton (1969, 1971) model as a vehicle for our analysis. Economic agents allocate their lifetime resources between two assets, a risky asset and a risk-free asset. Accordingly, the risky asset yields high but uncertain rates of return whereas the risk-free asset offers low but secure capital returns. In our model economy, the disposable lifetime resources differs across agents. Furthermore, agents are constrained in their preferences by subsistence costs.

Since different definitions of subsistence consumption have been studied in the literature, we investigate two concepts of subsistence consumption in regard to their promise to explain the empirical patterns detailed above. The first concept assumes a constant level of subsistence consumption, so called bread-and-butter needs. Here, we incorporate a time-invariant subsistence component, $\chi$, into a standard time-separable power utility function and obtain the following Stone-Geary form $u(c(t)) = \left(\frac{c(t) - \chi}{1 - 1/\eta} - 1\right)^{1 - 1/\eta}$, where $\chi, \eta > 0$. The second concept accounts for the external habit, which reflects the value agents attach to the average consumption in a certain community. This utility specification comprises the notion of the “keeping-up-with-the-Joneses” and thus takes into account an agent’s con-
stant desire to catch up with his neighbors. The utility function takes the following form
\[ u(c(t), C(t)) = \left\{ \left[ c(t) - \gamma C(t) \right]^{1-1/\eta} - 1 \right\} / (1 - 1/\eta), \] where \( \gamma, \eta > 0 \). The subsistence level of consumption changes over time in accord with the shifts in the average consumption, \( C(t) \). The parameter \( \gamma \) is a weight agents assign to the average consumption.

Our analytical results to the model with “bread-and-butter” consumption correspond well with the empirical patterns at the micro level. To be precise, time-invariant Stone-Geary utility function produces saving rates, high-risk portfolio shares, consumption growth volatility and endogenous IES, all positively related to the initial and current financial wealth. Yet, it does not generate a stationary relative distribution of wealth with heterogeneous IES across wealthy and non-wealthy agents. On the contrary, “keeping-up-with-the-Joneses” utility function generates a stationary relative distribution of wealth with varying IES across household wealth distribution over the entire equilibrium path, but nevertheless produces saving rates, high-risk portfolio shares and volatility of consumption growth, all independent of financial wealth and thus equal across heterogenous agents. As a result, the rich and the poor save and allocate to risky assets the same fractions of financial wealth and experience equally volatile consumption growth. In summary, the Stone-Geary specification with time-invariant subsistence consumption reflects qualitatively better the data.

In our final step, we assign values to model variables and parameters. We show quantitatively that the model with time-invariant Stone-Geary utility function is capable of producing value ranges of stock holding shares (6% for the poor and 22% for the rich) that correspond well with the fractions in the Survey of Consumer Finances, ranges of saving rates (6% for the poor and 17% for the rich) that are close to the estimated values in Dynan et al. (2004, Figures 1 and 2) and relative risk aversion coefficients (14 for the poor and 4.5 for the rich) that coincide quite well with the documented values in Malloy et al. (2009). Furthermore,
our results are robust to different levels of subsistence consumption.

Although all households along wealth and income distribution dynamics face great opportunities of quickly improving their living standards through large capital gains on high-risk assets, the poor tend to allocate lower fractions of financial wealth to stocks. Also, their savings decisions are subdued in comparison to the savings of the rich. So, even though the poor could considerably (and relatively more) benefit from both risky asset investments and greater saving rates, they prefer to get out of poverty at a relatively slow pace. We show that the time-invariant subsistence constraint imposed on the agents is the answer to that “unanticipated” behavior of the poor in the data. To be precise, the model’s results indicate the following. The non-rich may need to squeeze their consumption to extremely low levels when choosing the risky portfolio fractions and saving rates of the rich. Also, high volatility of risky investments may lead to existential risks among poorer agents. Consequently, the non-wealthy faced by time-invariant subsistence costs choose to smooth the disutility of consuming too close to subsistence level through foregoing large savings and risky asset investments. On the contrary, the “keeping-up-with-the-Joneses” subsistence consumption that grows with aggregate consumption does not leave the poor any space to save and invest lower fractions of wealth. Even though the poor are less willing to substitute intertemporally, they choose saving rates and risky asset shares of the rich in order to catch up with their wealthier peers. The pressure of increasing subsistence needs induces the poor to take the same risks as the wealthy do.

The introduction of subsistence consumption conforms with the approach of Wachter and Yogo (2010) and Browning and Crossley (2001), who distinguish between luxuries and basics. They show that the wealthy take more risks because they risk mainly the consumption of luxuries. In other words, risky asset investments mainly jeopardize the subsistence
consumption of the poor and not of the rich whose wealth holdings greatly exceed the subsistence level. The inclusion of existential needs into a single utility function seems to be more parsimonious than the introduction of several goods as the latter requires an extensive work with micro consumer data.

An important technical contribution to the two-asset Merton (1969, 1971) model is the consideration of bread-and-butter needs. The ability of the standard Merton (1969, 1971) model with constant subsistence consumption to provide explicit solutions has been already recognized in the literature. For example, Weinbaum (2005) makes use of the analytical tractability to investigate investment decisions of conservative investors who aspire to consume above a certain positive threshold level over time. One of his findings indicates that conservative agents consume less than the non-conservative ones in good times. The reason is that the conservative investors allocate some proportions of wealth to long-term bonds with the aim of ensuring coupon payments for future bad times. The conservative agents’ desire to achieve a certain level of standard of living resembles the situation of the poor who need to consume a certain level of wealth in order to ensure their existence. The poor agents will also prefer safe return investments that secure the satisfaction of the subsistence needs. Furthermore, Sethi et al. (1992) mention closed-form solutions to the consumption and portfolio decision problem that takes account of the time-invariant subsistence consumption and bankruptcy.

Our analytical results to the two-asset Merton (1969, 1971) model with time-invariant subsistence consumption are obtained through the method of undetermined coefficients (see Proposition 1 and Proof in Appendix B). The explicit solutions to the model with “keeping-up-with-the-Joneses” preferences require aggregation of a time-variant economy-wide subsistence constraint. On that account, we follow the result provided by Koulova-
tianos (2005, Theorem 3). With the aggregation result at hand, we solve analytically the two-asset portfolio-choice problem. To the best of our knowledge, Achury et al. (2012) is the first study to tackle the analytical derivation of the standard Merton (1969, 1971) model with external habits. Yet, “keeping-up-with-the-Joneses” utility function has already been introduced in the studies of Abel (1990), Chan and Kogan (2002), modeled as a ratio of individual consumption to external habits, and in the studies of Campbell and Cochrane (1999), Wachter (2005, 2006), modeled as a consumption minus habits.

Chan and Kogan (2002) numerically investigate portfolio policies in the model with external habits in agent’s preferences, whereas Abel (1990), Campbell and Cochrane (1999) and Wachter (2006) introduce external habits to consumption based asset pricing models with the aim of matching empirical phenomena on asset pricing, rates of return and consumption growth dynamics\textsuperscript{37}. The numerical analyses of portfolio choice with internal habits have been carried out by Gomes and Michaelides (2003), Polkovichenko (2007), and Brunnermeier and Nagel (2008).

5.2 Bread-and-Butter Needs (time-invariant subsistence consumption)

The main vehicle for our analysis is the two-asset portfolio model of Merton (1969, 1971). Time is continuous, \( t \in [0, \infty) \). Infinitely-lived agents allocate their initial wealth endowments, \( k_0 > 0 \), between two assets, namely a proportion \( \phi \) of wealth to risky asset and \( (1 - \phi) \) to non-risky asset.

Economic agents choose the paths of consumption \( (c(t))_{t \geq 0} \) and portfolio composition

\textsuperscript{37}Some of the empirical patterns: high equity premium, low consumption growth volatility, excess asset return volatility, low and stable risk-free interest rate.
\((\phi(t))_{t \geq 0}\) over time that maximize their expected life-time utility,

\[
E_0 \left\{ \int_0^\infty e^{-\rho t} \frac{[c(t) - \chi]^{1 - \frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} \, dt \right\},
\]

with \(\rho, \chi, \eta > 0\). The budget constraint of an investor consists of two parts. The first part accounts for the deterministic change in capital over time, whereas the second part captures random fluctuations in the amount of risky invested capital. The stochastic differential budget constraint is,

\[
dk(t) = \left\{ [\phi(t) R + (1 - \phi(t)) r_f] k(t) - c(t) \right\} dt + \sigma \phi(t) k(t) dz(t),
\]

where \(R\) and \(r_f\) are average rates of return of the risky and risk-free investments respectively, with \(R > r_f\), \(\sigma\) is standard deviation of the risky asset, and \(dz(t)\) is a Brownian motion, i.e. \(dz(t) = \varepsilon(t) \sqrt{dt}\) and \(\varepsilon(t) \sim N(0, 1)\).

### 5.2.1 Optimal consumption and portfolio decision rules

In this section, we solve the agent’s dynamic stochastic optimization problem in continuous time in order to derive optimal decision rules for consumption and portfolio allocation. The Hamilton-Jacobi-Bellman equation (HJB) captures the investor’s optimization problem. Since the optimization problem is the same every instant over the infinite-time horizon, we can omit the variables’ dependence on time. The HJB equation is as follows,

\[
\rho J(k) = \max_{\phi \geq 0} \left\{ \frac{(c - \chi)^{1 - \frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} + J'(k) \left\{ [\phi R + (1 - \phi) r_f] k - c \right\} + \frac{(\sigma \phi k)^2}{2} J''(k) \right\}.
\]

Next, we take first-order conditions (FOCs) with respect to the control variables. In the Proposition 1 we show that the value function is strictly concave, which secures that the first-order conditions constitute necessary and sufficient conditions for a global maximum. The FOCs with respect to consumption and portfolio share are respectively:
\[ (c - \chi)^{-\frac{1}{\hat{\sigma}}} = J'(k), \quad (37) \]

and

\[ \phi = \frac{J'(k)}{-J''(k) k} \frac{R - r_f}{\sigma^2}. \quad (38) \]

Before describing solutions to the optimization problem, we need to guarantee the existence and interiority of the results. For this reason, we assume two restrictions, namely one initial value restriction and one parameter restriction. The role of these technical assumptions becomes apparent once the functional form of the value function and the decision rule for consumption are derived. Furthermore, the presence of subsistence consumption induces additional constraints, which will appear in the course of this chapter.

**Assumption 1** Initial lifetime resources must strictly exceed lifetime subsistence needs:

\[ k_0 > \frac{\chi}{r_f}. \]

**Assumption 2** Parameter \( \eta \), which is directly linked to the intertemporal elasticity of substitution, namely \( IES = \eta (1 - \chi/c) \), must be strictly lower than the positive parameter value \( \bar{\eta} \):

\[ 0 < \eta < \bar{\eta} \equiv \frac{1 - \frac{r_f - \rho}{\frac{1}{2} (R - r_f)} + \left\{ \frac{r_f - \rho}{\frac{1}{2} (R - r_f)} \right\}^2 + 2 \cdot \frac{r_f + \rho}{\frac{1}{2} (R - r_f)} + 1}{2}. \]
Notice that values $\sigma, \rho, R, r_f$ are all positive, which ensures that the parameter $\bar{\eta}$ is indeed greater than zero.\textsuperscript{38} Also, it is worth mentioning that the upper bound on parameter $\eta$ does not imply IES values that would deviate from the deduced/estimated ones in the literature. As an example we cite some literature. Barsky et al. (1997) based on a survey with hypothetical situations provides a range of reasonable numbers for the intertemporal substitution parameter. Blundell et al. (1994) estimates suggest up to three times higher IES among the wealthiest as compared with the lowest percentile of income distribution. Also, Guvenen (2009) studying a macroeconomic asset pricing model assigns IES values respectively 0.1 and 0.3 for poor non-stockholders and wealthy stockholders. Due to the presence of $\chi$ our model is capable of reproducing the preference parameter values reported in the literature. Following the parameterization based on the U.S. data provided by Guvenen (2009, Table II), namely $R = 8\%$, $r_f = 2\%$, $\sigma = 20\%$, the implied upper bound on parameter $\eta$ takes on a reasonable range of values, e.g. $\bar{\eta} \simeq 1.25$ for $\rho = 1.5\%$ and $\bar{\eta} \simeq 1.8$ for $\rho = 4.5\%$. So, a plausible model parameterization is not hindered by the upper bound restriction on parameter $\eta$.

The analytical solution to the model with time-invariant subsistence consumption is presented in Proposition 1.

**Proposition 1**

*Respecting Assumptions 1 and 2, the optimization problem stated by HJB equation*

(36) delivers an optimal consumption decision rule,

\[ \left( \frac{1 - \frac{r_f - \rho}{\frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2}}{\frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2} \right)^2 < \left( \frac{\frac{r_f - \rho}{\frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2}}{\frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2} + 2 \frac{\frac{r_f - \rho}{\frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2}}{\frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2} + 1 \right). \]

\textsuperscript{38}Notice that
\[ c^* = C(k) = \xi k + \psi, \]  

where,
\[ \xi = \rho \eta + (1 - \eta) r_f - \frac{\eta (\eta - 1)}{2} \left( \frac{R - r_f}{\sigma} \right)^2, \]  

and,
\[ \psi = \eta \chi \frac{r_f - \rho + \frac{\eta - 1}{2} \left( \frac{R - r_f}{\sigma} \right)^2}{r_f}, \]

an optimal portfolio selection decision rule,
\[ \phi^* = \Phi(k) = \eta \frac{R - r_f}{\sigma^2} \left( 1 - \frac{\chi}{k} \right), \]  

and the value function form is,
\[ J(k) = -\frac{1}{\rho \left( 1 - \frac{1}{\eta} \right)} + \xi^{-\frac{1}{\eta}} \frac{\left( k - \frac{\chi}{r_f} \right)^{1-\frac{1}{\eta}}}{1 - \frac{1}{\eta}}. \]

**Proof** We provide the derivation of the optimization problem in Appendix B. □

Once proposition 1 is stated, we can contemplate the roles of Assumption 1 and 2. The importance of Assumption 1 is apparent when we look at the functional form of the value function, \( J(k) \). It secures a well-defined problem with interior solutions. The role of Assumption 2 becomes obvious when we set \( \xi > 0 \) with the aim of ascertaining that the gradient, \( \xi \), is strictly positive and so is the dependence of consumption on wealth.

Furthermore, rearranging terms in the decision rule for consumption implies the following expression, \( c = \xi (k - \chi/r_f) + \chi \), and points to another relevant aspect of Assumptions 1 and 2, which is to ascertain that \( c \geq \chi > 0 \). Consequently, although the constraint \( c \geq \chi \) should be imposed from the first, we can safely omit it as long as Assumptions 1 and 2...
hold. In particular, the upper bound on the parameter $\eta$ and thus on the investor’s willingness to substitute consumption intertemporally\textsuperscript{39}, in Assumption 2, prevents the agent’s consumption, $c$, from hitting the bottom subsistence level, $\chi$, in equilibrium. In addition, Assumption 2 imposes a limit on the agent’s tolerance for fluctuations in consumption even if $\chi = 0$ as in the original Merton (1969) model. In this event, the standard assumption of non-negative consumption, $c \geq 0$ is ensured.

5.2.2 Household financial wealth dynamics

In proposition 2 we provide the investor’s financial wealth dynamics, which ascertain that agent’s lifetime resources are, in equilibrium, always greater than his lifetime existential needs, namely $k^\ast (t) > \chi/r_f$ for $t \geq 0$.

**Proposition 2**

Respecting Assumptions 1 and 2, the investor’s financial wealth dynamics are,

$$k^\ast (t) - \frac{\chi}{r_f} = e^{\eta \left[ r_f - \rho + \frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2 \right] t + \eta \left( \frac{R - r_f}{\sigma} \right) z(t) \left( k_0 - \frac{\chi}{r_f} \right)},$$

(44)

where $z(t) = \int_0^t dz(s)$ and $\int$ is the Itô stochastic integral.

**Proof** The derivation of wealth dynamics can be found in Appendix B.\textsuperscript{\textsection}

In the following subsections, we continue with the description of saving rates, risky asset investments and consumption dynamics, and analyze their relations to wealth.

5.2.3 Household savings decisions

In this section we derive an optimal saving rate and provide necessary and sufficient conditions for the saving rate to be strictly positively related to wealth. The main condition is presented in Proposition 3. The proof is given right after proposition 3.

\textsuperscript{39}Notice that IES converges to $\eta$ when consumption goes to infinity, $\lim_{c \to \infty} \eta (1 - \chi/c) = \eta$. 

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Proposition 3

Respecting Assumptions 1 and 2, household saving rate is strictly positively correlated with wealth if and only if parameter $\eta$ is strictly greater than the value $\eta_1$.

$$\eta > \eta_1 \equiv -1 - \frac{r_f - \rho}{\frac{1}{2} \left( \frac{R-r_f}{\sigma} \right)^2} . \quad (45)$$

**Proof** For any point in time $t \geq 0$, the decision rules for consumption, $C(k)$, and portfolio selection, $\Phi(k)$, from Proposition 1, are inserted into the investor’s budget constraint (35).

After some algebraic steps, we obtain the equilibrium level of savings at time $t$, namely,

$$S^*(t) = \zeta \left[ k^*(t) - \frac{\lambda}{r_f} \right] , \quad (46)$$

where

$$\zeta \equiv \eta \left[ \frac{\eta + 1}{2} \left( \frac{R-r_f}{\sigma} \right)^2 + r_f - \rho \right] . \quad (47)$$

The ratio of savings level to overall investment gains constitutes the equilibrium saving rate at time $t$, namely,

$$s^*(t) = \frac{S^*(t)}{\phi^*(t) (R-r_f) k^*(t) + r_f k^*(t)} . \quad (48)$$

Inserting the solutions for savings level, $S^*(t)$, and portfolio choice, $\Phi(k)$, into (48) and rearranging terms enables one to deduce the dependence of saving rate on wealth. The optimal saving rate is as follows,

$$s^*(t) = \frac{\zeta}{\eta \left( \frac{R-r_f}{\sigma} \right)^2 + \frac{r_f}{1-\rho}} . \quad (49)$$

The saving rate is strictly positively related to wealth if and only if the nominator, given by $\zeta$, is positive. More precisely, $ds^*(t)/dk^*(t) > 0 \iff \zeta > 0 \iff (45)$.
Combining the restrictions on parameter $\eta$ from Assumption 2 and Proposition 3 we obtain the following constraint $\max \{ 0, \eta \} < \eta < \bar{\eta}$. It can be easily shown that $\eta < \bar{\eta}$ for all possible parameter values $\rho$, $\sigma$, $R$, and $r_f$.

Moreover, we know from equation (44) that the expected household wealth increases at a strictly positive rate if and only if,

$$r_f - \rho + \frac{1+\eta}{2} \left( \frac{R-r_f}{\sigma} \right)^2 > 0,$$

which is exactly in line with the parameter restriction in equation (45). Hence, the requirement for wealth to grow coincides with the necessary and sufficient condition for the saving rate to be strictly increasing in wealth. Corollary 1 summarizes the relation between the monotonicity of wealth and monotonicity of savings.

**Corollary 1**

*Respecting Assumptions 1 and 2, investor’s expected wealth increases at a strictly positive growth rate if and only if his saving rate is strictly rising in wealth.*

Corollary 1 provides intuition on how the model works. The wealthy recognize a great opportunity of having wealth growing over time, which secures their future consumption to be well above the subsistence level. The poor choose lower saving rates as their resources are lower and may be just sufficient to cover subsistence costs today. High saving rates of the rich would require the poor to squeeze their consumption even further to extremely low levels, which is painful. To be precise, economic agents face a trade-off between today’s pressure of living close to subsistence level (lower saving rates today) and pressure of existential needs in the future (high saving rates today). This trade-off comprises intertemporal household decisions. The poor burdened with the satisfaction of today’s basic consumption decide to get out of the poverty in a slow but least painful way, whereas the rich afford to secure their future
consumption to be highly above the subsistence level. Nevertheless, whether heterogenous agents indeed make such decisions hinges on two parameter values that influence agents’ preference for time and their attitude towards risk. The values of time preference rate, \( \rho \), and parameter, \( \eta \), that directly affects the IES determine whether the inequality (45) is satisfied. Inequality (45) restricts the parameter, \( \eta \), from below, and since \( \eta \) is tightly linked to the IES, the restriction encourages monotonic consumption and wealth paths. More tolerable or desirable consumption and wealth dynamics are in turn associated with higher saving rates over the infinite time horizon. Nonetheless, higher values of time preference rate, \( \rho \), make savings less desirable and since \( \rho \) enters the inequality (45), greater preference for time will necessitate a higher value for parameter \( \eta \). Yet, the parametric restriction (45) allows to pick from a broad range of plausible parameter values. Furthermore, if the inequality \( \eta \leq \eta \) were not satisfied, the equilibrium savings would be always negative, as \( \zeta \leq 0 \).

Proposition 4 refers to the relationship of the risky portfolio share with wealth. The risky portfolio share is defined as the fraction of wealth allocated to a risky asset.

**Proposition 4**

*Respecting Assumptions 1 and 2, the risky portfolio share is strictly positive and rising in household’s wealth.*

**Proof** The immediate evidence follows from the decision rule \( \Phi(k) \) in the Proposition 1. \( \square \)

Proposition 5 outlines the relationships between the variation coefficient of consumption and household’s initial wealth and between consumption growth volatility and investor’s current financial wealth.
Proposition 5

Respecting Assumptions 1 and 2, the variation coefficient of consumption is strictly positively related to initial wealth and the volatility of consumption growth rate is positively related to current wealth.

Proof Evidence on both relationships is provided in Appendix B. □

The theoretical result of Proposition 4 conforms with the risky asset holdings evident in the data (see for instance Carroll (2002) and Wachter and Yogo (2010)), whereas the statement of Proposition 5 corresponds well with the empirical results of Ogaki and Atkeson (1997) and Malloy et al. (2009, Table I) that wealthier households experience more volatile consumption growth. Also, as Corollary 1 explains the model-implied saving rates are increasing in lifetime resources, which coincides with the empirical findings of Dynan et al. (2004).

In summary, a single utility function of Stone-Geary form with time-invariant subsistence consumption is capable of reconciling stylized facts on risky asset holdings, saving rates and consumption growth volatility.

5.2.4 Dynamics of wealth inequality

In our partial-equilibrium model with price-taking agents the gap between rich and poor expands over time given that each household’s wealth increases over time as outlined in Corollary 1 and which is guaranteed only when \( \eta > \bar{\eta} \).

In this section, we examine the dynamics of relative wealth inequality. We consider two heterogeneous households with respect to their initial wealth holdings, more precisely \( k_{r,0} > k_{p,0} \) (subscript “r” stands for rich and “p” for poor). From equation (44), we can

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deduce the following,

\[
\frac{k_{p}^{*}(t) - \frac{\chi}{r_{f}}}{k_{p}^{*}(t)} = \frac{k_{r,0} - \frac{\chi}{r_{f}}}{k_{p,0} - \frac{\chi}{r_{f}}},
\]

which after rearranging terms gives the relative wealth ratio,

\[
\frac{k_{r}^{*}(t)}{k_{p}^{*}(t)} = \frac{k_{r,0} - \frac{\chi}{r_{f}}}{k_{p,0} - \frac{\chi}{r_{f}}} \times \frac{k_{r,0} - k_{p,0}}{k_{p,0} - \frac{\chi}{r_{f}}} \times \frac{1}{k_{p}^{*}(t)}. \tag{50}
\]

Equation (50) indicates that as the current wealth of a poorer household, \(k_{p}^{*}(t)\), grows over time, the relative wealth gap widens over time.

Hence, the model with constant subsistence consumption generates a non-stationary relative wealth distribution in the steady state. On the contrary, in the next section we find that the utility specification in the context of “keeping-up-with-the-Joneses” produces a stationary distribution of wealth. Also, Chan and Kogan (2002) show that the introduction of external habit into agent’s preferences guarantees an invariant wealth distribution across different states of the economy (ranging from higher-than-average, average to lower-than-average relative consumption) in a long-run growth model. Guvenen (2009) assumes a stationary economy, but suggests in his conclusions that the utility with external habit is able to generate an endogenous stationary wealth distribution in the long run\(^{40}\).

Nonetheless, the next section’s examination of the “keeping-up-with-the-Joneses” preferences points to some limitations of this utility specification. The external habit indeed generates a stationary distribution of relative wealth and produces the IES that is positively related to wealth, but it fails to reconcile the empirical patterns on savings, portfolio choice and consumption growth.

\(^{40}\)See footnote 7, p. 1723, and the paper’s conclusion in Guvenen (2009).
5.3 External Habit Formation (time-variant subsistence consumption)

In this section, subsistence consumption takes the form of external habit formation. The expected lifetime utility function is as follows,

$$E_0 \left\{ \int_0^\infty e^{-\rho t} \left[ \frac{c(t) - \gamma \bar{C}(t)}{1 - \frac{1}{\eta}} - 1 \right] dt \right\}, \quad (51)$$

where $\bar{C}(t)$ stands for the aggregate economy-wide consumption in period $t$ and $\gamma \in (0, 1)$ is the weight agents place on the average consumption.\(^{41}\) It is important to impose a restriction on parameter $\gamma$ in order to ensure that household consumption is $c(t) \geq \gamma \bar{C}(t)$. However, the only way to specify such a parametric constraint, which is determined by the initial wealth distribution, is to firstly solve the model under the practical assumption that an interior solution exists.\(^{42}\)

The utility formulation in equation (51) takes the “benchmark consumption” into account, which is defined as the average consumption in the economy. Agents derive utility from consuming above the average consumption level that is moderated by the importance agents give to the aggregate consumption. As a result, economic agents keep track of the aggregate consumption flow over time and adjust their individual consumption so as to keep up with the consumption of the Joneses.\(^{43}\) The concept of “keeping-up-with-the-Joneses”

\(^{41}\)Koulovatianos (2005, Theorem 3) shows that the utility function with time-varying subsistence levels permits exact linear aggregation across households in a non-stochastic environment. This implies that, on condition that all model’s results are interior, also in the event of aggregate shocks the aggregate consumption level, $\bar{C}(t)$, coincides with the decision of a (fictitious) household that is endowed with the average (aggregate) amount of wealth holdings, $\bar{K}(t)$.

\(^{42}\)Notice that the restriction guaranteeing a well-defined problem and interiority of solutions, namely $k_0 > \chi/r$ (where $k_0$ is the initial wealth of the poorest agent), in the previous model with constant subsistence consumption could also only be recognized once we had solved the problem under the practical assumption that solutions were interior.

\(^{43}\)Notice that the empirical investigation of our partial-equilibrium model with exogenous prices would not distinguish between the notions of internal and external habits. The reason is, as we show below, household’s consumption paths, in equilibrium, evolve in parallel. Also, the intertemporal elasticity of substitution given
utility function, for portfolio choice analysis, is also introduced by Chan and Kogan (2002). Yet, Chan and Kogan (2002) conduct a numerical investigation of portfolio policies and, in contrast to our notion of external habit as a flow variable, they use the history of aggregate consumption to obtain a stock of external habit.

In summary, the idea is that a group of price-taking investors, who live together in a certain area, but invest globally in an international stock market, monitor the average consumption level, \( \bar{C} \), in their residential area. In order to determine the average consumption flow over time, \( \bar{C}(t) \), we assume a fictitious utility-maximizing agent who is endowed with an average initial financial wealth \( \bar{K}_0 \). The concept of “representative consumer” who is endowed with average asset holdings and whose utility function comprises preferences of all agent types (or in the case of nonheterogeneous preferences, as in the present model, his utility is identical to everyone else’s) is presented in Caselli and Ventura (2000) and Koulovatianos (2005). The fictitious household’s decision rules for consumption and portfolio choice are consistent with the aggregated decisions made in the residential area. This result is possible due to linear aggregation that applies in the current setup of the model, which is confirmed once the solutions to the problem are derived.

The stochastic differential budget constraint of an individual investor is,

\[
dk(t) = \{[\phi(t) R + (1 - \phi(t)) r_f] k(t) - c(t)\} \, dt + \sigma \phi(t) k(t) \, dz(t) .
\]

(52)

Furthermore, since an individual agent keeps track of the average consumption path, \( \bar{C}(t) \), and the representative (fictitious) agent with \( \bar{K}(t) \) units of financial wealth also solves by,

\[
IES = \eta \left( 1 - \gamma \frac{\bar{C}}{c} \right)
\]

reduces to \( IES = (1 - \gamma) \) after having substituted \( c = \bar{C} \). The implied ratio of consumptions \( \bar{C}/c = (1/\gamma)(1 - IES/\eta) \) can match well the data for empirically reasonable \( IES \) values.

The relative relevance of internal versus external consumption habits has been recently empirically investigated by Grishchenko (2009).
for optimal decision rules, the individual investor will take account of both his current wealth level, \( k(t) \), and the aggregate wealth level, \( \bar{K}(t) \), when forming his value function. Hence, the individual agent’s value function is \( J(k, \bar{K}) \). Also, the individual investor with \( k \) will monitor the budget constraint dynamics of the average household with \( \bar{K} \), even though \( \bar{C}(t) \) (or \( \bar{K}(t) \)) is beyond the control of the individual.

The average household’s budget constraint is,

\[
d\bar{K}(t) = \left\{ \left[ \tilde{\Phi}(t) R + (1 - \tilde{\Phi}(t)) r_f \right] \bar{K}(t) - \bar{C}(t) \right\} dt + \sigma \tilde{\Phi}(t) \bar{K}(t) dz(t) . \tag{53}
\]

The maximization problem of a household with \( k \) units of wealth subject to both his and average household’s budget constraints is stated by the Hamilton-Jacobi-Bellman (HJB) equation,

\[
\rho J(k, \bar{K}) = \max_{c \geq 0, \phi} \left\{ \left( \frac{c - \gamma \bar{C}}{1 - \frac{1}{\bar{n}}} \right)^{\frac{1}{\bar{n}}} - 1 + J_k(k, \bar{K}) \left\{ [\phi R + (1 - \phi) r_f] k - c \right\} + 
\right. \\
+ J_K(k, \bar{K}) \left\{ [\tilde{\Phi} R + (1 - \tilde{\Phi}) r_f] \bar{K} - \bar{C} \right\} + \\
+ \frac{(\sigma \phi k)^2}{2} J_{kk}(k, \bar{K}) + \frac{(\sigma \tilde{\Phi} \bar{K})^2}{2} J_{\bar{K} \bar{K}}(k, \bar{K}) + \sigma^2 \phi \tilde{\Phi} k \bar{K} J_{k \bar{K}}(k, \bar{K}) \right\} \tag{54}
\]

where \( J_x \) and \( J_{xx} \) are first and second partial derivatives with respect to \( x \in \{k, \bar{K}\} \) while \( J_{k \bar{K}} \) is the cross-derivative. The average consumption and portfolio choice trajectories \( \bar{C}(t) \) and \( \tilde{\Phi}(t) \) of a non-average individual coincide with the aggregate decision rules \( \bar{C}(t) = C(\bar{K}(t)) \) and \( \tilde{\Phi}(t) = \Phi(\bar{K}(t)) \), which are solutions to the optimal control problem of an individual with \( \bar{K} \) units of wealth and whose budget constraint is given by (53), for all \( t \geq 0 \).

The first-order conditions to the maximization problem are,

\[
(c - \gamma \bar{C})^{-\frac{1}{\bar{n}}} = J_k(k, \bar{K}) , \tag{55}
\]
\[ \phi = \frac{R - r_f}{\sigma^2} \frac{J_k (k, \bar{K})}{-J_{kk} (k, \bar{K}) k} + \Phi \cdot \frac{J_{k\bar{K}} (k, \bar{K})}{-J_{kk} (k, \bar{K}) k}. \]  

(56)

In Proposition 6 we present analytical results for the problem stated above. Prior to that, however, we impose in Assumption 3 a parametric restriction on parameter \( \gamma \) in order to ensure the existence of interior solutions.

**Assumption 3** The parameter \( \gamma \) is restricted to be strictly lower than the ratio of the poorest household's initial wealth, \( k_0 \), to the average initial wealth, \( \bar{K}_0 \).

\[ \gamma < \frac{k_0}{\bar{K}_0}. \]  

(57)

With Assumption 3 at hand, we characterize in Proposition 6 interior solutions to the problem.

**Proposition 6**

Respecting Assumptions 2 and 3, the optimization problem stated by the HJB equation (54) delivers the optimal consumption decision rule,

\[ c^* = C (k) = \xi k, \]  

(58)

where

\[ \xi = \rho \eta + (1 - \eta) r_f - \frac{\eta (\eta - 1)}{2} \left( \frac{R - r_f}{\sigma} \right)^2, \]  

(59)

and the optimal portfolio selection decision rule,

\[ \phi^* = \Phi (k) = \eta \frac{R - r_f}{\sigma^2}, \]  

(60)

with the value function given by,

\[ J (k, \bar{K}) = -\frac{1}{\rho \left( 1 - \frac{1}{\eta} \right)} + \xi^{-\frac{1}{\eta}} \frac{(k - \gamma \bar{K})^{1 - \frac{1}{\eta}}}{1 - \frac{1}{\eta}}. \]  

(61)
Proof Derivation of the analytical solutions is given in Appendix B.\(\square\)

The role of Assumption 3 becomes apparent once we look at the value function form, \(J(k, \bar{K})\). More precisely, the assumption ensures a well-defined problem with interior solutions.

Corollary 2 summarizes first observations based on the decision rules in Proposition 6 and household wealth dynamics, which we provide in Appendix B.

Corollary 2

Respecting Assumptions 2 and 3, the solution to the maximization problem given by the HJB equation (54) yields identical across wealthier and poorer agents saving rates, portfolio allocation shares, and the same variation coefficients of consumption and volatilities of consumption growth rate.

Proof The results are straightforward once noticed that household wealth dynamics evolve over time according to the standard geometric Brownian motion process with identical coefficients across richer and poorer households, independently from initial restrictions.\(\square\)

In the economy with agents who are not exposed to idiosyncratic risks, like, for example, labor income risk, but face the same aggregate risks associated with the volatility of stock index returns, the relative wealth gap will be expanding over time. Furthermore, the movements of stock returns follow a Gaussian random walk and economic agents internalize that the effects of aggregate shocks on wealth dynamics are permanent.

The empirical facts on saving behaviors, portfolio allocation and consumption volatility are qualitatively reconciled by the model economy with constant existential needs, as shown in Propositions 3 through 5. The saving rates, risky portfolio shares and volatility of consumption growth are all strictly positively related to wealth. Also, endogenously
derived IES are rising in wealth. The Corollary 1 reveals that agents who face constant bread-and-butter needs, \( \chi \), can escape future pressure of living near the subsistence level by accumulating wealth, namely by choosing positive saving rates. So, the poor foresee moving away from the subsistence bound. Yet, since the non-rich may feel already today the pressure of consuming too near the subsistence level that remains equal over time, they can pursue a strategy of lower positive saving rates and lower risky portfolio shares with the aim to relieve today’s existential pressures and to avoid excessive risks that could endanger their (future) survival. In summary, agents can afford to get out of the poverty at a slower pace and thus to rebalance today and future subsistence pressures.

This strategy, however, cannot be pursued by \( \gamma \hat{C}(t) \)-type household whose subsistence needs are increasing over time. Since, the pressure of the existential needs rises and agents do not foresee moving away from the pressing necessities, the poor cannot afford to save at a lower rate and undertake low-return but less risky investments. The poorer \( \gamma \hat{C}(t) \)-type agents need to keep up with the decisions the Joneses make so as to maintain their relative living standards. Corollary 2 provides a strong result of identical saving rates and portfolio shares across wealthier and poorer households producing parallel wealth paths. Hence, the model with “keeping-up-with-the-Joneses” preferences generates a stationary relative wealth distribution. So, although agents differ with respect to IES, they keep track of the average consumption flow and update accordingly their individual consumption producing a parallel to average consumption paths. Also, this implies that the equilibrium path is equivalent to the equilibrium outcome of the maximization problem with homothetic preferences, where external habit is not present, i.e. \( \gamma = 0 \).

The linear aggregation feature contributes to the strong result of Corollary 2. The

\[44\text{Notice that parameter } \gamma \text{ does not enter any of the decision rules, but only the value function in Proposition 6, which guarantees the same equilibrium paths for both utility specifications, namely with habit, } \gamma > 0, \text{ and without habit, } \gamma = 0.\]
“keeping-up-with-the-Joneses” utility function implies a rebalancing effect of saving rates and risky portfolio shares among wealthier and poorer agents. This effect is possible because the benchmark level of subsistence needs is in aggregate units. More precisely, individual’s subsistence consumption is determined by the consumption of the “representative consumer” with average wealth holdings\textsuperscript{45} and by the taste parameter $\gamma$ that is identical across all agents. Also, even if the taste parameter were heterogeneous across wealthier and poorer households, Caselli and Ventura (2000) show that the weighted average sum of taste parameters can, without loss of generality, be normalized to 1, which allows for linear aggregation.\textsuperscript{46}

Some former studies on aggregation, see, for example, Chatterjee (1994), forego aggregation of subsistence consumption and assume exogenously given minimum consumption that implies a positive dependence of savings on wealth. This finding contradicts the result of Corollary 2 but corresponds well with that of Corollary 1. Hence, the importance of time-invariant exogenously given minimum consumption is here once more underlined.

In our analysis, the remarkable result of Corollary 2 is obtained due to Theorem 3 in Koulovatianos (2005) that states necessary conditions for the linear aggregation in an economy with time-varying subsistence consumption.

The major contribution of this chapter that bases on Achury et al. (2012) is the application of exact linear aggregation property to portfolio-choice analysis.

\textsuperscript{45}For more details on the concept of the “representative consumer”, see Caselli and Ventura (2000) and Koulovatianos (2005).

\textsuperscript{46}In addition, Caselli and Ventura (2000) agents derive utility from consuming public goods and they place a weight on public consumption according to their “taste”. In a similar manner, the poorer households, in our analysis, could place a higher weight on subsistence consumption as the satisfaction of basic needs or the catching up with others is relatively more essential for them.
5.4 Quantitative exercise

We have demonstrated that the empirical facts can be qualitatively reconciled by the model economy with time-invariant subsistence consumption. In this section, we assign values to model parameters and variables in order to obtain model-implied IES, stockholding and saving behaviors across wealth-heterogeneous agents. Our target is to investigate whether this model also has a potential to quantitatively match the stylized facts. More precisely, the implied stock holding shares are confronted with the fractions derived from the Survey of Consumer Finances (SCF, 2007). Furthermore, the model’s results for saving rates are compared to the values estimated by Dynan et al. (2004). Finally, the model-implied elasticities of intertemporal substitution are checked against the estimated levels reported in the finance literature. The main goal of this quantitative analysis is to examine whether our implied results can simultaneously match the targeted empirical patterns by using a single, identical across investors, utility function. In a nutshell, the purpose of the calibration is to challenge our analytical solution to the model with time-invariant subsistence consumption.

The model parameters are calibrated according to the values commonly used in the finance literature. All parameter values are reported in Table 2.1. Yet, the calibration of lifetime resource variable \( k \) and subsistence parameter \( \chi \) requires a cautious consideration. The variable \( k \) is defined as total household lifetime resources and hence comprises both household wealth holdings and lifetime labor earnings. We construct variable \( k \) by income classes using the Survey of Consumer Finances (SCF) data. Our proposed value for the subsistence parameter \( \chi \) may be debatable and controversial, as one can argue that our achieved results are subject to the choice of \( \chi \). For this reason, we perform a sensitivity analysis with respect to different subsistence levels.

First, we discuss the calibration of the variable, \( k \), and parameter, \( \chi \), and then we report
our benchmark findings and results of the sensitivity analysis.

5.4.1 Definition and construction of the resource variable “$k$”

In this subsection, we define our resource variable $k$ and explain how we construct the variable from the data. The budget constraint (35) comprises the dynamics of the variable $k$. We can integrate both sides of the equation (35), take time zero conditional expectations and use the transversality condition to obtain,

$$
k_0 = \int_0^\infty E_0 \left[ e^{-\int_0^t r(s) \, ds} \, c(t) \right] \, dt \equiv EPVC,
$$

(62)

where $EPVC$ stands for the expected present value of consumption.

Our notion of variable $k$ comprises both wealth endowments and labor earnings. If we distinguished explicitly between asset holdings, $a$, and labor income, $y$, the household’s budget constraint provided by equation (35) would take the following form,

$$
da(t) = \{ [\phi(t) \, R + (1 - \phi(t)) \, r_f] \, a(t) + y(t) - c(t) \} \, dt + \sigma \phi(t) \, a(t) \, dz_a(t),
$$

(63)

and

$$
dy(t) = \nu(y,t) \, dt + \theta(y,t) \, dz_y(t),
$$

(64)

where $z_a(t)$ and $z_y(t)$ are standard Brownian motion processes, which are restricted as follows: $z_y(t) = \rho_{a,y} z_a(t) + \sqrt{1 - \rho_{a,y}^2} z(t)$. Notice that $z(t)$ is another standard Brownian motion that is not related to $z_a(t)$ and $\rho_{a,y} \in (-1, 1)$ is the coefficient of correlation between equity asset returns and income growth.\(^{47}\)

\(^{47}\)Such a formulation of budget constraint is examined by Henderson (2005). Also, Duffie et al. (1997) model the budget equation in a similar manner, but they additionally take account of risky asset’s dividends. In the next chapter, we also distinguish between asset gains and uncertain labor earnings. However, our budget equation will be even more specific due to the possibility of investment in multiple risky assets. Koo (1998) is another study that introduces stochastic labor income and multiple risky assets.
In a similar manner, we can solve the budget constraint (63), namely we integrate and take conditional expectations to obtain,

\[ EPVC = a_0 + EPVY, \]  

where \( EPVY \) stands for the expected present value of labor earnings,

\[ EPVY = \int_0^\infty E_0 \left[ e^{-\int_0^t (r(s)ds) y(t)} \right] dt, \]

and \( y(t) \) follows Brownian motion with a drift as expressed by equation (64), while \( r(t) \equiv \phi(t)R + (1 - \phi(t))r_f \) is the total return to asset investments determined by portfolio selection.

Combining equations (62) and (65) leads to \( k_0 = a_0 + EPVY. \) So, the variable \( k \) is a comprehensive measure of lifetime resources.

We use the Survey of Consumer Finances data from 2007 on family net worth and before-tax family income in order to construct our resource variable \( k \). The SCF data covers incomes and wealth of stockholding households only. We average amounts of net worth and income

\footnote{The initial wealth condition given in Assumption 1, which ensures satisfaction of the transversality condition, combined with equation (62) yields,}

\[ EPVC > \frac{\chi}{r_f}. \]  

So, the expected present discounted value of lifetime consumption stream must be strictly larger than the expected present discounted value of lifetime subsistence costs.

Combining equation (65) with condition (66) gives,

\[ EPVC = a_0 + EPVY > \frac{\chi}{r_f}. \]

Hence, the initial wealth must satisfy the following restriction,

\[ a_0 > -\left( EPVY - \frac{\chi}{r_f} \right). \]  

The expression in brackets, \( EPVY - \chi/r_f \), accounts for the expected present discounted value of discretionary lifetime labor earnings. If lifetime discretionary labor earnings are strictly greater than zero, the net worth of financial assets, \( a \), in (67) can be negative, which corresponds to negative asset holdings of some households in SCF data.
over the period 1989-2007 by income percentiles and report their values in thousands of 2007 US dollars in columns (1) and (2) of Table 2.2. The effective marginal tax rates by income percentiles, given in column (3) of Table 2.2, base on the tax and transfer estimates of Grant et al. (2010). With marginal tax rates at hand we can calculate after-tax labor incomes, which we provide in column (4) of Table 2.2.

Total lifetime resources of each income class, $k_y$, can be obtained by summing up a net worth, $a_y$, and expected present value of after-tax labor income, $EPVY_y$ of the respective class (note: subscripts “y” denotes variables belonging to an income class (percentile)). The discount factor calculating $EPVY_y$ is determined by the portfolio asset allocation of each income class, namely $r_y \equiv \phi_y R + (1 - \phi_y) r_f$. The values of interest rates, $R$ and $r_f$, are given in Table 2.1. In the definition of $\phi_y$ we follow Wachter and Yogo (2010) and construct $\phi_y$ as the percentage proportion of net worth, $a_y$, invested in stocks and other equity. To be precise, we use the SCF data on stockholding, business equity, equity in nonresidential property and other residential property to determine stocks/equity holdings of each income group and then we construct risky portfolio shares as $\phi_y \equiv (\text{stock and other equity holdings}) / a_y$. The entries of column (5) and (6) in Table 2.2 are respectively $\phi_y$ and $k_y$. In our calibration part, we assume that stocks and the remaining equity categories are equivalently risky. Finally, data on “other equity” fractions across different age and income classes reported in the Survey of Consumer Finances is prone to aggregation bias due to age-related effects. For instance, young households due to lack of credit history experience difficulties in obtaining credit and thus in initiating their own business or buying their own property. Consequently, the data for categories of residential property, business equity and equity in nonresidential property may not reflect the true equity holdings across the population. In contrast, a less restrictive access to stock markets assures more accurate fractions of stocks across the population. We tackle the aggregation issue by projecting holdings of “other equity” with the aid of the practical assumption that the stockholding pattern of each income group approximately resembles the pattern of other risky asset holdings over the entire lifetime of a household. We provide all details on how we construct the total stock/equity fractions (the entries of column (5) in Table 2) in Table 2A in the Data Appendix B. The obtained fractions of risky asset holdings across income percentiles, the entries of column (5) in Table 2, correspond quite well with the risky portfolio shares by wealth quartiles in Wachter and Yogo (2010, Table 8) reported for the years from 1989 to 2004. Although we confront here risky shares by income percentiles with portfolio fractions by net worth percentiles, we still find comparable asset holding results, confirming the effectiveness of our projection method.
we provide the proportions of total resources invested in stocks/equity, which are derived as
\[ \Phi_y(k_y) = \frac{\phi_y a_y}{k_y}, \]
in column (7) of Table 2.2. Accordingly, the entries of column (7) are risky portfolio shares from the data we aim to match.

5.4.2 Calibration of subsistence parameter “\( \chi \)”

Some recent studies document the presence of subsistence consumption in the utility function. Donaldson and Pendakur (2006) identify household-type basic/subsistence needs under an equivalence scale constraint, namely “Generalized Absolute Equivalence Scale Exactness (GAESE)”. Donaldson and Pendakur (2006) show, using Canadian consumer data, that the equivalent expenditure functions have a significant fixed and an important varying, dependent on reference expenditure, components. Furthermore, Koulovatianos et al. (2007, 2008) provide strong evidence for the GAESE formulation and the existence of subsistence consumption across different household types in six examined countries, namely Botswana, China, Cyprus, France, Germany and India. More precisely, Koulovatianos et al. (2007) pursue a survey approach and collect responses on income levels for different family types taking as a reference a childless individual with several reference incomes. Based on the survey responses Koulovatianos et al. (2007) identify subsistence incomes across family types in the six examined countries, and find statistical support for the positive levels of subsistence income in all investigated cases. Their estimated monthly per capita subsistence values range from 111 to 302 US 2004 dollars. These estimates are comparable to the subsistence results based on Canadian micro data in Donaldson and Pendakur (2006, p. 262, Table 4).

In the calibration of the subsistence parameter \( \chi \) we follow the estimates of subsistence incomes provided by Koulovatianos et al. (2007, Table 4). In order to comply with the SCF 2007 data that we use in our analysis we calibrate our benchmark per capita monthly
subsistence consumption to 230 USD\textsuperscript{50}, which gives an annual value of 2,760 USD. Furthermore, we consider lower and higher per capita subsistence costs for the sensitivity analysis, namely 150 USD per month (1,800 USD per annum) and 300 USD per month (3,600 USD per annum). In addition, since the SCF data is collected at the household level, we calculate the parameter $\chi$ by multiplying yearly subsistence cost per person with family size. The household size is set to 2.5 persons in accordance with 2.54, the US family size averaged over the period ranging from 1986 to 2004, according to the Luxembourg Income Study data. As a result, the benchmark annual household subsistence consumption amounts to 6,900 USD, which is reported in Table 2.1.

The lifetime subsistence consumption of an infinitely-lived individual amounts to 92,000 USD in the benchmark case due to our risk-free rate value of 3%. Accordingly, the lifetime subsistence needs per capita are 60,000 USD and 120,000 USD for lower and higher values of $\chi$ in the sensitivity analysis.

5.4.3 Quantitative findings

In this subsection we describe the calibration of the remaining parameter values and discuss our main findings. The benchmark parameter values are reported in Table 2.1. The interest rates $R$, $r_f$ and standard deviation $\sigma$ are calibrated to values of 7%, 3% and 20% respectively.

In the calibration of these parameters we follow values computed from the data (see, for instance, empirical statistics on expected U.S. stock and bond annual returns, $R \simeq 8\%$, $r_f \simeq 2\%$, and standard deviation of stock, $\sigma \simeq 20\%$, in Guvenen (2009, Table II)) and values commonly used in the portfolio selection literature (see, for example, Gomes and Michaelides (2003, 2005) or Wachter and Yogo (2010) who assign values $R = 6\%$, $r_f = 2\%$, and $\sigma = 18\%$ in their benchmark calibration). Our main target is to stick to the equity

\textsuperscript{50}The abbreviation “USD” means 2007 US Dollars throughout the chapter, unless indicated otherwise.
risk premium of 4% as opposed to the historical risk premium of 6%. The equity premium of 4% is prevalent and widely used in the portfolio selection literature (see, for example, Cocco et al. (2005), Gomes and Michaelides (2005), Yao and Zhang (2005), Wachter and Yogo (2010)). We set the risk-free interest rate to a slightly higher value of 3% with the aim of achieving empirically reasonable saving rates. This is due to our target that aspires to match three empirical regularities at the same time. If we pursued to conform only one of the empirical patterns, we could freely assign values of 2% and 6% for \( r_f \) and \( R \) and pick on preference parameters \( \rho \) and \( \eta \) from an acceptable parameter range.

Our aspiration is to show that the standard two-asset Merton (1969, 1971) model with time-invariant subsistence consumption calibrated according to finance literature is capable of quantitatively reproducing three empirical patterns, namely: (a) risky portfolio shares, (b) saving rates, and (c) intertemporal elasticities of substitution across different income classes. The multiple target calibration exercise provides best results when the preference parameters are assigned the following values: \( \eta = 0.23 \) and \( \rho = 0.025 \).

In summary, through the quantitative exercise we want to demonstrate that the introduction of Stone-Geary preferences with time-invariant subsistence consumption into the portfolio choice problem succeeds in quantitatively matching multiple data series. Our objective is to emphasize the role of time-invariant subsistence consumption in the utility function, which can act as an important modelling ingredient in resolving consumption, savings and portfolio selection puzzles. Consequently, since our innovative contribution to the standard portfolio choice literature is the utility function with constant subsistence consumption, we do not perform comparative statistic for the parameters \( R, r_f, \sigma, \rho, \) and \( \eta \) as they are standard in the finance literature. Instead, we concentrate on a more plausible, for our model, sensitivity analysis, which is to examine the robustness of our model results for different
values of subsistence consumption, the parameter $\chi$.

Our quantitative results produced by benchmark parameterization and sensitivity analysis are plotted in Figures 2.1 and 2.2. The horizontal axis of both figures represents income percentiles by which stockholding households are categorized, namely from the lowest to the highest incomes, as in the Table 2.2.

Figure 2.1 depicts the proportions of total resources allocated to stocks by income percentiles, namely our data and model values of $\Phi (k_y)$. The solid line plots the stock holdings as a fraction of wealth derived from the SCF data, namely the column (7) of Table 2.2. The remaining lines are produced by our model for different values of subsistence consumption. The dashed line depicts our benchmark stock holding shares, which conforms with column (8) in Table 2.2.

We observe in Figure 2.1 that both the stock holding shares implied by the benchmark calibration and the corresponding fractions in the data increase from about 6% in the lowest income class to 22% in the highest income class. The low fractions of wealth in stocks are an outcome of the poor facing relatively more costly subsistence consumption goods, whereas high stock holding shares are a result of the rich facing larger lifetime resources, namely as $k \to \infty$ the fraction of wealth invested in stocks $\Phi (k_y)$ converges to $\eta (R - r_f) / \sigma^2$, the model’s upper limit of stock holdings. So, the larger the resources of an investor, the less influential is the subsistence restriction for the portfolio decision. Subsequently, different values for subsistence consumption, our parameter $\chi$, affect especially the risky portfolio decisions, $\Phi (k_y)$, of the poorest households, as apparent in Figure 2.1. Nonetheless, the overall sensitivity of the risky asset share, $\Phi (k)$, to varying values of parameter $\chi$, is low, which validates the correctness of the model’s setup.

Although we do not obtain a perfect match of our simple portfolio choice model to the
data, the model reproduces the observed stock holding shares of the extreme income classes, namely of the poorest and the richest investors. Furthermore, since our model framework does not imply any limiting restrictions in order to produce the risky portfolio shares across various income percentiles, we believe that the concave pattern of the model-induced portfolio choice curve $\Phi(k)$ could be reduced when more real-world features (for example diverse investment possibilities, such as the ownership of housing or private business) are accounted for.

With equation (48) at hand, we plot the saving rates for all income classes in Figure 2.2 (see also column (9) of Table 2.2 for the benchmark saving rates). Here, once again, the low sensitivity of model-implied saving rates indicates that our results are robust to different values of the parameter $\chi$. The poorest households’ saving decisions are especially affected by varying subsistence levels. This is due to low wealth/income resources of the poor that may be just hovering around the subsistence costs, not leaving much space for savings. The implied benchmark saving rates increase from about 6% to 17%, comparing well with the range of saving rate estimates in Dynan et al. (2004, Figure 1, page 419).

The model coefficients of relative risk aversion are calculated as an inverse of the $IES$, namely $RRA = 1/ [\eta (1 - \chi/c)]$. The implied consumption-dependent coefficients of relative risk aversion are reported in Table 2.3. These values correspond quite well to the estimates of relative risk aversion in the finance literature. For example, Malloy et al. (2009), using consumption of stockholders, estimate the risk aversion coefficients to be even as low as 5-7 for the richest third of stockholding households and 6-9 for the remaining stockholders. Also, our results match empirical estimates of Barsky et al. (1997). Barsky et al. (1997) use a survey approach to identify four distinct groups subject to willingness to gamble. The survey participants are asked whether they were willing to change their current secure job if they
could either double their income or experience a cut of one third, and based on that response the participants are confronted with the next risky opportunity: double income/cut of one half if the first answer was positive and double income/cut of one fifth if the first answer was negative. Based on the survey responses, Barsky et al. (1997, Table I) provide the expected values of relative risk aversion across four groups. Their estimated risk coefficients range from 3.8 for the least risk averse to 15.7 for the most risk averse respondents. These values comply well with our implied relative risk aversion (RRA) coefficients for the benchmark parameterization that range from 4.5 for the richest who are more willing to take on gambles to 14.35 for the poorest who are burdened with the satisfaction of the basic consumption. Notice that increasing the parameter $\chi$ in our analysis affects enormously the poorest agents who become extremely risk averse with the coefficient of relative risk aversion rising to 57.85. Furthermore, Barsky et al. (1997) find that more risk tolerant respondents hold higher fractions of financial wealth in stocks, namely 4.1% higher on average between the most-tolerant and least-tolerant respondents.

Our simple quantitative exercise shows that incorporating time-invariant subsistence consumption into the two-asset Merton (1969, 1971) model allows for reproducing multiple empirical regularities without imposing any restrictive bounds.

5.5 Concluding remarks

We introduce common across agents utility function that is of Stone-Geary form and examine its properties to explain endogenously three stylized facts, namely: (i) larger risky portfolio shares, (ii) greater saving rates, and (iii) more volatile consumption growth, of the wealthy. The fourth stylized fact we aim to match relates to the IES, as it is widely recognized that the rich are less risk averse with respect to wealth/consumption fluctuations. Yet, we argue that a priori assumption of exogenously different preferences for richer and poorer households
may directly lead to any desired results. For instance, assuming that the rich are more risk tolerant automatically signifies that they are more willing to take on risky investments.

We use the standard two-asset Merton (1969, 1971) model to analytically examine consumption/savings and portfolio choice decisions. Our innovative contribution to portfolio choice literature is the use of Geary-Stone preferences with subsistence consumption to provide analytical solutions. More precisely, we investigate two types of subsistence consumption within the Stone-Geary framework, namely: (a) a constant level of basic consumption goods (called as a bread-and-butter consumption, given by parameter $\chi$), and (b) a time-varying level of subsistence consumption (which captures the process of external habit formation over time, namely the “keeping-up-with-the-Joneses” effect, defined as $\gamma \bar{C}(t)$, where $\bar{C}(t)$ stands for the average consumption in a certain residential area at time $t$, $\gamma > 0$). We oppose the two concepts by studying the promise of each for simultaneously resolving/reproducing the consumption, savings and portfolio choice empirical regularities.

The Stone-Geary formulation with time-invariant subsistence consumption is successful in generating results that comply with all the empirical stylized facts stressed above. The decision rule for portfolio share, optimal saving rate formula, variance of consumption growth and consumption-dependent IES, all depend positively on the level of wealth once the assumptions guaranteeing a well defined model and interior solutions are stated. Moreover, the calibration of the simple two-asset model with constant subsistence consumption shows that the model also has a great potential of quantitatively explaining the empirical data. More precisely, implied stock portfolio shares for extreme low and high incomes, 6% versus 22%, match the SCF data between 1989-2007. Model-implied saving rates increase from 6% to 17% with rising income percentiles in accordance with the estimates provided by Dynan et al. (2004). Finally, the coefficients of relative risk aversion range from 14 for the
poorest to 4.5 for the richest households corresponding well with the estimates in Barsky et al. (1997) and Malloy et al. (2009). The sensitivity analysis with respect to different values of subsistence consumption reaffirms the robustness of our results.

On the contrary, the “keeping-up-with-the-Joneses” specification generates identical, irrespective of household wealth, saving rates and portfolio shares, even though it also produces heterogeneity in agents’ risk attitudes.

The notable result of the model with constant subsistence consumption can be only achieved by imposing a loose parametric constraint that guarantees that the expected wealth grows at a strictly positive rate if, and only if, the saving rate is strictly rising with wealth (see Corollary 1). Since the wealth holdings of the poor are low, high saving rates chosen by the rich may be painful for the poor. Subsequently, since poorer investors foresee their wealth to grow over time, they feel less pressure to squeeze their current consumption to the most minimum subsistence level with the aim of saving for the future. In summary, poorer households choose to accumulate their wealth in a slower but less painful way. Furthermore, poor investors choose a low return but secure investment strategy allocating a higher fraction of financial wealth to a less risky asset.

In contrast, an investor with subsistence needs varying over time cannot anticipate a possibility for moving away from the subsistence bound. Since the pressure of consuming at least at the subsistence level increases over time, the investor with external habit cannot choose lower saving rates and less risky portfolio shares with secure but low rates of return. To be precise, poor investors are pressured to get out of poverty in a fast and painful way, accumulating wealth at the same pace as the rich do. The sharp result of equivalent across agents saving rates and portfolio shares can be obtained due to the exact linear aggregation feature that time-varying Stone-Geary preferences satisfy (see Koulovatianos (2005, Theorem
3)).

In summary, our examination provides a strong support for the use of Stone-Geary preferences with constant subsistence consumption in portfolio choice analysis. Using a single common across agents Stone-Geary utility function, our model reproduces three empirical facts at the household level and implies consumption-dependent IES increasing in wealth.
Research Article 3

6. Household Investment in Risky Assets: Recursive Preferences with Subsistence Consumption

This chapter is based on the paper “Analytical Guidance for Fitting Parsimonious Household-Portfolio Models to Data”, co-authored with Christos Koulovatianos and Jian Li.

6.1 Introduction

In this chapter we analyze the multi-asset Merton model (1969, 1971) with stochastic labor income modified to include subsistence consumption. The savings/consumption and portfolio decisions are modelled in a more realistic environment that reaffirms the essential role of existential needs, consistent with the previous chapter’s outcomes. In Figure 3.1 we depict the holdings of stocks and business equity across U.S. households classified by income and wealth percentiles. The increasing risky asset shares with higher income and wealth levels are evident for the U.S. data. These monotonic patterns together with the evidence of saving rates rising in incomes, as documented by Dynan et al. (2004), are simultaneously confronted in the present model with N risky assets and uncertain labor earnings. The failure of the standard portfolio-choice models to explain these monotonic behaviors motivated our further examination of households’ portfolio and savings decisions.

Our study delivers two important results. Subsistence consumption produces saving rates that are rising in wealth on the condition that total future resources (capital and labor

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51 See Data Appendix C for details on the construction of variables in Figure 3.1. The data source is the Survey of Consumer Finances (SCF) 2007. All variables in Figure 3.1 are reported per equivalent adult, as we correct for the effects of different household sizes. We use √n as the equivalence scale, where n is the household size. The size n contains household members of all ages. See Tables C5 and C6 in the Data Appendix C for the comparison between the data for the full sample (as in Figure 3.1) and those that result from restricting the age group to 25-59. Here we learn that demographic and/or life-cycle biases do not play a significant role. As a result, we do not restrict the age group throughout the chapter.
incomes) are predicted to increase over time. This is in line with the previous chapter’s outcome. The second result suggests that the coefficient of correlation between returns on business and stock equity needs to be low in order to assure that both risky asset shares are rising in wealth. Two inequalities in a model, which include the Sharpe ratios of portfolio assets, provide bounds for the coefficient of correlation. In brief, the existential needs play a primary role in securing that a fraction of a risky asset is increasing in wealth. Nevertheless, the low correlation coefficient between asset returns is essential to secure that both asset categories are simultaneously rising in wealth. A low coefficient of correlation enables the diversification of investment risks guaranteeing balanced asset portfolios as wealth rises.

Some literature adopted heterogeneous preferences across richer and poorer agents in order to match the allocations of assets and saving rates in the data. This literature has assumed various elasticities of intertemporal substitution (IES) across heterogeneous agents.\footnote{See, for instance, Guvenen (2009) and De Graeve et al. (2009) who employ utility functions with constant IES but varying IES parameters. Agents that have higher IES are prone to save more and to invest higher fractions of wealth in risky assets. Since their saving rates are higher these agents become wealthier.}

We, in turn, have aspired to obtain consumption/savings and portfolio choices using a common across agents utility function. Our approach distinguishes through the consideration of subsistence costs. In addition, we use a more adequate utility function for the portfolio-choice problem, namely the Epstein-Zin-Weil form of preferences.\footnote{See the studies of Epstein and Zin (1989) and Weil (1989) for more details on recursive utility function in discrete time. For the recursive preferences in continuous time that we employ in this chapter see Duffie and Epstein (1992a,b).} The separation of intertemporal elasticity of substitution (IES) from the coefficient of relative risk aversion (RRA) allows for the trade-off between stability and safety. This feature is desirable in the portfolio selection literature as both time and risk play significant roles in portfolio decisions. More precisely, investors may wish to gain large returns in a short time period but high-return assets are associated with large risks that may jeopardize investors’ consumption. On
the contrary, low-return assets do not endanger investors’ consumption but guarantee only a slow wealth accumulation over time. So, agents need to find a trade-off between the risk they accept and the time over which the returns will materialize.

For our examination we need to specify the amount of an individual’s subsistence consumption. The econometric findings of Atkeson and Ogaki (1996) and Donaldson and Pendakur (2006) confirm the existence of subsistence needs. Yet, we do not rely on the econometric studies to calibrate the subsistence level, as the robustness of subsistence consumption estimates may be questionable among applied-theory researchers due to the issues the econometric models face.\textsuperscript{54} Consequently, as in the previous chapter, we follow the survey estimates on living standards across different household’s sizes conducted by Koulovatianos et al. (2007, 2010) and argue that a minimum annual amount to survive equals to about 3,000 US dollars per adult.\textsuperscript{55}

We show that subsistence consumption generates not only rising portfolio shares and saving rates in wealth but also a plausible heterogeneity in IES across richer and poorer households.\textsuperscript{56} In the previous chapter we have demonstrated that the IES variation across wealth-heterogeneous agents does not suffice to generate saving rates that are increasing in wealth. Specifically, we have shown that a portfolio selection model with external habit preferences produces IES that rise with wealth and at the same time it generates identical across agents saving rates and risky portfolio shares. Consequently, we have concluded that it is essential to consider a time-invariant subsistence component in the specification of

\textsuperscript{54}Examples of econometric issues are identification, exogeneity, simultaneity.

\textsuperscript{55}Our parameterization refers to 2007 US dollars. In the calibration we follow the survey of Koulovatianos et al. (2007, 2010). The survey is based on the data from six countries and is conducted using the method of Koulovatianos et al. (2005)).

\textsuperscript{56}Wachter and Yogo (2010) suggest a similar idea by using non-homothetic preferences. They employ the same approach as Browning and Crossley (2001) and distinguish between necessities and luxuries. Wachter and Yogo (2010) obtain IES heterogeneity across richer and poorer agents after aggregating micro consumer data into a basket of consumer goods. Our approach, on the other hand, uses a-priori a basket of consumer goods, which may be more appealing to macro researchers.
minimum survival needs in order to obtain savings and portfolio-selection patterns in line with the observed rates.

Positive savings increase households’ wealth giving them an opportunity to escape poverty and to consume further above the minimum level of survival needs in the future. As shown in the previous chapter, the poor see this potential of growing resources over time and decide to deal with their subsistence concerns at a slower pace. The trade-off between today’s and tomorrow’s sacrifices makes the poor choose lower saving rates than the rich do. More precisely, the poor smooth out their suffering over time. This mechanism induces saving rates to be rising in wealth if returns on capital investments are the only source of income. Yet, exogenous innovations to labor earnings shift future available resources that may influence savings and portfolio decisions. We show that the pattern of saving rates rising in wealth is maintained when a stochastic labor income process, in accord with the permanent income hypothesis, is considered in the portfolio/savings model.

Moreover, introducing two types of risky assets, namely stocks and business equity, constitutes another challenge for our investigation. The differing risks associated with two categories makes the aggregation of both assets into a broader asset category a difficult task. The common stock shares of companies traded on capital markets are much easier to buy and sell than private businesses. This friction in trading business equity entails additional idiosyncratic risks for the business-equity holders. Yet, these risks are difficult to quantify using price realizations from the past.57 As a result, the coefficients of correlation between returns on stocks and business equity as well as between innovations to labor income and business equity are mostly unknown from the individual household’s perspective. Consequently, the second main goal of this chapter is to investigate the stochastic structure of business-equity risks faced by equity-holding households. To do that we match the holding shares of stocks

in the model with fractions in the data and study implications for the variance-covariance matrix between business equity and stocks.

Furthermore, the separation of stocks from business equity imposes a challenge that is technical in nature. In a model with two risky portfolio assets and uncertain labor income there are three exogenous shocks. This high stochastic dimensionality makes the computation of the model difficult.\textsuperscript{58} Furthermore, the simulation of such a complex model economy may deliver black-box results. Subsequently, in order to handle the model’s complexity we pursue an analytical approach to problem solving by assuming a perfect insurability of labor-income risk. More precisely, we restrict the sum of squares of correlations between risky asset gains and labor income growth to unity. Such a parametric restriction does not lead to implausible correlations. Also, the exogenous role of labor income to shift future resources is retained. Thanks to this assumption we are able to obtain closed-form results, which allow for analytical examination of our goals. Analytically tractable solutions are attained despite the recursive form of preferences in our model, which are modified to include existential consumption. We argue that the ability to insure labor-income risk may not be far from the real world, in particular the US. A recent work of Guvenen and Smith (2010) suggests that macroeconomic models adopt too high values of income-risk uninsurability. Since the fluctuations in labor income may be predicted to some extent, economic agents can insure against these variations by increasing both precautionary savings and allocation of capital to risky assets.

To summarize, we show that subsistence consumption in the multi-asset model of Merton (1969, 1971) with stochastic labor earnings and Epstein-Zin-Weil preferences can explain empirical facts of rising savings, stocks and business equity shares in wealth. In addition,\textsuperscript{58} Noteworthy, Garlappi and Skoulakis (2010) propose a computational method for a numerical analysis that seems to be promising to overcome the curse-of-dimensionality issues.
our analysis indicates that the correlation coefficient between stock and business equity is an important factor in enabling a successful parameterization despite the complex wealth and substitution effects brought by subsistence consumption.

6.2 Model

We develop an extended version of the previous chapter’s partial-equilibrium model in continuous time. To be precise, the model is augmented to include more real-world characteristics that may be very important in the portfolio-choice and consumption/savings analysis.

First, we assume that economic agents receive income from two sources, which are capital gains and labor earnings. The returns on capital are endogenously created, whereas labor income shifts exogenously the resource constraint. Also, in this chapter, agents face a broad range of risky assets and one risk-free asset as opposed to the more simple two-asset model of the previous article. Second, we use the Epstein-Zin-Weil recursive preferences instead of the standard time-additive (time-separable) expected utility function. The advantage of the recursively defined utility function is the separation of the relative risk aversion from the intertemporal elasticity of substitution, which enables us to disentangle the influence of each on the portfolio choice and consumption/savings decisions. This feature is particularly valuable in the portfolio-choice literature, as time and risk play a significant role in financial markets. More precisely, investors are able to independently express the financial risk they accept and their willingness for fluctuations in wealth and consumption. We believe these extensions create a more realistic model economy, which, we show, supports the findings of the previous chapter.

We turn now to the model description. At any continuous time instant $t \in [0, \infty)$ an economic household obtains a labor income, $y(t)$. The stochastic process of labor income is modelled as a geometric Brownian motion with the average labor income growth rate, $\mu_y$,
as a drift,
\[ \frac{dy(t)}{y(t)} = \mu_y dt + \sigma_y dz_y(t) , \] (68)
with \( \mu_y \geq 0, \sigma_y > 0, \) and \( z_y(t) = \varepsilon(t) \cdot \sqrt{t}, \) where \( \varepsilon(t) \sim N(0,1), \) i.e. \( z_y(t) \) is a Brownian process, for an initial value of \( y(0) = y_0 > 0. \)

The economic household is also endowed with an initial wealth, \( a_0 \in \mathbb{R}, \) which he allocates between one risk-free and \( N \geq 1 \) high-risk assets. The price process of a high-risk asset \( i \in \{1, \ldots, N\}, \) is given by a geometric Brownian diffusion process with drift,
\[ \frac{dp_i(t)}{p_i(t)} = R_i dt + e_i \sigma dz_i^T(t) , \] (69)
where \( p_i \) and \( R_i \) are price and average rate of return of an asset \( i \in \{1, \ldots, N\}, \) and \( z(t) \equiv [z_1(t) \ z_2(t) \ \cdots \ z_N(t)] \) is a row vector comprising Brownian motions of \( N \) risky assets. In order to obtain the \( N \times N \) matrix \( \sigma \) we decompose the variance-covariance matrix \( \Sigma \) that governs risks of \( N \) risky assets. More precisely, \( \Sigma = \sigma \sigma^T. \) The \( 1 \times N \) vector \( e_i \) contains the value 1 in the position of an asset \( i \in \{1, \ldots, N\} \) and zeros in the remaining positions. As such, the stochastic term takes only into account the risk of an asset \( i \in \{1, \ldots, N\}. \)

The price process of risk-free asset reduces to the secure rate of return, \( r_f, \) of this asset over time, namely,
\[ \frac{dp_f(t)}{p_f(t)} = r_f dt . \] (70)

The risks to labor income and financial assets correlate with each other. The coefficient \( \rho_{y,i} \) captures the magnitude of the correlation between labor income and risky asset \( i \in \{1, \ldots, N\}. \) Specifically,
\[ z_y(t) = \sqrt{1 - \rho_{y,1}^2 - \cdots - \rho_{y,N}^2} z_0(t) + \rho_{y,1} z_1(t) + \cdots + \rho_{y,N} z_N(t) , \] (71)

where \( z_0(t) \) is a Brownian motion that captures movements other than risky assets and labor income. Hence, if \( \rho_{y,1}^2 + \ldots + \rho_{y,N}^2 \neq 1 \), the risk to labor income cannot be eliminated by buying/shorting financial assets only. If, however, the squares of correlations between income and financial assets can be summed to 1, namely \( \rho_{y,1}^2 + \ldots + \rho_{y,N}^2 = 1 \), the labor risk can be fully insured by trading risky assets.

The model with multiple exogenous shocks to assets and labor income can be solved numerically. Yet, the simulation analysis is a difficult task and the high dimensionality of the model may make the numerical investigation a black-box. Hence, in our analysis, we aspire to facilitate the derivation of an analytical solution and thus make an assumption of perfectly insurable labor income risk, namely \( \rho_{y,1}^2 + \ldots + \rho_{y,N}^2 = 1 \).

The household wealth holdings evolve according to the stochastic differential budget constraint:

\[
da(t) = \left\{ \phi(t) R_T^T + \left[ 1 - \phi(t) 1^T \right] r_f \right\} a(t) + y(t) - c(t) \right\} dt + a(t) \phi(t) \sigma d z_T(t), \tag{72}
\]

where \( R = [R_1 \cdots R_N] \) and \( \phi(t) = [\phi_1(t) \cdots \phi_N(t)] \) are row vectors comprising respectively all risky asset rates of return and all portfolio shares allocated to risky assets at any time instant \( t \geq 0 \) (the superscript \( T \) of any matrix \( A \), given as \( A^T \), denotes the transpose of \( A \)). We allow for short selling by not imposing any restrictions on the risky asset portfolio shares \( \phi(t) \).

Economic households maximize their expected lifetime utility function subject to the budget constraint (72) and labor income process (68). In the specification of the utility function we follow Duffie and Epstein (1992a,b) who extend the discrete-time formulation of recursive preferences of Epstein and Zin (1989) and Weil (1989) to continuous time. Our contribution to the continuous-time specification of Epstein-Zin-Weil preferences is the incorporation of a positive constant subsistence consumption, \( \chi \), into the utility function.
As shown in the previous chapter, the time-invariant subsistence consumption has a great potential to explain empirical facts at the micro level.

The expected lifetime recursive utility function is defined in accordance with the formulation and parameterization of Epstein-Zin-Weil preferences in continuous time by Duffie and Epstein (1992a,b), namely,

$$J(t) = E_t \left[ \int_t^{\infty} f(c(\tau), J(\tau)) \, d\tau \right],$$  \hspace{1cm} (73)

where $f(c, J)$ is a normalized “aggregator” function of current consumption, $c$, and continuation utility, $J$. The normalized aggregator is defined as,

$$f(c, J) \equiv \rho (1 - \gamma) \cdot J \cdot \frac{\left( \frac{c - \chi}{(1 - \gamma) J^{1 - \gamma}} \right)^{1 - \frac{1}{\eta}}}{1 - \frac{1}{\eta}} - 1,$$  \hspace{1cm} (74)

where $\chi \geq 0$ is the subsistence parameter and $\rho, \eta, \gamma > 0$. For $\chi = 0$ which conforms with the standard utility formulation, the parameter $\eta$ is the intertemporal elasticity of substitution (IES) and $\gamma$ denotes the relative risk aversion measure. Yet, this is no longer true when the subsistence parameter is positive, $\chi > 0$, which is a crucial case in our investigation. In Appendix C we find that the IES equals $\eta (1 - \chi/c)$ regardless of whether $\gamma \neq 1/\eta$ or not. Hence, parameter $\eta$ places an upper limit on the IES in the presence of a positive subsistence level (remember that $c \geq \chi$). So, as consumption converges to infinity, namely $c \to \infty$, the parameter $\eta$ converges to IES.

Furthermore, and very interestingly, in Appendix C we prove that setting $\gamma = 1/\eta$ in the case of $\chi > 0$ reduces the more general recursive utility to the time-additive Hyperbolic-Absolute-Risk-Aversion utility function.$^{59}$

$^{59}$In particular, we illustrate in Appendix C that $f(c, J)|_{\eta = 1/\gamma}$ transforms the continuation utility function into
6.2.1 Analytical solution to the household’s optimization problem

Continuation utility in equilibrium, \( J^* (t) \), is the household’s value function, which depends on financial assets and labor earnings, so \( J^* (t) = V(a(t), y(t)) \) for any time instant \( t \geq 0 \). The optimization problem is expressed by the Hamilton-Jacobi-Bellman equation (HJB). In the model economy with infinitely-lived households and constraints that are of time-invariant state space form, we can omit the time index from the HJB optimization equation. The HJB is given by,

\[
0 = \max_{c \geq x} \left\{ f(c, V(a, y)) + \left\{ [\phi R^T + (1 - \phi 1^T) r_f] a + y - c \right\} V_a(a, y)
+ \frac{1}{2} \alpha^2 \phi \sigma \phi^T \cdot V_{aa}(a, y) + \mu_y \cdot V_y(a, y)
+ \frac{1}{2} (\sigma_y y)^2 \cdot V_{yy}(a, y) + \sigma_y a y \phi \sigma \rho_y^T \cdot V_{ay}(a, y) \right\}, \quad (76)
\]

where \( V_x \) and \( V_{xx} \) are respectively first and second partial derivatives with respect to variable \( x \in \{ a, y \} \), \( V_{ay} \) denotes the cross-derivative, and \( \rho_y \) gives a row vector that comprises all coefficients of correlation between each risky asset rate of return and the labor income growth, namely \( \rho_y = [\rho_{y,1} \cdots \rho_{y,N}] \).

Once the problem is stated by the HJB in (76), we can derive the first-order conditions with respect to the control variables, consumption and portfolio selection. The first-order conditions are respectively,

\[
f_c(c, V(a, y)) = V_a(a, y), \quad (77)
\]

\[
J(t) = \rho E_t \left\{ \int_t^\infty e^{-\rho (\tau - t)} \left[ \frac{c(\tau) - \chi}{1 - \frac{1}{\psi}} - 1 \right] d\tau \right\}. \quad (75)
\]

Koo (1998) proposes a similar model economy to ours. Yet, his utility specification conforms with the additively time-separable utility function provided in (75) with \( \chi = 0 \).
\[ \phi^T = \left( \sigma \sigma^T \right)^{-1} \left( R^T - r_f 1^T \right) \frac{V_a(a, y)}{-a \cdot V_{aa}(a, y)} - \sigma_y \frac{y}{a} \left( \rho_y \sigma^{-1} \right)^T \frac{V_{ay}(a, y)}{V_{aa}(a, y)}. \] (78)

Also, in this chapter, we need to make assumptions that will guarantee that an analytical analysis of the problem is possible and that solutions are interior. Once the optimal decision rule for consumption and the formula for value function are derived, the role of assumptions that are technical in nature becomes apparent.

**Assumption 1** Initial total lifetime resources are restricted to be greater than lifetime subsistence needs,

\[ a_0 + \frac{y_0}{r_y} > \frac{X}{r_f}, \]

where the discount factor for lifetime labor earnings is given by,

\[ r_y \equiv r_f - \mu_y + \sigma_y \left( R - r_f 1 \right) \left( \rho_y \sigma^{-1} \right)^T. \]

**Assumption 2** Parameter \( \eta \) is restricted from above as follows,

\[ \eta < \frac{1}{1 - \frac{\rho}{r_f + \frac{\nu}{\nu}}}, \]

where \( \nu \) denotes the squared reward-to-variability ratio,

\[ \nu \equiv \left( R - r_f 1 \right) \left( \sigma \sigma^T \right)^{-1} \left( R^T - r_f 1^T \right). \]

In proposition 1 we provide the model’s analytical results, namely decision rules for consumption and portfolio selection and the value function formula.

**Proposition 1**

In the above model economy with perfectly insurable labor income \( \rho_{y,1}^2 + \ldots + \rho_{y,N}^2 = 1 \) and short selling possibilities, in which Assumptions 1 and 2 are respected, the optimization problem stated by the HJB equation (76) delivers the solution to the problem, namely an optimal decision rule for portfolio selection.
\[ \phi^* = \Phi(a, y) = \frac{1}{\gamma} \left( \mathbf{R} - r_f \mathbf{1} \right) \left( \sigma \sigma^T \right)^{-1} \left( 1 - \frac{\mu}{r_f} \frac{\nu}{a} \right) \]

\[ + \left[ \frac{1}{\gamma} \left( \mathbf{R} - r_f \mathbf{1} \right) \left( \sigma \sigma^T \right)^{-1} - \sigma_y \rho_y \sigma^T \right] \frac{\nu}{\gamma} \frac{r_y}{a}, \tag{79} \]

an optimal decision rule for consumption,

\[ c^* = C(a, y) = \xi \left( a + \frac{y}{r_y} - \frac{X}{r_f} \right) + \chi, \tag{80} \]

where

\[ \xi = \rho \eta + (1 - \eta) r_f - \frac{(\eta - 1)}{2} \nu, \tag{81} \]

and the formula for the value function is,

\[ V(a, y) = \rho^{\gamma \frac{y}{r_y} - \frac{X}{r_f}} \xi^{\frac{1 - \gamma}{1 - \gamma}} \frac{a + \frac{y}{r_y} - \frac{X}{r_f}}{1 - \gamma}. \tag{82} \]

**Proof** The derivation of the analytical formulas is given in Appendix C. □

We elaborate on some individual terms appearing in the above solution. The present value of lifetime expected labor income at time \( t \geq 0 \) is given by \( y(t)/r_y \). From equation (68) we know that labor income grows at an exponential rate, namely \( y(t) = y_0 \cdot e^{\mu t + \sigma y_z(t)} \). Also, our restriction of perfectly insurable labor income risk, \( \rho_y \rho_y^T = 1 \), combined with equation (71) implies \( z_y(t) = \rho_y \cdot z^T(t) \). As a result, the effective discount rate, which is used to compute the present value of lifetime labor earnings, consists of three ingredients, namely \( r_y = r_f - \mu_y + \sigma_y \left( \mathbf{R} - r_f \mathbf{1} \right) \left( \sigma^{-1} \right)^T \rho_y^T \). The first ingredient is the risk-free interest rate, \( r_f \), that accounts for the opportunity cost of not having invested in the risk-free asset. The second ingredient reduces the overall opportunity cost by the average income growth,
\( \mu_y \). The last ingredient is associated with the opportunity cost of foregoing investments in risky assets, which may be easier to see when we assume for a moment that the two first cost ingredients are equal to zero and the effective discount factor is \( r'_y = \sigma_y R (\sigma^{-1})^T \rho^T_y \). The new discount rate amounts to the risky assets' rates of return adjusted for risks of the risky assets, multiplied by the labor income volatility, \( \sigma_y \), which together with the correlation coefficients, \( \rho^T_y \), increase the opportunity cost. Accordingly, in the initial case, the opportunity cost is the foregone excess return adjusted for risk, namely \((R - r_f 1) (\sigma^{-1})^T\), multiplied by labor income volatility (risk that we accept) and correlation coefficients. Similarly, the present value of total lifetime resources is given by the sum \((a + y/r_y)\), whereas the present value of lifetime subsistence consumption needs is comprised by the term \( \chi/r_f \). In order to clarify why the risk-free rate, \( r_f \), alone acts as a discount factor for lifetime subsistence needs we consider a particular case of a household with debt or minimum wealth, \( a \), such that its total lifetime resources just suffice to cover the lifetime subsistence needs, namely \( a + y/r_y = \chi/r_f \). In this instance, the household's optimal portfolio decision in equation (79) reduces to \( \phi^* \cdot a = -\sigma_y y/r_y \rho_y \sigma^{-1} \). This indicates that the special household trades risky assets only in order to completely offset the labor-income risk and thus to secure its equilibrium consumption profile to be always at the subsistence level, namely \( c^* (t) = \chi \) for any time \( t \geq 0 \). This strategy prevents against fluctuations in consumption and ensures that the requirement \( c(t) \geq \chi \) is met with equality for all \( t \geq 0 \). Since this particular household does not have the ability to save (its total income is at the subsistence level), the risk-free rate, \( r_f \), alone determines the intertemporal opportunity cost for the subsistence needs. Finally, the whole term \((a + y/r_y - \chi/r_f)\) determines the discretionary total lifetime resources.

The formula for the value function in equation (82) explains the role of Assumption 1, which restricts the discretionary lifetime total resources to be strictly greater than zero. To be
precise, Assumption 1 secures a well-defined problem and an interior solution to this problem. Also, the optimal consumption decision rule, (80), is a function of discretionary lifetime total resources, \((a + y/N - \chi/r_f)\). The role of Assumption 2 is to assure that the slope of the decision rule for consumption, \(\xi\), is strictly greater than zero, ascertaining the positive dependence of consumption on lifetime resources. Notice that both parameters \(\gamma\) and \(\eta\) that drive respectively risk aversion and intertemporal elasticity of substitution influence the slope, \(\xi\). Interestingly, if the value of the parameter \(\eta\) lies below 1, which also implies that the \(IES = \eta \left(1 - \chi/c\right) < 1\), then the parameter \(\gamma\) affects negatively the propensity to consume, \(\xi\). Hence, if \(\eta < 1\), then greater uncertainty, a higher value of \(\gamma\), stimulates precautionary savings. Nevertheless, the exact effect of a change in risk aversion on household saving rate behavior is difficult to determine. Furthermore, the optimal portfolio selection decision and thus the expected gains on assets are influenced by the parameter \(\gamma\) and, interestingly, they are not affected by the parameter \(\eta\). In the following sections, we elaborate on the features of the saving rate and we characterize household’s optimal portfolio composition in the two risky asset case. Before turning to that we want to shortly emphasize another role of Assumption 1 and 2, which is to secure that the condition \(c(t) \geq \chi\) is satisfied at all times. Subsequently, we do not need to explicitly impose the requirement of \(c(t) \geq \chi\), which corresponds to the standard constraint of non-negative consumption \(c(t) \geq 0\) when \(\chi = 0\). Also, in the case when \(r_f + \nu/(2\gamma) > \rho\), Assumption 2 places an upper limit on the parameter \(\eta\) and thus on the household’s tolerance for intertemporal variation in consumption, which guarantees that the equilibrium consumption does not hit the subsistence level, namely \(c^*(t) > \chi\) for all \(t \geq 0\).\(^{60}\)

The next section deals with the features of the saving rate.

\(^{60}\)Notice that case \(r_f + \nu/(2\gamma) < \rho\) must be abandoned as in this case our restriction assuring a positive dependence of consumption on lifetime resources in Assumption 2 would require a negative value of \(\eta\).
6.2.2 Definition and features of the saving rate

The saving rate depends on the model’s state variables, assets and labor income, and it is calculated as follows,

\[ s(a, y) = 1 - \frac{C(a, y)}{I(a, y)}, \tag{83} \]

where \( s(a, y) \) denotes the saving rate, \( C(a, y) \) is the optimal consumption decision given by (80) and \( I(a, y) \) denotes total income, namely a household’s capital gains and labor earnings, in which we insert the vector of optimal portfolio shares given by the equilibrium decision rule \( \Phi(a, y) \) in equation (79).

We carry out some algebra in order to deduce a dependence of the saving rate on asset holdings, \( a \), or labor income, \( y \). Yet, the analytic result does not provide us with an obvious statement, namely the saving rate is as follows,

\[ s(a, y) = \left[ \eta (r_f - \rho) + \frac{a + 1}{2} \nu \right] \left( a + \frac{\nu}{r_y} - \frac{\chi}{r_f} \right) - \mu_y \frac{\nu}{r_y} \]
\[ \left( \frac{\nu}{r_f} + r_f \right) \left( a + \frac{\nu}{r_y} - \frac{\chi}{r_f} \right) + \chi - \mu_y \frac{\nu}{r_y}. \tag{84} \]

As the analytic form indicates, the subsistence parameter, \( \chi \), and the trend of income, \( \mu_y \), complicate determining of the saving rate’s dependence on state variables and/or other parameter values. Nevertheless, the subsistence consumption is an essential model component for our quantitative analysis. Hence, we elaborate gradually on qualitative characteristics of the saving rate in (84).

**Subsistence needs have a considerable impact on the saving rate.** The requirement to consume above the subsistence level may encourage households to save for the future. More precisely, economic agents may aspire to ensure the satisfaction of their basic consumption over time. Also, the saving incentives may be motivated by the fact that con-
suming near the subsistence level is painful. Hence, all households may pursue a strategy of high saving rates with the aim of moving further away from the subsistence level. In the previous chapter, examining a similar framework with \( \mu_y = 0 \) and \( \gamma = 1/\eta^6 \), we show that the wealthy pursue higher saving rates than the poor do, which may seem surprising at first.

We would anticipate that households living near the subsistence would aspire to get out of the miserable situation as soon as they can. Yet, the study of the previous chapter explains that the current period misery discourages the indigent households from choosing the same saving rates as the rich do (see Corollary 1 in Chapter 2). The indigent households live already today near the poverty level and the high saving rates of the wealthy would require the poor to lower their consumption today to the very minimum, which is painful. In the current framework with \( \mu_y \neq 0 \) the income trend exogenously changes the resource budget constraint over time, affecting household’s saving incentives (unlike the asset holdings, \( a \), that are endogenous). Subsequently, we examine two cases, \( \mu_y \neq 0 \) and \( \mu_y = 0 \), in order to determine how subsistence consumption, \( \chi \), influences the dependence of the saving rate on assets, \( a \), and labor income, \( y \).

**No trend in income (\( \mu_y = 0 \))** If we set the income trend to zero, the saving rate in equation (84) reduces to,

\[
\begin{align*}
\left. s \left( a, y \right) \right|_{\gamma > 0, \mu_y = 0} &= \frac{\eta \left( r_f - \rho \right) + \frac{\eta + 1}{2} \frac{\nu}{\gamma}}{\frac{\nu}{\gamma} + r_f + \frac{x}{a + \frac{x}{\nu} - \frac{x}{\gamma}}}. \\
\end{align*}
\]

So, in the absence of exogenous growth in income, the saving rate, \( s \left( a, y \right) \left|_{\gamma > 0, \mu_y = 0} \right. \), positively depends on both wealth, \( s_a \left( a, y \right) \left|_{\gamma > 0, \mu_y = 0} \right. > 0 \), and income, \( s_y \left( a, y \right) \left|_{\gamma > 0, \mu_y = 0} \right. > 0 \), if and only if, \( s \left( a, y \right) \left|_{\gamma > 0, \mu_y = 0} \right. \) is strictly greater than zero. This monotonicity result is consistent with our previous findings, which are outlined in Proposition 3 and Corollary 1.

\( ^{61} \) In Chapter 2 we study standard Merton (1969, 1971) model with time-additive (time-separable) HARA utility function and no labor earnings.
of Chapter 2. It is worth emphasizing that the model specification with no labor income of the previous chapter complies with the current one for $\mu_y = 0$. If we set $\mu_y = 0$, the assumption of perfectly insurable labor income risk, $\rho_y \rho_y^T = 1$, implies that the trendless stochastic income process is entirely absorbed by risky asset investments and fully integrated in future wealth holdings, $a$, that evolve endogenously over time. Nevertheless, our current specification is distinguished by the case of $\gamma \neq 1/\eta$, which indicates that the saving rate’s monotonicity is consistent between two cases, $\gamma = 1/\eta$ studied in the previous chapter and $\gamma \neq 1/\eta$.

Interestingly, if we additionally set $\chi = 0$, the saving rate in (84) simplifies to,

$$ s(a, y) \big|_{(\chi=0, \mu_y=0)} = \frac{\eta (r_f - \rho) + \frac{\rho + \nu}{\gamma} \frac{\nu}{\gamma + r_f}}{\frac{\nu}{\gamma} + r_f}, $$

and it becomes independent of wealth and income. Hence, the lack of requirement to maintain subsistence consumption implies that the rich and the poor save at the same rate.

As already stated, the above monotonicity result holds only under the condition of a strictly positive saving rate. Both saving rates (85) and (86) are strictly positive exactly when the numerator of both is strictly greater than zero. Subsequently, we carry out some algebra and discover that $s(a, y) \big|_{(\chi \geq 0, \mu_y=0)} > 0$ if and only if,

$$ \eta > \frac{-1}{1 + \frac{r_f - \rho}{\nu}}. $$

We combine two conditions on parameter $\eta$, namely condition (87) and Assumption 2 (88), and obtain,

$$ s(a, y) \big|_{(\chi \geq 0, \mu_y=0)} > 0 \Leftrightarrow \frac{r_f + \frac{\nu}{2\gamma}}{r_f + \frac{\nu}{2\gamma} - \rho} > \eta > \frac{-\frac{\nu}{2\gamma}}{r_f + \frac{\nu}{2\gamma} - \rho}. $$
In the event of \( r_f + \nu / (2\gamma) > \rho \), requirement in (88) is automatically guaranteed, and the saving rate is positive while \( \mu_y = 0 \). In this instance, upper and lower bounds on the willingness to substitute consumption intertemporally are assured. An upper bound takes care of a non-corner solution case, securing \( c^*(t) > \chi \) for \( t \geq 0 \), and a lower bound encourages positive savings. Both are guaranteed due to a relatively low risk aversion and/or relatively high rates of return on assets (high risk-free rate and Sharpe ratio) and a relatively low time preference rate, \( \rho \).

**Non-zero income trend** \((\mu_y \neq 0)\) We examine the saving rate characteristics for a more empirically reasonable case of \( \mu_y > 0 \). Performing some algebra on equation (84) results in,

\[
s(a, y) \big|_{(\chi > 0, \mu_y > 0)} = 1 - \frac{\xi + \frac{\chi}{\gamma} (r_f - \rho) + (\eta - 1) \frac{\nu}{\gamma}}{r_f - \frac{\gamma}{\gamma} \frac{\nu}{\gamma} a + \frac{\mu_y}{\gamma} - \frac{\mu_y}{1 + r_y}} .
\]

(89)

Equation (89) points to the importance of incorporating a subsistence parameter \( \chi \) into the model specification. If we set \( \chi = 0 \), equation (89) reduces to,

\[
s(a, y) \big|_{(\chi = 0, \mu_y > 0)} = 1 - \frac{\xi}{r_f - \frac{\gamma}{\gamma} \frac{\nu}{\gamma} a + \frac{\mu_y}{\gamma} - \frac{\mu_y}{1 + r_y}} .
\]

(90)

The use of homothetic preferences in a multiple-asset model with labor income implies a saving rate (90) that declines with increasing labor income when \( a > 0 \). We can summarize the monotonicity of \( s_a(a, y) \) with respect to labor earnings as follows,

\[
s_y(a, y) \big|_{(\chi = 0, \mu_y > 0)} < 0 \quad \text{if} \; a > 0 , \quad s_y(a, y) \big|_{(\chi = 0, \mu_y > 0)} > 0 \quad \text{if} \; a < 0 .
\]

(91)

In case of \( a > 0 \) there is a clear wealth effect on consumption implying the saving rate’s dependence on labor income to be negative. Household’s labor income grows over time at rate \( \mu_y > 0 \) shifting exogenously its future lifetime resources. An additional increase in labor earnings results in more resources today and in a larger amount of expected future resources.
Consequently, a household with positive wealth holdings, $a > 0$, anticipating its resources to increase over time can reduce its savings. To be precise, higher future consumption bundles can be obtained by fewer sacrifices today. Notice also that present and future consumption are normal goods in our economy, which reinforces this idea. If, however, a household is indebted, $a < 0$, a rise in labor income, which increases present and also future available resources through income’s long-term trend, is a possibility to repay household debt at a relatively lower cost. So, the indebted household seeing the opportunity of servicing its debt at the (relatively) reduced cost increases savings.

A ceteris-paribus rise in asset holdings, $a$, brings about a raise in the ratio $a/y$ boosting the saving rate in (90) and (89). Subsequently, an increase in $a/y$ induces a comparative disadvantage for the labor income, $y$, that exogenously shifts the overall resource constraint without making sacrifices (when $\mu_y > 0$). To be precise, the role of $a/y$ accounts for this comparative disadvantage by implying,

$$s_a(a, y) \bigg|_{ \chi > 0, \mu_y > 0 } > 0,$$

which reinforces our result of a saving rate increasing in wealth, $a$, if $\chi > 0$, obtained from equation (85) for the case of a zero income trend, $\mu_y = 0$.

So, the main finding of our analytical analysis is given by (92), which affirms that the saving rate is positively dependent on asset holdings, $a$. We cannot, however, conclude from equation (89) how the saving rate changes in response to an increase in labor income, $y$. In order to determine the sign of $s_y(a, y) \bigg|_{ \chi > 0, \mu_y > 0 }$ we would need to resort to numerical analysis, which is out of the scope of our investigation here. Also, the case of the negative income trend, $\mu_y < 0$, which is empirically less plausible, leads to unclear saving rate’s monotonicity with respect to wealth and income necessitating numerical examination.
6.2.3 Portfolio selection analysis in the two risky asset case

In this section we aspire to determine the dependence of risky portfolio shares, \( \phi \), on wealth holdings, \( a \), and labor income, \( y \). Yet, our analytical investigation of this dependence is complicated by the role of the variance-covariance matrix of high-risk assets. In the case of two risky assets (\( N = 2 \)) the variance-covariance matrix is,

\[
\Sigma = \begin{bmatrix}
\sigma_s^2 & \rho_{s,b} \sigma_s \sigma_b \\
\rho_{s,b} \sigma_s \sigma_b & \sigma_b^2
\end{bmatrix},
\]

where \( \sigma_i \) denotes the standard deviation of risky asset \( i \in \{s, b\} \) and \( \rho_{i,j} \) is the coefficient of correlation between two high-risk assets \( i, j \in \{s, b\} \). The subscript notation is respectively “s” and “b” for stocks and business equity. The stochastic structure of asset returns and labor earnings comprises three standard deviations, \( \sigma_s \), \( \sigma_b \), and \( \sigma_y \), and two coefficients of correlation, \( \rho_{s,b} \) and \( \rho_{y,s} \), as the correlation parameter \( \rho_{y,b} \) can be obtained from the restriction of perfectly insurable income risk, namely \( \rho_{y,s}^2 + \rho_{y,b}^2 = 1 \).

In Appendix C we make use of the decomposition of the variance-covariance matrix for the two risky asset case (\( N = 2 \)) and perform some algebra on equation (79).\(^{62}\) The portfolio share of stocks becomes,

\[
\phi_s^* = \frac{1}{\gamma} - \frac{1}{\gamma - \rho_{s,b}^2} \cdot \frac{R_s - r_f}{\sigma_s} - \rho_{s,b} \cdot \frac{R_b - r_f}{\sigma_b} \cdot \left( 1 - \frac{\chi}{a} \right)
\]

\(^{62}\) Also, in Appendix C we demonstrate that the discount rate of labor income, \( r_y \), equals,

\[
r_y = r_f - \mu_y + \sigma_y \left[ \frac{R_s - r_f}{\sigma_s} \cdot \left( \frac{\rho_{y,s}}{\sqrt{1 - \rho_{s,b}^2}} \right) + \frac{R_b - r_f}{\sigma_b} \cdot \frac{\rho_{y,b}}{\sqrt{1 - \rho_{s,b}^2}} \right]. \tag{93}
\]

From equation (93) we conclude that the magnitude of \( r_y \) is determined by \( r_f, \mu_y, \sigma_y \), and also by a linear association of two Sharpe ratios that are multiplied by expressions containing the correlation coefficients \( \rho_{y,s} \) and \( \rho_{s,b} \), all contributing to the level of lifetime labor earnings \( y/r_y \).
\[ + \left[ \frac{1}{\gamma} \cdot \frac{1}{1 - \rho_{s,b}^2} \cdot \frac{R_{s} - r_f}{\sigma_s} - \rho_{s,b} \frac{R_{s} - r_f}{\sigma_s} \right] - \sigma_y \left( \frac{\rho_{y,s}}{\sigma_s} - \sqrt{1 - \rho_{y,s}^2} \cdot \frac{\rho_{s,b}}{\sigma_b} \right) \right] \frac{y}{v} \frac{v_y}{a}, \tag{94} \]

while the share of business equity is,

\[ \phi^*_b = \frac{1}{\gamma} \cdot \frac{1}{1 - \rho_{s,b}^2} \cdot \frac{R_{b} - r_f}{\sigma_b} - \rho_{s,b} \frac{R_{b} - r_f}{\sigma_b} \right] (1 - \frac{\chi}{a}) \]

\[ + \left[ \frac{1}{\gamma} \cdot \frac{1}{1 - \rho_{s,b}^2} \cdot \frac{R_{b} - r_f}{\sigma_b} - \rho_{s,b} \frac{R_{b} - r_f}{\sigma_b} \right] - \sigma_y \left( \frac{\rho_{y,s}}{\sigma_s} - \sqrt{1 - \rho_{y,s}^2} \cdot \frac{\rho_{s,b}}{\sigma_b} \right) \right] \frac{y}{v} \frac{v_y}{a}. \tag{95} \]

We can conclude from equations (94) and (95) that the parameter \( \eta \) that is tightly connected with the IES does not influence the portfolio selection decisions. In contrast, the parameter \( \gamma \) that drives the measure of relative risk aversion affects risky portfolio shares in (94) and (95). Specifically, the response of the portfolio shares to an increase in the parameter \( \gamma \) depends on how the correlation coefficient, \( \rho_{s,b} \), relates to the ratio of two Sharpe ratios, \( [(R_i - r_f)/\sigma_i] / [(R_j - r_f)/\sigma_j] \), \( i, j \in \{s, b\} \). The risky asset shares, \( \phi^*_s \) and \( \phi^*_b \), may be either both decreasing in \( \gamma \) or one of the shares may be declining and the other increasing in \( \gamma \). Since the correlation between two risky assets, \( \rho_{s,b} \), is smaller than 1, the result of both shares, \( \phi^*_s \) and \( \phi^*_b \), rising in \( \gamma \) cannot be obtained.

The dependence of risky asset shares, \( \phi^*_s \) and \( \phi^*_b \), on asset holdings, \( a \), and labor earnings, \( y \), in (94) and (95) is difficult to deduce due to complex combinations of parameters that relate to asset returns, their variance-covariance matrix and their correlation with labor income risk. For instance, if \( \rho_{s,b} < [(R_i - r_f)/\sigma_i] / [(R_j - r_f)/\sigma_j] \), \( i, j \in \{s, b\} \), both portfolio shares, \( \phi^*_s \) and \( \phi^*_b \), are rising in wealth, \( a \), according to the first term of equations (94) and (95) given that \( \chi > 0 \). The subsistence parameter, \( \chi > 0 \), is crucial to make asset holdings, \( a \), in the first term of (94) and (95) positively related to shares of risky assets, \( \phi^*_s \) and \( \phi^*_b \), in accordance with the data. Nonetheless, a more complicated combination of parameters in front of the ratio of expected lifetime income to assets, \( \frac{\nu}{v_y}/a \), that plays an additional role
in portfolio decision making, as captured by the second term of (94) and (95), makes the analysis inconclusive.

Nevertheless, portfolio shares of risky assets in equations (94) and (95) point to two main ingredients that facilitate an appropriate calibration of parameters in the two risky asset case. A positive level of subsistence needs, $\chi > 0$, and a low value of correlation coefficient between two high-risk assets, in particular such that $\rho_{s,b} < [(R_i - r_f) / \sigma_i] / [(R_j - r_f) / \sigma_j]$, $i, j \in \{s, b\}$, strongly contribute to the result of portfolio shares rising in wealth matching the data.

6.3 Calibration

6.3.1 Benchmark Calibration

Parameter values are given in Table 3.1. The idiosyncratic risk to labor income, $\sigma_{\gamma}$, is calibrated to 8.21%, which falls within the standard parameter range in the micro literature (see, for instance, Gomes and Michaelides (2003, p. 736)). Also, we assign a value of 1.15% for the mean of labor-income growth (see, for example, Heaton and Lucas (2000)). Next, the stock index rate of return and its volatility are calibrated to their long-run values. Accordingly, we assign parameter values of 7.56% and 21% for $R_s$ and $\sigma_s$, which correspond well with the values in Guvenen (2009) who uses 8% and 20% and Gomes and Michaelides (2003) who in turn employ 6% and 18%.

We have chosen a rather generous value of 3.56% for the risk-free rate as opposed to the standard value of 2% (see, for instance, Gomes and Michaelides (2003) and Guvenen (2009)). Also, in the previous chapter we have chosen a higher value, namely a rate of 3%. In both model economies, risk-free rate values of 3% or above work better in matching real-world characteristics. Furthermore, Roussanov (2010) uses an even larger rate of 5% for the non-risky asset. Yet, our target is to stick to the equity risk premium of 4%, which is not
unusual in portfolio-choice literature. For instance, an equity risk premium of 2.5% is one of the values investigated by Gomes and Michaelides (2003).

We calibrate the time preference parameter, \( \rho \), in line with the previous chapter. The distinction of intertemporal elasticity of substitution, \( \eta \), from relative risk aversion, \( \gamma \), allows one to assign values without the requisite of parameter reciprocity. In the previous chapter we have calibrated the parameter \( \eta \) to 0.23 in order to obtain a degree of risk aversion of 4.35. Here we are able to calibrate the parameters independently. We assign a lower value for the parameter \( \eta \) in light of the empirical results of Hall (1988). Hall (1988) finds that intertemporal elasticity of substitution is not likely to be higher than 0.1 and “may well be zero”. At the same time Hall (1988) does not advocate excessive risk aversion levels. In the current setup of the model, the elasticity value amounts to 0.08 and it does not imply unreasonable high risk aversion levels. To be precise, we calibrate the coefficient of relative risk aversion to 4.78 against 12.5, which were the reciprocal result in the model with additive time-separable preferences.

Furthermore, there is a small difference in the calibration of subsistence consumption as compared to the previous chapter. We increase the monthly amount of existential needs per person from USD 230 to USD 245. Yet, this amount is still within the bounds of survey results on subsistence consumption needs provided by Koulovatianos et al. (2007, 2008), which are USD 111 and 302.

Once we have calibrated the above parameter values we conduct an exhaustive search exercise based on a minimum-distance approach in order to obtain the best fit of the model economy to the evidence.\(^{63}\) The resulting shares of risky assets, depicted in Figure 3.2, account for our benchmark calibration of \( \Phi(a, y) \). The fractions of business equity by income percentiles seem to be matched only imperfectly. Yet, the failure of the standard portfolio-

\(^{63}\) More details on the minimum-distance-approach are provided in the Data Appendix C.
choice literature to obtain an increasing risky asset share in wealth makes our result successful and notable. The model and data business equity shares for the lowest and highest income percentiles meet at around the 5th and 95th percentiles. So, the model’s range of actions correspond quite well with the data. The increasing business shares in the level of income encourage further future work with the aim of achieving an even better fit to the data. On the contrary, the implied stockholding shares fit almost perfectly the data. The minimum-distance exercise delivers the parameter values for business equity, for which the best fit of the model to the data is achieved. More precisely, the exhaustive search implies the mean, \( R_o \), and standard deviation, \( \sigma_o \), of business-equity returns to be 18% and 42.07%. The rate of 18% lies close to the average value estimates provided by Moskowitz and Vissing-Jorgensen (2002, Table 4, p. 756). The model’s implication for the standard deviation of business-equity return is not far away from the value assigned by Roussanov (2010), who, similarly to our setup, distinguishes three investment technologies (risk-free, private and public equities). The private equity in his model is associated with idiosyncratic risk and thus resembles our business equity. Roussanov (2010) uses the private equity rate of 11% and calibrates its volatility to 45%, which comes quite close to our implied volatility value. Also, Moskowitz and Vissing-Jorgensen (2002, p. 765) point out: “ [...] the annual standard deviation of the smallest decile of public firm returns is 41.1 percent. A portfolio of even smaller private firms is likely to be as volatile.”

It is not easy to assess idiosyncratic individual-specific risks from the perspective of a household. There are unobservable frictions in the choice of outside options. For instance, relocating of private business, if wished, cannot be done easily. Also, the insurances against fire and/or theft are only imperfect. These and other frictions related to having one’s own business justify volatility values above 40%, which may even be much higher if all risks could
be taken into account.

Furthermore, the minimum-distance exercise provides a value for the correlation coefficient between unexpected stock index returns and labor income innovations. The implied correlation value, $\rho_{ys}$, amounts to 48.93%. Although the value seems to be high, we find some support in the literature. For example, Luo (2011) uses a coefficient of 35%, but increases the correlation value to 50% in his experiments (see Luo (2011, Figures 3, 5, 7)). The parametric constraint $\rho_{ys}^2 + \rho_{yb}^2 = 1$, which guarantees analytical tractability, together with the implied correlation value $\rho_{ys} = 48.93\%$ gives immediately the correlation coefficient of business equity returns and labor income growth, namely $\rho_{yb} = \sqrt{1 - (48.93\%)^2} = 87.21\%$. It seems to be plausible that business returns are highly correlated with family income growth as households owning a business are prone to employ family members. Consequently, the business returns account for a high fraction of family income.

Having a closer look at equations (94) and (95) and in particular the Sharpe ratios we came to the conclusion that a right and acceptable value for the key correlation between business-equity and stock index returns, $\rho_{sb}$, is 1.74%.

In summary, our extensive analysis indicates that the above model is capable to match the observed fractions of wealth allocated to risky assets on the condition that business-equity gains are very volatile and only slightly correlated with stock index returns. Since the idiosyncratic individual-specific business risks can be considerable the model’s implications seem to be plausible. Notably, the model reproduces the estimates of saving rates provided by Dynan et al. (2004). The implied saving rates across all income groups are depicted in Figure 3.3.
6.3.2 Robustness Analysis

First, we perform sensitivity analysis with respect to the level of subsistence consumption. We vary the amount of monthly subsistence necessities, namely USD 245, by $12 - 18\%$. As Figures 3.4 and 3.5 show lowering and increasing the parameter $\chi$ by $12 - 18\%$ does not imply any significant changes to the fitted risky portfolio shares and saving rates. Nevertheless, the crucial point is to investigate the impact of removing the subsistence consumption parameter from the model. For this reason we set $\chi = 0$. The outcome distinguishes notably from the remaining sensitivity results as shown in Figures 3.4 and 3.5. Risky portfolio fractions of wealth as well as saving rates are almost equal across all income groups. The u-shaped pattern of saving rates in Figure 3.5 arises as a result of the cross-sectional behavior of income-to-asset ratio, $y/a$, along all income groups in the data, which we present in Figure 3.6. The ratio of income to assets, $y/a$, influences both savings and portfolio decisions as shown in equations (89), (94), and (95).

Next, we conduct robustness check on the value of the correlation coefficient between business-equity and stock index returns, $\rho_{ab}$. In Figures 3.7 and 3.8 we plot respectively risky asset shares and saving rates across all income classes for various correlation values. Slight variations in $\rho_{ab}$ can have significant impacts on the fractions of risky assets and particularly on the saving rates. So, the implication of the model for the benchmark correlation $\rho_{ab}$ to amount to 1.74\% seems to be robust.

Finally, we check the sensitivity of implied saving rates to changes in the intertemporal elasticity of substitution, $\eta$. We remind that the parameter $\eta$ influences only the savings decisions and it does not affect any of the portfolio shares as equations (89), (94) and (95) show. It is not a surprise that even small variations in parameter $\eta$ have a substantial impact on the saving rates. Notably, however, their monotonicity is not affected as shown in Figure
3.9. The saving rates shift in a parallel manner.

Lastly, we point out that the saving-rate behavior in the case of additive time-separable preferences, namely when \( \eta = 1/\gamma \), is also quantitatively plausible. Yet, the separation of risk aversion, \( \gamma \), from willingness to substitute over time, \( \eta \), enables the exclusion of the parameter \( \eta \) from portfolio decisions, which is a revolutionary step in portfolio choice analysis. Consequently, since Epstein-Zin-Weil preferences offer an additional valuable degree of freedom, the recursive utility may prove beneficiary for the extensions of the model, such as the addition of liquidity constraints.

6.4 Conclusion

We have investigated households’ portfolio, savings and consumption decisions in a standardized portfolio choice model with recursive preferences of Epstein-Zin-Weil modified to include subsistence consumption. We have shown analytically that subsistence consumption is a crucial parameter in reconciling the model’s predictions with stylized facts on portfolio and savings decisions. Most importantly, we have demonstrated that subsistence consumption is capable of quantitatively attaining saving rates and fractions of wealth invested in risky assets that are jointly increasing in wealth. Our analytical and simulated results have revealed a crucial ingredient that enables the mechanism in terms of which subsistence consumption reconciles the data: private returns on business equity must be associated with considerable idiosyncratic risks, which in turn dampens the correlation between individual business and stock equity returns from the perspective of a household; conversely, this feeble correlation induces both risky portfolio shares to rise with wealth in an even and smooth way thanks to diversification motives.

We have obtained analytical solutions due to the simplicity of our model, in which we have excluded liquidity constraints and assumed a case of perfectly insurable labor-income
risk. So, a possible extension would be to break these presumptions and yet to pursue a parsimonious and at the same time robust theory of household portfolio that does not depend on assumptions of heterogeneous preferences and does not require one to resort to behavioral approaches, which are extremely difficult to quantify.

The curse of dimensionality will make it challenging to simulate such extended model economics. Advanced works similar to that of Garlappi and Skoulakis (2010) can be helpful in conducting these extensions. Yet, we have shown that the calibration of multi-asset portfolio models is demanding and challenging. As a result, our current analysis can be used as a guide for model simulations. Finally, our investigation has demonstrated that the subsistence consumption parameter incorporated in household portfolio-choice models is promising for resolving several household-finance puzzles.

7. Thesis Conclusions

Our focus has been on the widening gap between the rich and the poor. In the spirit of capitalism unequal incomes and wealth endowments imply unequal chances and opportunities. Inequality brings about conflicts across wealth/income classes, hampering economic growth and prosperity. We have learnt that on one hand poor agents, if subsidized financially by a government, tend to behave selfishly and to act as rent-seekers. On the other hand we have shown that not subsidizing and leaving economic poorer agents on their own is not the solution either. Although faced by high-return risky investment possibilities poorer agents allocate lower fractions of wealth to these assets. Also, the non-wealthy choose lower saving rates than the rich do. Consequently, the non-wealthy remain poor and the wealthy become richer. Furthermore, we have shown that consumption growth of the poor is less volatile than that of the rich. Less risky asset investments imply less volatile consumption growth
among the poor. Our results point to high risk aversion of poor agents as they cannot afford volatile consumption growth due to a bread-and-butter constraint. The need to satisfy subsistence costs constrains, in particular, the poor whose wealth holdings may just suffice to cover existential expenditures.

Both our investigations of fiscal transfers and subsistence costs have tackled the issue of unequal resources and thus of unequal chances and opportunities. We argue that transfer payments could be more promising in improving the situation of low-income classes if payments were made for educational purposes. We think that work disincentives and/or tax avoidance triggered by high taxes and elevated redistribution of resources would diminish as more people with an educational degree would imply higher technological development and growing financial prosperity in the long run. Also, the poor with higher human capital would be able to broaden their investment knowledge and thus to understand and accept higher investment risks ending up richer. The decreasing inequality in wealth would be a natural outcome of a policy that enables an easier access to education for non-wealthy households.

In summary, we recommend the redistribution of resources in the form of educational subsidies to the low-income population. Such a policy creates not only better living standards for the poor but improves the overall growth and employment potential of the economy.
Appendix A: Political and economic decisions

Description of the model

Production

The production function has a functional form:

\[ Y_t = z_{1,t} \left[ \alpha (K_t H_t)^{1 - \frac{1}{\beta}} + (1 - \alpha) L_t^{1 - \frac{1}{\beta}} \right]^{\frac{\beta}{\beta - 1}}, \]

in which \( z_{1,t} \) is a hicks-neutral shock that improves productivity of both input factors, utilized capital, \( K_t H_t \), and labor supply, \( L_t \). The shock follows a first-order autoregressive process:

\[ \ln (z_{1,t+1}) = \rho_1 \ln (z_{1,t}) + \varepsilon_{1,t+1}, \]

with \( \varepsilon_{1,t+1} \sim N (0, \sigma^2_{1,e}) \), i.e. it is i.i.d. over time.

The second innovation, \( z_{2,t} \), hits investment, \( I_t \) (see, for example, Greenwood, Hercowitz, and Krusell (EER, 2000)):

\[ K_{t+1} = z_{2,t} \cdot I_t + (1 - \delta(H_t)) K_t, \quad (96) \]

in which

\[ \ln (z_{2,t+1}) = \rho_2 \ln (z_{2,t}) + \varepsilon_{2,t+1}, \]

with \( \varepsilon_{2,t+1} \sim N (0, \sigma^2_{2,e}) \), i.e. it is i.i.d. over time. Investment specific technological innovation \( z_2 \) increases production efficiency of capital goods. The component \( z_{2,t} \cdot I_t \) in (96) acts as an additional sector of production, in which the shock \( z_{2,t} \) measures the efficiency of the sector to generate intermediate goods. Consequently, both investments in new capital and maintenance (replacement) of old capital are less costly in terms of production of consumer goods. One can better see the sectoral production efficiency when we re-arrange equation (96) solving for \( I_t \),

\[ I_t = e^{-x_{2,t}} [K_{t+1} - (1 - \delta(H_t)) K_t], \quad (97) \]
in which

\[ x_{2,t} = \ln (z_{2,t}) \ . \]

We plug equation (97) into the economy’s resource constraint,

\[ Y_t = C_t + I_t + G_t \xrightarrow{(97)} \]

\[ e^{-x_{2,t}} K_{t+1} = Y_t + e^{-x_{2,t}} (1 - \delta(H_t)) K_t - C_t - G_t \]  \hspace{1cm} (98)

We can once again observe that the investment innovation \( x_2 \) changes the replacement cost of current and next period capital. As \( x_2 \) is positive, the price of capital is lower in real terms. Also, the maintenance cost of capital decreases for a positive \( x_2 \), as the usual depreciation rate \( \delta(H_t) \) becomes \( e^{-x_2} \delta(H_t) \). As a result, a balanced fiscal budget with tax-exempt depreciation becomes,

\[ T_t = (\tau_t - ge^{x_{3,t}}) Y_t - \tau_t \delta(H_t)e^{-x_{2,t}} K_t \]  \hspace{1cm} (99)

in which \( T_t \) and \( \tau_t \) stand respectively for lump-sum fiscal transfers and marginal tax rate in period \( t \), \( g \) is an exogenous government consumption share and \( x_3 \) is a government spending shock. Hence, the total government consumption is given by,

\[ G_t = ge^{x_{3,t}} Y_t \]  \hspace{1cm} (100)

in which

\[ x_{3,t} = \ln (z_{3,t}) \ , \]

where

\[ \ln (z_{3,t+1}) = \rho_3 \ln (z_{3,t}) + \varepsilon_{3,t+1} , \]

with \( \varepsilon_{3,t+1} \sim N(0, \sigma_{3,\varepsilon}^2) \).

Plugging (99) and (100) into (98) gives,

\[ e^{-x_{2,t}} K_{t+1} = (1 - ge^{x_{3,t}}) Y_t + (1 - \delta(H_t)) e^{-x_{2,t}} K_t - C_t \]  \hspace{1cm} (101)
The production function has constant returns to scale, namely it is,

\[ Y_t = R_t K_t H_t + w_t L_t , \]  

(102)

in which \( R_t \) is the rental capital cost and \( w_t \) is the wage rate. Plugging (102) into (101) gives,

\[ e^{-z_{2,t}} K_{t+1} = [1 + (1 - ge^{z_{3,t}}) e^{z_{2,t}} R_t H_t - \delta(H_t)] e^{-z_{2,t}} K_t + (1 - ge^{z_{3,t}}) w_t L_t - C_t . \]  

(103)

The setup of the problem and analytical results for steady-state values

Throughout, lower-case letters denote individual variables and upper-case letters denote aggregate variables (the notation of prices is an exception). Households are endowed with capital, \( k \), which they lease to firms. Consequently, households, as capital owners, are in charge of capital maintenance. Since the depreciation rate depends on the utilization of capital goods, households suggest and thus control the capacity utilization level, \( h \). Firms, as a counterpart of the leasing contract, need to accept the utilization rates. So, the representative firm maximizing its profits decides on the amounts of capital and labor supply it wants to hire, and on the rate of capital utilization rate it accepts. The (representative) firm’s problem is,

\[
\max_{K_t, L_t, H_t} \left[ \alpha(K_t H_t)^{1 - \frac{1}{v}} + (1 - \alpha) L_t^{1 - \frac{1}{v}} \right]^{\frac{v}{v-1}} - R_t K_t H_t - w_t L_t
\]

(it is important to mention that since the production function is linearly homogeneous and there are no externalities in production, we use aggregate labor supply and capital — yet, in general, we should first look at a small firm and then aggregate the supply and demand for labor and capital, but we omit this simple step). Maximizing the production function gives the rental capital cost, \( R_t \),

\[ R_t = \alpha z_{1,t}^{1-\frac{1}{v}} \left( \frac{Y_t}{K_t H_t} \right)^{\frac{1}{v}} \Rightarrow R_t = \alpha z_{1,t} \left[ \alpha + (1 - \alpha) \left( \frac{K_t H_t}{L_t} \right)^{\frac{1}{v}-1} \right]^{\frac{1}{v-1}} , \]  

(104)

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and the wage rate, \( w_t \),
\[
 w_t = (1 - \alpha) z_{1,t}^{1 - \frac{1}{\theta}} \left( \frac{Y_t}{L_t} \right)^{\frac{1}{\theta}} \implies w_t = (1 - \alpha) z_{1,t} \left[ \alpha \left( \frac{K_t H_t}{L_t} \right)^{1 - \frac{1}{\theta}} + 1 - \alpha \right]^{\frac{1}{\theta}}. 
\] (105)

The utility function of a household \( i \) is given by,
\[
 u(c_{i,t}, 1 - l_{i,t}) = \left[ \theta c_{i,t}^{1 - \frac{1}{\theta}} + (1 - \theta) (1 - l_{i,t})^{1 - \frac{1}{\theta}} \right]^{\frac{x}{x-1}} - 1 
\] (106)
\[
 \bar{\tau}_t \equiv (1 - \tau_t) \left( R_t H_{t,t} - e^{-x_{2,t} \delta(H_{i,t})} \right), \quad t = 0, 1, \ldots, 
\] (107)
\[
 \bar{w}_t \equiv (1 - \tau_t) w_t, \quad t = 0, 1, \ldots. 
\] (108)

The Lagrangian of an individual price-taking household’s problem is,
\[
 L = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_{i,t}, 1 - l_{i,t}) + \sum_{t=0}^{\infty} \lambda_t \left[ (e^{-x_{2,t}} + \bar{\tau}_t) a_{i,t} + \bar{w}_t \omega_i l_{i,t} + T_t - c_{i,t} - e^{-x_{2,t}} a_{i,t+1} \right] \right\}. 
\]

The individual households are also policy-rule takers despite that they express their preferences over fiscal policy.

The first-order conditions are,
\[
 \frac{\partial L}{\partial c_{i,t}} = 0 \Rightarrow \beta^t u_1(c_{i,t}, 1 - l_{i,t}) = \lambda_t \Rightarrow \beta^t \left[ \frac{x}{x-1} \right]^{1 - \frac{1}{\theta}} \theta c_{i,t}^{\frac{1}{\theta}} = \lambda_t 
\] (109)
\[
 \frac{\partial L}{\partial l_{i,t}} = 0 \Rightarrow \beta^t u_2(c_{i,t}, 1 - l_{i,t}) = \lambda_t \bar{w}_t \omega_i \Rightarrow \beta^t \left[ \frac{x}{x-1} \right]^{1 - \frac{1}{\theta}} (1 - \theta) (1 - l_{i,t})^{-\frac{1}{\theta}} = \lambda_t \bar{w}_t \omega_i 
\] (110)
\[
 \frac{\partial L}{\partial h_{i,t}} = 0 \Rightarrow \frac{\partial \bar{\tau}_t}{\partial h_{i,t}} = 0 \Rightarrow R_t = e^{-x_{2,t} \delta'(h_{i,t})} 
\] (111)
\[
 \frac{\partial L}{\partial a_{i,t+1}} = 0 \Rightarrow e^{-x_{2,t}} \lambda_t = E_t \left[ \lambda_{t+1} \left( e^{-x_{2,t+1}} + \bar{\tau}_{t+1} \right) \right] 
\] (112)
\[
\frac{\partial L}{\partial \lambda_t} = 0 \Rightarrow e^{-z_{i,t} \lambda} a_{i,t+1} = (e^{-z_{i,t} \lambda} + r_t) a_{i,t} + \alpha_i \omega_i l_t + T_t - c_{i,t}
\]

for all \( t \in \{0, 1, \ldots\} \), together with the initial condition \( k_0 > 0 \) and the transversality condition. From (109) and (110) we obtain,

\[
\frac{1 - \theta}{\theta} \left( \frac{c_{i,t}}{1 - l_{i,t}} \right)^{\frac{1}{\eta}} = \bar{w}_i \omega_i
\]

(114)

**Steady-state calculations**

Due to that \( u(c_{i,t}^{ss}, 1 - l_{i,t}^{ss}) = u(c_{i,t+1}^{ss}, 1 - l_{i,t+1}^{ss}) \) in the deterministic steady state, conditions (109) and (112) imply,

\[
\frac{1 - \beta}{\beta} = \tau^{ss},
\]

while (19) gives,

\[
\frac{1 - \beta}{\beta (1 - \tau^{ss})} = R^{ss} H^{ss} - \delta (H^{ss}),
\]

(115)

and (111) implies,

\[
\frac{1 - \beta}{\beta (1 - \tau^{ss})} = \delta' (H^{ss}) H^{ss} - \delta (H^{ss}).
\]

(116)

Notice that capacity utilization is expressed in aggregate form as all individuals, irrespective of skills and/or initial wealth holdings, agree upon the utilization rate given by (116). The depreciation rate,

\[
\delta (h) = \delta_c + \frac{b_h}{1 + \xi} h^{1+\xi}
\]

(117)

combined with equation (116) gives,

\[
H^{ss} = \left[ \frac{1 - \beta}{\beta (1 - \tau^{ss})} + \delta_c \right]^{\frac{1}{1+\xi}} \left( 1 - \frac{1}{1+\xi} b_h \right)
\]

(118)

Plugging (118) into the depreciation rate (117) gives,

\[
\delta (H^{ss}) = \delta_c + \frac{b_h}{1 + \xi} (H^{ss})^{1+\xi} = \delta_c \left( 1 + \frac{1}{\xi} \right) + \frac{1 - \beta}{\beta \xi (1 - \tau^{ss})}.
\]

(119)
Equation (107) combined with (104) implies,

\[ \frac{Y^{ss}}{K^{ss}H^{ss}} = \left[ \frac{\alpha H^{ss}}{\beta(1 - \gamma^{ss}) + \delta(H^{ss})} \right]^{-\nu} \Rightarrow \]

\[ \Rightarrow \left[ \alpha + (1 - \alpha) \left( \frac{K^{ss}H^{ss}}{L^{ss}} \right)^{\frac{1}{\gamma^{ss}}} \right]^{-\nu} = \left[ \frac{\alpha H^{ss}}{\beta(1 - \gamma^{ss}) + \delta(H^{ss})} \right]^{-\nu} \Rightarrow \]

\[ \Rightarrow A^{ss} \equiv \frac{K^{ss}H^{ss}}{L^{ss}} = \left\{ \frac{\left[ \frac{\alpha H^{ss}}{\beta(1 - \gamma^{ss}) + \delta(H^{ss})} \right]^{1-\nu} - \alpha}{1 - \alpha} \right\}^{\frac{1}{1-\nu}} , \tag{121} \]

on the condition that \( E(z_1) = E(z_2) = E(z_3) = 1 \). We use quadratic approximation to solve the problem. The certainty equivalence property allows us to consider the steady deterministic state of capital, taking \( E(z_1) = E(z_2) = E(z_3) = 1 \) as the long-run value of the shocks.

Household’s optimality conditions are:

\[ a_{i,t+1} = (1 + \bar{r}_t) a_{i,t} + \bar{w}_i \omega_i L_{i,t} + T_t - c_{i,t} , \tag{122} \]

\[ \frac{1 - \theta}{\theta} \left( \frac{c_{i,t}}{1 - l_{i,t}} \right)^{\frac{1}{\xi}} = \bar{w}_i \omega_i . \tag{123} \]

We have pinned down the aggregate steady state capital-labor ratio in (121). Next, we aspire to calibrate the relative-wealth and vector \([\omega_r, \omega_m, \omega_p]^T\). The equation (122) in the steady state becomes,

\[ c_i^{ss} = r^{ss} a_i^{ss} + \bar{w}^{ss} \omega_i l_i^{ss} + T^{ss} . \tag{124} \]

Furthermore,

\[ (123) \Rightarrow c_i^{ss} = \left( \frac{\theta}{1 - \theta} \right)^X (\bar{w}^{ss} \omega_i)^X (1 - l_i^{ss}) \tag{124} \]

\[ \Rightarrow r^{ss} a_i^{ss} + \bar{w}^{ss} \omega_i l_i^{ss} + T^{ss} = \left( \frac{\theta}{1 - \theta} \right)^X (\bar{w}^{ss} \omega_i)^X (1 - l_i^{ss}) \Rightarrow \]

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\[ \Rightarrow \bar{w}^{ss} \omega_i l_i^{ss} \left[ 1 + \left( \frac{\theta}{1 - \theta} \right)^X (\bar{w}^{ss} \omega_i)^{X - 1} \right] = \left( \frac{\theta}{1 - \theta} \right)^X (\bar{w}^{ss} \omega_i)^X - \bar{r}^{ss} a_i^{ss} - T^{ss}. \] \quad (125)

We solve equation (125) for \( l_i^{ss} \):

\[ l_i^{ss} = \frac{\left( \frac{\theta}{1 - \theta} \right)^X (\bar{w}^{ss} \omega_i)^X}{\bar{w}^{ss} \omega_i + \left( \frac{\theta}{1 - \theta} \right)^X (\bar{w}^{ss} \omega_i)^X} - \frac{\bar{r}^{ss} a_i^{ss} + T^{ss}}{\bar{w}^{ss} \omega_i + \left( \frac{\theta}{1 - \theta} \right)^X (\bar{w}^{ss} \omega_i)^X}. \] \quad (126)

If we knew \([\omega_r \, \omega_m \, \omega_p]^T\) and \(K^{ss}\), we could obtain steady state values of \(l_i^{ss}\) for \(i \in \{r, m, p\}\) from equation (126). Yet, we only know Table 2 in Krusell and Rios-Rull (1999) with distributional statistics for the U.S. We employ vectors \([A_{r}^{\text{relative}} \, A_{m}^{\text{relative}} \, A_{p}^{\text{relative}}]^T\) and \([Y_{L,r}^{\text{relative}} \, Y_{L,m}^{\text{relative}} \, Y_{L,p}^{\text{relative}}]^T\) from Table 2 in Krusell and Rios-Rull (1999), to calculate the normalized distribution values:

\[
\begin{bmatrix}
A_{r}^{\text{norm}} \\
A_{m}^{\text{norm}} \\
A_{p}^{\text{norm}}
\end{bmatrix}
= \begin{bmatrix}
A_{r}^{\text{relative}} \\
A_{m}^{\text{relative}} \\
A_{p}^{\text{relative}}
\end{bmatrix}
\cdot \frac{1}{[\mu_r \, \mu_m \, \mu_p]}.
\]

Obviously it is,

\[
[\mu_r \, \mu_m \, \mu_p] \cdot \begin{bmatrix}
A_{r}^{\text{norm}} \\
A_{m}^{\text{norm}} \\
A_{p}^{\text{norm}}
\end{bmatrix} = 1,
\]

and

\[
\begin{bmatrix}
A_{r}^{\text{norm}} \\
A_{m}^{\text{norm}} \\
A_{p}^{\text{norm}}
\end{bmatrix}
\cdot K^{ss} = \begin{bmatrix}
A_{r}^{ss} \\
A_{m}^{ss} \\
A_{p}^{ss}
\end{bmatrix}. \quad (127)
\]

133
In the same way,

\[
\begin{bmatrix}
Y_{L,r}^{\text{norm}} \\
Y_{L,m}^{\text{norm}} \\
Y_{L,p}^{\text{norm}}
\end{bmatrix}
= 
\begin{bmatrix}
Y_{L,r}^{\text{relative}} \\
Y_{L,m}^{\text{relative}} \\
Y_{L,p}^{\text{relative}}
\end{bmatrix}
\cdot 
\frac{1}{\begin{bmatrix}
\mu_r \\
\mu_m \\
\mu_p
\end{bmatrix}}
\cdot 
\begin{bmatrix}
Y_{L,r}^{\text{relative}} \\
Y_{L,m}^{\text{relative}} \\
Y_{L,p}^{\text{relative}}
\end{bmatrix}.
\]

Notice that \(Y_{L,i}^{\text{relative}}\) hinges on both \(\omega_i\) and \(l_i^{ss}\) and \(l_i^{ss}\) is a function of \(\omega_i\). We use equation (125) to numerically calculate \([\omega_r \omega_m \omega_p]\). In addition, \(l_i^{ss}\) also hinges on \(A_t^{ss}\), so we must co-determine the steady state level of aggregate capital, \(K^{ss}\). Here we have,

\[
\begin{bmatrix}
Y_{L,r}^{\text{norm}} \\
Y_{L,m}^{\text{norm}} \\
Y_{L,p}^{\text{norm}}
\end{bmatrix}
= 
\begin{bmatrix}
\omega_r L_r^{ss} \\
\omega_m L_m^{ss} \\
\omega_p L_p^{ss}
\end{bmatrix}
\cdot 
\frac{1}{L^{ss}}.
\]

(128)

From (126) and (99) together with (102) we get,

\[
\omega_i l_i^{ss} = \frac{\left(\frac{\theta}{1-\theta}\right)^X \left(\bar{w}^{ss}\right)^X \omega_i^{X+1}}{\bar{w}^{ss} \omega_i + \left(\frac{\theta}{1-\theta}\right)^X \left(\bar{w}^{ss} \omega_i\right)^X} \omega_i l_i^{ss} A_t^{norm} + \omega_i \left[(\tau^{ss} - g) \left(R_t^{ss} H^{ss} + \frac{w^{ss}}{L^{ss}} H^{ss}\right) - \delta (H^{ss}) \tau^{ss}\right] K^{ss}
\]

(129)

and

\[
L^{ss} = \sum_i \mu_i \omega_i l_i^{ss} = \sum_i \frac{\left(\frac{\theta}{1-\theta}\right)^X \left(\bar{w}^{ss}\right)^X \omega_i^{X+1}}{\bar{w}^{ss} \omega_i + \left(\frac{\theta}{1-\theta}\right)^X \left(\bar{w}^{ss} \omega_i\right)^X} - K^{ss} \sum_i \mu_i \omega_i l_i^{ss} A_t^{norm} + \omega_i \left[(\tau^{ss} - g) \left(R_t^{ss} H^{ss} + \frac{w^{ss}}{L^{ss}} H^{ss}\right) - \delta (H^{ss}) \tau^{ss}\right].
\]

(130)

We need to keep in mind that

\[
\sum_i \mu_i = 1
\]

(131)

134
\[
\sum_i \mu_i \omega_i = 1 \quad (132)
\]

With the aim of obtaining vector \([\omega_r^s, \omega_m^s, \omega_p^s]^T\) that complies with both \([Y_{L,r}^{\text{norm}}, Y_{L,m}^{\text{norm}}, Y_{L,p}^{\text{norm}}]^T\) and \([A_{r}^{\text{norm}}, A_{m}^{\text{norm}}, A_{p}^{\text{norm}}]^T\) we follow the algorithm:

**Step 1** We make a guess for the vector \([\omega_r^{(0)}, \omega_m^{(0)}, \omega_p^{(0)}]^T\). In the \(n\)-th step the guess becomes \([\omega_r^{(n)}, \omega_m^{(n)}, \omega_p^{(n)}]^T\).

**Step 2** From equation (130) we get,

\[
1 = \frac{1}{L^{ss}} \sum_i \mu_i \left( \frac{\theta}{1-\theta} \right)^x \left( \bar{w}^{ss} \omega_i \right)^x \omega_i^{x+1} - \frac{K^{ss}}{L^{ss}} \sum_i \mu_i \frac{\tau^{ss} A_{m}^{\text{norm}} + \omega_i^x}{\bar{w}^{ss} \omega_i + \left( \frac{\theta}{1-\theta} \right)^x \left( \bar{w}^{ss} \omega_i \right)^x} \left( \tau^{ss} - g \right) \left( R^{ss} H^{ss} + \frac{w^{ss}}{K^{ss}} H^{ss} \right) - \delta \left( H^{ss} \right)^{\tau^{ss}} \quad \text{(133)}
\]

and we insert \(K^{ss} \cdot H^{ss}/L^{ss} \equiv \Lambda^{ss}\) into (133) and solve for \(L^{ss}\),

\[
L^{ss(n)} = \frac{\sum_i \mu_i \left( \frac{\theta}{1-\theta} \right)^x \left( \bar{w}^{ss} \omega_i \right)^x \omega_i^{(n)x+1}}{1 + \frac{\Lambda^{ss}}{L^{ss}} \sum_i \mu_i \frac{\omega_i^{(n)ss} A_{m}^{\text{norm}} + \omega_i^{(n)x}}{\bar{w}^{ss} \omega_i^{(n)} + \left( \frac{\theta}{1-\theta} \right)^x \left( \bar{w}^{ss} \omega_i^{(n)} \right)^x} \left( \tau^{ss} - g \right) \left( R^{ss} H^{ss} + \frac{w^{ss}}{K^{ss}} H^{ss} \right) - \delta \left( H^{ss} \right)^{\tau^{ss}}} \quad \text{(144)}
\]

Once we have obtained \(L^{ss(n)}\) we calculate \(K^{ss(n)} = \Lambda^{ss} \cdot L^{ss(n)}/H^{ss}\). And using equation (126) we calculate the levels of individual labor hours:

\[
l_i^{ss(n)} = \frac{\left( \frac{\theta}{1-\theta} \right)^x \left( \bar{w}^{ss} \omega_i \right)^x \omega_i^{(n)}}{\bar{w}^{ss} \omega_i^{(n)} + \left( \frac{\theta}{1-\theta} \right)^x \left( \bar{w}^{ss} \omega_i^{(n)} \right)^x} - \frac{\bar{w}^{ss} A_{m}^{\text{norm}} + \left( \tau^{ss} - g \right) \left( R^{ss} H^{ss} + \frac{w^{ss}}{K^{ss}} H^{ss} \right) - \delta \left( H^{ss} \right)^{\tau^{ss}}}{\bar{w}^{ss} \omega_i^{(n)} + \left( \frac{\theta}{1-\theta} \right)^x \left( \bar{w}^{ss} \omega_i^{(n)} \right)^x} K^{ss(n)} \quad \text{(135)}
\]

135
For the update of vector \( \begin{bmatrix} \omega_r^{(n)} & \omega_m^{(n)} & \omega_p^{(n)} \end{bmatrix}^T \) we need to perform an intermediate step. For all \( i \in \{r, m, p\} \) we know the following,

\[
\omega_i^n \cdot l_i^{ss*} = Y_{Li}^{norm} \cdot L^{ss*}.
\]

(136)

So, using (135) and (134) we calculate,

\[
\tilde{\omega}_i^{(n+1)} = \frac{Y_{Li}^{norm} \cdot L^{ss(n)}}{l_i^{ss(n)}},
\]

(137)

for all \( i \in \{r, m, p\} \).

**Step 3** if the difference

\[
\left\| \begin{bmatrix} \tilde{\omega}_r^{(n+1)} \\ \tilde{\omega}_m^{(n+1)} \\ \tilde{\omega}_p^{(n+1)} \end{bmatrix} - \begin{bmatrix} \omega_r^{(n)} \\ \omega_m^{(n)} \\ \omega_p^{(n)} \end{bmatrix} \right\| < \varepsilon_\omega,
\]

(138)

where \( \varepsilon_\omega > 0 \) is an arbitrary chosen tolerance criterion, then the \( n \)-th omega vector

\[
\begin{bmatrix} \omega_r^{(n)} \\ \omega_m^{(n)} \\ \omega_p^{(n)} \end{bmatrix}
\]

is the solution to the loop, on the condition that

\[
\begin{bmatrix} \mu_r & \mu_m & \mu_p \end{bmatrix} \cdot \begin{bmatrix} \omega_r^{(n)} \\ \omega_m^{(n)} \\ \omega_p^{(n)} \end{bmatrix} = 1.
\]

(139)

If the above difference is greater than \( \varepsilon_\omega > 0 \), an overshooting correction occurs,

\[
\hat{\omega}_i^{(n+1)} = \lambda_\omega \omega_i^{(n)} + (1 - \lambda_\omega) \tilde{\omega}_i^{(n+1)},
\]

(140)

for a predetermined over-shooting parameter \( \lambda_\omega \in (0, 1) \). The updating correction process does not ensure that the vector \( \begin{bmatrix} \omega_r^{(n+1)} & \omega_m^{(n+1)} & \omega_p^{(n+1)} \end{bmatrix}^T \) is normalized, namely it does not
guarantee that \( \begin{bmatrix} \mu_r & \mu_m & \mu_p \end{bmatrix} \cdot \begin{bmatrix} \hat{\omega}_r^{(n+1)} & \hat{\omega}_m^{(n+1)} & \hat{\omega}_p^{(n+1)} \end{bmatrix}^T = 1 \). So, we normalize the updated vector,

\[
\begin{bmatrix}
\hat{\omega}_r^{(n+1)} \\
\hat{\omega}_m^{(n+1)} \\
\hat{\omega}_p^{(n+1)}
\end{bmatrix} = \frac{1}{\begin{bmatrix} \mu_r & \mu_m & \mu_p \end{bmatrix} \cdot \begin{bmatrix} \hat{\omega}_r^{(n+1)} \\
\hat{\omega}_m^{(n+1)} \\
\hat{\omega}_p^{(n+1)} \end{bmatrix}} \cdot \begin{bmatrix} \mu_r \\
\mu_m \\
\mu_p 
\end{bmatrix} \cdot 
\begin{bmatrix} \hat{\omega}_r^{(n+1)} \\
\hat{\omega}_m^{(n+1)} \\
\hat{\omega}_p^{(n+1)} \end{bmatrix} .
\] (141)

We return to Step 2 and proceed until convergence.

After having obtained values for \( K^{ss} \) and \( L^{ss} \) for any \( \tau^{ss} \) we continue with the algorithm. Yet, first, we make a comment about calibration. In order to guarantee a well-behaved model we set the consumption share of GDP, \( C^{ss}/Y^{ss} \), as our target.

\[
C^{ss} = (1 - g) \left( R^{ss} K^{ss} H^{ss} + w^{ss} L^{ss} \right) - \delta(H^{ss}) K^{ss} \Rightarrow
\]

\[
\Rightarrow \frac{C^{ss}}{Y^{ss}} = 1 - g - \frac{\delta(H^{ss}) K^{ss} H^{ss}}{H^{ss}} \overset{(120)}{\Rightarrow} \]

\[
\overset{(120)}{\Rightarrow} \frac{C^{ss}}{Y^{ss}} = 1 - g - \frac{\delta(H^{ss})}{H^{ss}} \left[ \frac{\alpha H^{ss}}{\beta(1 - \tau^{ss}) + \delta(H^{ss})} \right]^\nu
\] (142)

In a nutshell, the parameters \( \alpha, \beta, \nu, \delta \) and \( g \) are the only ones affecting the ratio \( C^{ss}/Y^{ss} \). Except for parameter \( g \), we are not likely to change any of these parameters in a sensitivity analysis.

Algorithm for computing the steady-state tax rate and the decision rules when the next period policy is of the form \( \tau_{t+1} = \Psi(X_t, A_t, \tau_t) \)

The current tax rate \( \tau_t \) is taken as given (so, \( \tau_t \) is assumed to be one of the state variables), while the next period’s tax rate is determined by the politico-economic process, namely by a policy rule, which is specified to be of the form \( \tau_{t+1} = \Psi(X_t, A_t, \tau_t) \).
Step 1

This outer loop begins with an initial guess for $\tau^{ss}$, which we denote as $\tau^{ss,(0)}$. The iteration index is given by $n_r$, so we are using the symbol $\tau^{ss,(n_r)}$ for an updated steady-state tax rate. We use this guess to calculate $K^{ss}$, which depends on $\tau^{ss}$ (see equations (121) and (134)). Then we calculate vectors around which we will construct the quadratic form of the (momentary) utility function for each problem. Specifically, we use $K^{ss}$ together with the vector $[A_r^{\text{norm}} \quad A_m^{\text{norm}} \quad A_p^{\text{norm}}]^T$ and equation (127) to calculate the vector of steady-state wealth levels $A^{ss} = [A_r^{\text{norm}} \quad A_m^{\text{norm}} \quad A_p^{\text{norm}}]^T \cdot K^{ss}$, which hinge on $\tau^{ss,(n_r)}$. Accordingly, we calculate $L^{ss} = [L_r^{ss} \quad L_m^{ss} \quad L_p^{ss}]^T$ using equation (135) and the above calculated vector $[\omega_r \quad \omega_m \quad \omega_p]^T$. For notational ease, we define the vectors as follows,

$$
A \equiv \begin{bmatrix} A_r \\ A_m \\ A_p \end{bmatrix},
$$

$$
H \equiv \begin{bmatrix} H_r \\ H_m \\ H_p \end{bmatrix},
$$

$$
L \equiv \begin{bmatrix} L_r \\ L_m \\ L_p \end{bmatrix},
$$

and

$$
X \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.
$$
Furthermore, we define six matrices that we will use to aggregate decision rules in several steps below. Let $j$ be,

$$
    j \equiv \begin{cases} 
    1 , & \text{if } i = r \\
    2 , & \text{if } i = m \\
    3 , & \text{if } i = p 
    \end{cases}
$$

We define a $9 \times 8$ matrix $M_i$ for all $i \in \{r, m, p\}$,

$$
    M_{i,[1:7,1:7]} = I_7
$$

in which $I_7$ is a $7 \times 7$ identity matrix,

$$
    M_{i,[9,8]} = 1 \quad \text{while the remaining elements of the row 9 are 0 ,}
$$

and

$$
    M_{i,[8,4\rightarrow j]} = 1 \quad \text{while the remaining elements of the row 8 are 0 .}
$$

The reason why we formulate the matrix $M_i$, $i \in \{r, m, p\}$ in this way will become clear below.

In addition, we need to formulate a second matrix to facilitate aggregation of the intermediate equilibrium decision rules. We define a $10 \times 9$ matrix $\hat{M}_i$ in which,

$$
    \hat{M}_{i,[1:7,1:7]} = I_7 ,
$$

$$
    \hat{M}_{i,[9:10,8:9]} = I_2 \quad \text{while the remaining elements of rows 9 and 10 are 0 ,}
$$

where $I_2$ is a $2 \times 2$ identity matrix, and

$$
    \hat{M}_{i,[8,4\rightarrow j]} = 1 \quad \text{while the remaining elements of row of 8 are 0 .}
$$
Step 2

We begin a second loop with an initial guess for the policy rule \( \Psi \). We denote the initial guess as \( \Psi^{(0)} \), while the iteration index is \( n_\psi \). So, the symbol \( \Psi^{(n_\psi)} \) stands for the policy rule update. The form of the policy rule \( \Psi^{(n_\psi)} \) is given by,

\[
\tau_{t+1} = \tilde{\Psi}^{(n_\psi)}, \quad \begin{bmatrix}
1 \\
X_t \\
A_t \\
\tau_t
\end{bmatrix} = \begin{bmatrix}
\psi^{(n_\psi)}_c \\
\psi^{(n_\psi)}_X \\
\psi^{(n_\psi)}_A \\
\psi^{(n_\psi)}_r
\end{bmatrix} \cdot \begin{bmatrix}
1 \\
X_t \\
A_t \\
\tau_t
\end{bmatrix}, \quad (143)
\]

with

\[
x_{1,t} = \ln(z_{1,t}),
\]

\[
x_{2,t} = \ln(z_{2,t}),
\]

\[
x_{3,t} = \ln(z_{3,t}),
\]

Notice that the vectors are defined in the following way,

\[
\psi^{(n_\psi)}_X \equiv \begin{bmatrix}
\psi^{(n_\psi)}_{z_1} \\
\psi^{(n_\psi)}_{z_2} \\
\psi^{(n_\psi)}_{z_3}
\end{bmatrix}
\]

and,

\[
\psi^{(n_\psi)}_A \equiv \begin{bmatrix}
\psi^{(n_\psi)}_{A_r} \\
\psi^{(n_\psi)}_{A_m} \\
\psi^{(n_\psi)}_{A_p}
\end{bmatrix}
\]

Step 3

In this step we investigate how the economy reacts to the guessed policy rule and to a one-period deviation in the committed tax rate. We divide this step into two substeps. First, we calculate the competitive equilibrium (CE) decision rules and, in particular, the value function of the representative household subject to \( \Psi^{(n_\psi)} \). Second, we calculate the “intermediate equilibrium” (IE) decision rules and, in particular, the value function of the
representative household subject to $\Psi^{(n_\psi)}$ and a potential deviation in the next period’s tax rate, $\tau_{t+1}$.

**Substep 3a**

In this substep we calculate the competitive equilibrium of the model economy subject to $\Psi^{(n_\psi)}$.

A guess for the law of motion of aggregate labor, capital and capacity utilization is given by,

\[
\begin{bmatrix}
L_t \\
A_{t+1} \\
H_t
\end{bmatrix} = \mathcal{A}^{(n_\kappa)} \left( X_t, A_t, \tau_t \mid \Psi^{(n_\psi)} \right) = \mathcal{R}^{(n_\kappa)}_{|\Psi^{(n_\psi)}} \cdot \begin{bmatrix}
1 \\
X_t \\
A_t \\
\tau_t
\end{bmatrix},
\]

(144)

in which $\mathcal{R}^{(n_\kappa)}_{|\Psi^{(n_\psi)}}$ is a $9 \times 8$ matrix (here we use an iteration index $n_\kappa$). An individual household’s Bellman equation is,

\[
V^i \left( X, A, a_i, \tau \mid \Psi^{(n_\psi)} \right) = \max_{a_i' \in [a_i, a_i], l_i \in [0, 1], \kappa_i \geq 0} \left\{ u \left( e^{-z_2} + \bar{r} \right) a_i + \bar{\omega} \omega_i l_i + T - e^{-z_2} a_i' \right\} + \beta V^i \left( X', A', a_i', \tau' \mid \Psi^{(n_\psi)} \right)
\]

(145)

subject to,

\[
\begin{bmatrix}
L' \\
A' \\
H
\end{bmatrix} = \mathcal{A}^{(n_\kappa)} \left( X, A, \tau \mid \Psi^{(n_\psi)} \right),
\]

(146)

\[
\tau' = \Psi^{(n_\psi)} \left( X, A, \tau \right),
\]

(147)

whereas $\bar{r}$ and $\bar{\omega}$ we have defined above.

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Subsubstep 3a(i) The fixed point of the above Bellman equation is calculated using quadratic approximation. We make a guess, $Q_{\psi}^{(0)}(n_{\psi})$, on the value function’s quadratic form and use the right-hand-side (RHS) of the above Bellman equation as a mapping. A good guess on the quadratic form is a negative definite matrix, namely $Q_{\psi}^{(0)}(n_{\psi}) = -I_9$, in which $I_9$ is a $9 \times 9$ identity matrix. This guess enables the existence of a global maximum. In brief, the Bellman equation acts as a contraction mapping, which, for the $n_{CE}$-th step of the updating process, can be written as,

$$s_i^T Q_{\psi}^{(n_{CE}-1)} s_i = \max_{i, a_i, h_i} \left\{ r_i^T Q_{\psi}^{(n_{CE})} r_i + \beta (s'_i)^T Q_{\psi}^{(n_{CE})} s'_i \right\},$$  \hspace{1cm} (148)$$

subject to the laws of motion (143) and (144), with $s_i^T \equiv [1 \ X \ A \ a_i \ \tau]$ and

$$r_i^T \equiv [1 \ X \ A \ a_i \ \tau \ l_i \ a'_i \ h_i \ L \ H].$$

Next, we incorporate the constraints (143) and (144) into the right-hand-side of (148).
For this reason we must express the RHS of equation (148) in a matrix form,

\[ r^T Q_u r + \beta (s')^T Q^{(ncE)}_{V_i | \phi} s' = \begin{bmatrix} 1 \\ X \\ A \\ a_i \\ \tau \\ l_i \\ a_i' \\ h_i \\ L \\ H \\ 1 \\ X' \\ A' \\ a_i' \\ \tau' \end{bmatrix}^T \begin{bmatrix} 1 \\ X \\ A \\ a_i \\ \tau \\ l_i \\ a_i' \\ h_i \\ L \\ H \\ 1 \\ X' \\ A' \\ a_i' \\ \tau' \end{bmatrix}, \]

and specify a $27 \times 12$ matrix $P$, 

143
$$P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
R^{(n_e)}_{\psi^{(n_e)}_{[4, 3, 1]}} & R^{(n_e)}_{\psi^{(n_e)}_{[4, 4, 2, 2]}} & R^{(n_e)}_{\psi^{(n_e)}_{[4, 4, 5, 7]}} & 0 & R^{(n_e)}_{\psi^{(n_e)}_{[4, 6, 8]}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
and build a 12 × 12 matrix

$$W_i^{(nCE)} = P^T \cdot \begin{bmatrix} Q_i^1 & 0 \\ 0 & \beta Q_{V|\psi}^{(nCE)} \end{bmatrix} \cdot P.$$  

As a result, the right-hand-side of the Bellman equation has become an unconstrained maximization problem, namely,

$$\max_{l_i, a_i', h_i, \tau} \begin{bmatrix} 1 \\ X \\ A \\ a_i \\ \tau \\ l_i \\ a_i' \\ h_i \end{bmatrix}^T \cdot W_i^{(nCE)} \cdot \begin{bmatrix} 1 \\ X \\ A \\ a_i \\ \tau \\ l_i \\ a_i' \\ h_i \end{bmatrix}.$$  

The first-order conditions are summarized by a 3 × 9 matrix

$$B_i^{(nCE)} = - \left( W_i^{(nCE)} \right)_{i,[10:12,10:12]}^{-1} W_i^{(nCE)}_{i,[10:12,1:9]},$$

while the update of the value function’s quadratic form is,

$$Q_{V|\psi}^{(nCE+1)} = P_{i,1}^T \cdot W_i^{(nCE)} \cdot P_{i,1},$$

in which

$$P_{i,1} = \begin{bmatrix} I_9 \\ B_i^{(nCE)} \end{bmatrix}.$$  

We continue until we have found the fixed point, $Q^*_{V|\psi}(\nu_v)$, and the individual decision rules, which are expressed by $B_i^*$.  

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**Substep 3a(ii)** The above loop for finding the fixed point of the value function is built within a loop for the fixed point of the aggregate law of motion, $A^{(n_e)}$. The update for $A^{(n_e+1)}$ is,

$$\bar{R}^{(n_e+1)}_{\Psi(n_e)} = \lambda_e R^{(n_e)}_{\Psi(n_e)} + (1 - \lambda_e) K^{(n_e)}_{\Psi(n_e)},$$

in which $\lambda_e$ acts as an overshooting parameter, and $K^{(n_e)}_{\Psi(n_e)}$ is a $9 \times 8$ matrix, which we define below.

Let us remind that,

$$j \equiv \begin{cases} 1, & \text{if } i = r \\ 2, & \text{if } i = m \\ 3, & \text{if } i = p \end{cases}$$

and for all $i \in \{r, m, p\},$

$$K^{(n_e)}_{\Psi(n_e),[j,i]} = B^*_{i,[1,j]} \cdot M_i$$

$$K^{(n_e)}_{\Psi(n_e),[5+j,i]} = B^*_{i,[2,j]} \cdot M_i$$

$$K^{(n_e)}_{\Psi(n_e),[6+j,i]} = B^*_{i,[3,j]} \cdot M_i$$

in which $M_i$ is a matrix which we have defined in **Step 1** and $B^*_i$ has been defined in **Substep 3a(i)**. To be precise, $[L \ A' \ H]^T = K^{(n_e)}_{\Psi(n_e)} \cdot [1 \ X \ A \ \tau]^T$ is obtained through aggregating individual decision rules, $[l_i \ a'_i \ h_i]^T = B^* \cdot [1 \ X \ A \ a_i \ \tau]^T$, and thus considering market-clearing conditions, namely $L_i = l_i$, $A_i = a_i$ and $H_i = h_i$ for all $i \in \{r, m, p\}$. If the difference between $K^{(n_e)}_{\Psi(n_e)}$ and $\bar{R}^{(n_e)}_{\Psi(n_e)}$ is not arbitrarily small, we return to **Substep 3a(i)** and proceed until convergence resulting in $R^*_{\Psi(n_e)}$.

**Substep 3b**

In this substep we calculate the *intermediate equilibrium (IE)* of the economy subject to $\Psi(n_e)$. We aspire to investigate how the economy reacts to a single deviation from the
committed tax rate. There is only one loop in this step. So, the following equation is a non-Bellman equation:

\[
\hat{V}^i \left( \mathbf{X}, \mathbf{A}, a_i, \tau, \tau' \mid \psi^{(n_\psi)} \right) = \max_{a_i' \in [a_{iL}, a_{iU}], l_i \in [0, 1], \delta_t} \left\{ u_i ((1 + \bar{r}) a_i + \bar{\omega} \omega_i l_i + T - a_i'), 1 - l_i \right\} + \beta V^{**} \left( \mathbf{X}', \mathbf{A}', a_i', \tau' \mid \psi^{(n_\psi)} \right)
\]

subject to,

\[
\begin{bmatrix}
\mathbf{L} \\
\mathbf{A}' \\
\mathbf{H}
\end{bmatrix} = \hat{\mathcal{A}}^{(n_{1E})}_{\psi^{(n_\psi)}} \left( \mathbf{X}, \mathbf{A}, \tau, \tau' \mid \psi^{(n_\psi)} \right) = \hat{\mathcal{R}}^{(n_{1E})}_{\psi^{(n_\psi)}} \cdot
\begin{bmatrix}
1 \\
\mathbf{X} \\
\tau \\
\tau'
\end{bmatrix},
\]

in which \(\hat{\mathcal{A}}^{(n_{1E})}_{\psi^{(n_\psi)}}\) is a new decision rule, which takes into account a change in the next period’s tax rate, \(\tau'\) (notice that \(\hat{\mathcal{R}}^{(n_{1E})}_{\psi^{(n_\psi)}}\) is a \(9 \times 9\) matrix). The fixed point for \(V^{**}\) complies with \(\hat{\mathcal{R}}^{*}_{\psi^{(n_\psi)}}\) and individual decisions. The quadratic form for \(V^{**}\) is \(Q^{**}_{\psi^{(n_\psi)}}\) while the quadratic approximation of the non-Bellman equation can be written as,

\[
\hat{s}^T_i Q^{(n_{1E}+1)}_{\psi^{(n_\psi)}} \hat{s}_i = \max_{k'} \left\{ r_i^T Q_i \mathbf{r}_i + \beta (s_i')^T Q^{**}_{\psi^{(n_\psi)}} s_i' \right\},
\]

subject to constraint (149) only, with \(s_i^T \equiv [1 \ X \ A \ a_i \ \tau], \hat{s}_i^T \equiv [1 \ X \ A \ a_i \ \tau \ \tau'],\) and \(r_i^T \equiv [1 \ X \ A \ a_i \ \tau \ \tau \ l_i \ a'_i \ \delta_t \ \mathbf{L} \ \mathbf{H}].\)

We incorporate the only constraint (149) into the right-hand-side of (150). For this
reason we express the RHS of equation (150) in a form of matrix,

\[ r_i^T Q_i^r r_i + \beta (s_i')^T Q_{V_i|\Phi(n_c)}^{**} s_i' = \\
\begin{bmatrix}
1 \\
X \\
A \\
a_i \\
\tau \\
l_i \\
ad_i' \\
h_i \\
L \\
H \\
1 \\
X' \\
A' \\
ad_i' \\
\tau'
\end{bmatrix}^T \\
\begin{bmatrix}
1 \\
X \\
A \\
a_i \\
\tau \\
l_i \\
ad_i' \\
h_i \\
L \\
H \\
1 \\
X' \\
A' \\
ad_i' \\
\tau'
\end{bmatrix},
\]

then we construct a $27 \times 13$ matrix $\mathbf{P}$,
\[ \hat{P} = \]

\[
\begin{array}{cccccccccccc}
1 & X & A & a_i & \tau & \tau' & l_i & a'_i & h_i \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

and build a 13 × 13 matrix

\[ \hat{W}^{(n_{1E})} = \hat{P}^T \begin{bmatrix} Q^i_u & 0 \\ 0 & \beta Q^{**} \end{bmatrix} \hat{P} \]
As a result, the RHS of the non-Bellman equation has become an unconstrained maximization problem, namely,

$$\max_{\tau, a_i, h_i} \begin{bmatrix} 1 \\ X \\ A \\ \tau \\ \tau' \\ l_i \\ a_i' \\ h_i \end{bmatrix}^T \cdot \begin{bmatrix} 1 \\ X \\ A \\ \tau \\ \tau' \\ l_i \\ a_i' \\ h_i \end{bmatrix} \rightarrow \tilde{W}_i^{(n, r, E)}.$$  

The first-order conditions are comprised by a $3 \times 10$ matrix,

$$\hat{B}_i^{(n, r, E)} \equiv -\left( \tilde{W}_i^{(n, r, E)} \right)_{[1:13, 11:13]}^{-1} \tilde{W}_i^{(n, r, E)}_{[1:11, 13:10]}, \quad (151)$$

The update for the intermediate aggregate law of motion $A_{r, E}^{(n, r, E+1)}$ is,

$$\hat{K}_{r, E}^{(n, r, E+1)} |_{\Phi \left( \psi \left( \sigma \right) \right)} = \hat{K}_{r, E}^{(n, r, E)} |_{\Phi \left( \psi \left( \sigma \right) \right)} + (1 - \tilde{\lambda}) \hat{K}_{r, E}^{(n, r, E)} |_{\Phi \left( \psi \left( \sigma \right) \right)},$$

in which $\tilde{\lambda}_k$ acts as an overshooting parameter and $\hat{K}_{r, E}^{(n, r, E)} |_{\Phi \left( \psi \left( \sigma \right) \right)}$ is a $9 \times 9$ matrix.

Let us remind that,

$$j \equiv \begin{cases} 1 & \text{if } i = r \\ 2 & \text{if } i = m \\ 3 & \text{if } i = p \end{cases},$$

and for all $i \in \{r, m, p\},$

$$\hat{K}_{r, E}^{(n, r, E)} |_{\Phi \left( \psi \left( \sigma \right) \right), \left[ j, j \right]} = \hat{B}_i^{(n, r, E)} \cdot \tilde{M}_i,$$

$$\hat{K}_{r, E}^{(n, r, E)} |_{\Phi \left( \psi \left( \sigma \right) \right), \left[ 3 + j, j \right]} = \hat{B}_i^{(n, r, E)} \cdot \tilde{M}_i.$$
\[ \hat{\kappa}^{(n, \psi)}_{\chi} = \hat{\tilde{B}}^{(n, \psi)}_{t, [3, \lambda]} \cdot \hat{M} \]

in which \( \hat{M} \) is a matrix we have defined in Step 1, whereas \( \hat{\tilde{B}}^{(n, \psi)}_{t} \) has been defined in equation (151). To be precise, \([L \ A' \ H]^T = \kappa^{(n, \psi)}_{\chi} \cdot [1 \ X \ A \ \tau \ \tau']^T\) reflects the aggregate decision rules (notice that individual decisions are \([l_i, a_i', h_i]^T = B^* \cdot [1 \ X \ A \ a_i \ \tau \ \tau']^T\)), which we obtain using the market-clearing conditions, \(L_i = l_i, A_i = a_i, H_i = h_i\) for all \(i \in \{ r, m, p \}\). If the difference between \(\hat{\kappa}^{(n, \psi)}_{\chi} \) and \(\hat{\kappa}^{(n, \psi)}_{\chi} \) is not arbitrarily small, we return to Substep 3b and proceed until convergence resulting in \(\hat{\kappa}^{(n, \psi)}_{\chi} \).

Step 4

Once we have obtained \(\hat{\kappa}^{(n, \psi)}_{\chi} \) we use the individual decision rules \(\hat{\tilde{B}}^*_i \) for \(i \in \{ r, m, p \}\) that comply with \(\hat{\kappa}^{(n, \psi)}_{\chi} \),

\[ \hat{\tilde{B}}^*_i \equiv -\left(\hat{\tilde{W}}_{t, [11:13,11:13]}^*\right)^{-1} \hat{\tilde{W}}_{t, [11:13,1:10]}^* \text{, for all } i \in \{ r, m, p \}, \]

and quadratically approximate \(\hat{\tilde{V}}^t(X, A, a_i, \tau, \tau' | \psi^{(n, \psi)})\). To be precise, we compute the quadratic form \(Q^*_{\psi, [\psi^{(n, \psi)}]}\), where

\[ \hat{s}_{t, i}^T Q^*_{\psi, [\psi^{(n, \psi)}]} \hat{s}_{t, i} = \max_{l_i, a_i', a_i} \left\{ r_i^T Q_a r_i + \beta(s_{i, t}^T Q_{\star, [\psi^{(n, \psi)}]} s_{i, t}) \right\}, \quad (152) \]

subject to,

\[
\begin{bmatrix}
L \\
A' \\
H
\end{bmatrix}
= \hat{\kappa}^{(n, \psi)}_{\chi}
\begin{bmatrix}
1 \\
X \\
A \\
\tau \\
\tau'
\end{bmatrix}.
\]

This can be quickly done through,

\[ Q^*_{\psi, [\psi^{(n, \psi)}]} = \hat{\tilde{P}}_{t, i, 1}^T \cdot \hat{\tilde{W}}^* \cdot \hat{\tilde{P}}_{t, i, 1}, \]

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where
\[
\hat{P}_{i,1} \equiv \begin{bmatrix} I_{10} \\ \hat{B}_i^* \end{bmatrix}.
\]
With the quadratic form \(Q^*_{\hat{\nu}_{i,1}\psi^*(\nu)}\) we can find policy preferences of an individual household that is representative for each \(i \in \{r, m, p\}\). Yet, since the median voter wins the elections, we only solve the unconstrained median voter’s problem to determine policy rule \(\Psi\),

\[
\max_{\tau'} \begin{bmatrix} 1 \\ X \\ A \\ a_m \\ \tau \\ \tau' \end{bmatrix}^T \cdot Q^*_{\hat{\nu}_{m}\psi^*(\nu)} \cdot \begin{bmatrix} 1 \\ X \\ A \\ a_m \\ \tau \\ \tau' \end{bmatrix}.
\]

The first-order conditions are comprised by a 1 x 9 vector,

\[
\psi^{m,\text{indiv.}} \equiv -\left( Q^*_{\hat{\nu}_{m}\psi^*(\nu)} \right)^{-1} Q^*_{\hat{\nu}_{m}\psi^*(\nu)} \mid_{[10,1]} \psi_{\hat{\nu}_{m}\psi^*(\nu)} \mid_{[10,9]} = \begin{bmatrix} \psi_{c}^{m,\text{indiv.}} \\ \psi_{X}^{m,\text{indiv.}} \\ \psi_{A}^{m,\text{indiv.}} \\ \psi_{x_m}^{m,\text{indiv.}} \\ \psi_{r}^{m,\text{indiv.}} \end{bmatrix},
\]

in which

\[
\psi_{X}^{m,\text{indiv.}} \equiv \begin{bmatrix} \psi_{x_1}^{m,\text{indiv.}} \\ \psi_{x_2}^{m,\text{indiv.}} \\ \psi_{x_3}^{m,\text{indiv.}} \end{bmatrix}
\]

and

\[
\psi_{A}^{m,\text{indiv.}} \equiv \begin{bmatrix} \psi_{A_r}^{m,\text{indiv.}} \\ \psi_{A_m}^{m,\text{indiv.}} \\ \psi_{A_p}^{m,\text{indiv.}} \end{bmatrix}.
\]

The policy rule update occurs as follows,

\[
\psi^{(n+1)} = \lambda_{\psi} \psi^{(n)} + (1 - \lambda_{\psi}) \psi^{(n+1)},
\]

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in which \( \lambda_\phi \) acts as an overshooting parameter and

\[
\Psi^{(n_\phi)} = \psi^{m,\text{indiv}} \cdot M_m ,
\]

where \( M_m \) is a matrix we have defined in \textbf{Step 1}. If the difference between \( \Psi^{(n_\phi)} \) and \( \Psi^{(n_\phi)} \) is not arbitrarily small, we return to \textbf{Step 3} and proceed until convergence, which results in \( \Psi^* \).

\textbf{Step 5}

Finally, it is reasonable to check whether the quadratic Taylor expansion is taken around the point that can be replicated by our approximated economy with accuracy. We calculate

\[
\tau^{ss,(n_\tau)} = \Psi^* \cdot \begin{bmatrix}
1 \\
X^{ss} \\
A^{ss} \\
\tau^{ss}
\end{bmatrix},
\]

and compare \( \tau^{ss,(n_\tau)} \) and \( \tau^{ss,(n_\tau)} \). If the difference is not arbitrary small, we update as follows,

\[
\tau^{ss,(n_\tau+1)} = \lambda_\tau \tau^{ss,(n_\tau)} + (1 - \lambda_\tau) \tau^{ss,(n_\tau)} ,
\]

in which \( \lambda_\tau \) acts as an overshooting parameter, and return to \textbf{Step 2}, proceeding until convergence to obtain \( \tau^{ss,*} \). In our final step we run a Monte Carlo simulation using \( \tau^{ss,*} \), \( \Psi^* \), \( A^{ss} \), and all decision rules.

\textbf{Impulse responses}

In this section we demonstrate the impulse responses and net contributions of each wealth group. Firstly, we plot impulse responses to 1% TFP, investment and government spending shocks in the aggregate economy. Secondly, we plot impulse responses for each income
group and analyze net contributions of each class to the social system. The comparison of the net contributions is particularly interesting as the transfer payments are the main research object of our paper. The median voter who is poorer than the mean voter will vote for higher taxation with the aim of larger direct transfers to his resources. By plotting net contribution responses we can monitor which class is a net contributor and which is a net receiver. The net contributions are plotted for both steady state transfers

\[ NetContr^{i,ss} = \frac{Contr^{i}(t) - T^{ss}}{T^{ss}} \]

and transfers in every period

\[ NetContr^{i}(t) = \frac{Contr^{i}(t) - T(t)}{T^{ss}} \]

for each class \(i = \{r, m, p\}\). The contribution of each income group is calculated based on equation (122) and is as follows:

\[ Contr^{i} = (\tau - g) \cdot (r_{i}a_{i,t} + \omega_{i}w_{i}l_{i,t}) = (\tau - g) \cdot ((R_{i}H_{i,t} - e^{-x_{2,i}}\delta(H_{i,t}))a_{i,t} + \omega_{i}w_{i}l_{i,t}) \]

Also, we plot deviations from the steady state levels for output, consumption, investment and labor for each income class. To plot the impulse responses for every group, we derive the individual’s decisions for consumption and investment according to the class \(i\) budget constraint. The budget constraint of group \(i\) is as follows:

\[(1 - \tau_{i})R_{i}H_{i,t}a_{i,t} + (1 - \tau_{i})w_{i}\omega_{i}l_{i,t} + T_{i} = e^{-x_{2,i}a_{i,t+1}} - e^{-x_{2,i}(1 - (1 - \tau_{i})\delta(H_{i,t}))a_{i,t} + c_{i,t}} \]

So the income of class \(i\) is
\[ y_{i,t} = (1 - \tau_t)(R_t H_{i,t} - e^{-z_{2,t} \delta(H_{i,t})}) a_{i,t} + (1 - \tau_t) w_{i,t} \omega_{i,t}, \]

the investment of class \( i \) is
\[ i_{i,t} = e^{-z_{2,t} a_{i,t+1}} - e^{-z_{2,t} (1 - (1 - \tau_t) \delta(H_{i,t})) a_{i,t}}, \]

and the consumption of class \( i \) is
\[ c_{i,t} = y_{i,t} + T_t - e^{-z_{2,t} (a_{i,t+1} - a_{i,t})}. \]
Appendix B: Proofs of Propositions

Proof of the statement in Proposition 1

First, we propose a functional form for the value function, \( J(k) \), namely,

\[
J(k) = a + b \frac{(k - \omega)^{\frac{1}{\eta} - \frac{1}{\eta}}}{1 - \frac{1}{\eta}}. \tag{153}
\]

The first derivative of the value function with respect to wealth is,

\[
J'(k) = b(k - \omega)^{-\frac{1}{\eta}}. \tag{154}
\]

while the second derivative is,

\[
J''(k) = -\frac{1}{\eta} b (k - \omega)^{-\frac{1}{\eta} - 1}. \tag{155}
\]

Next, we combine equations (154) and (37) to obtain,

\[
c = b^{-\eta} k + \chi - b^{-\eta} \omega. \tag{156}
\]

In addition, we insert equations (154) and (155) into (38), which results in,

\[
\phi = \eta \frac{R - r_f}{\sigma^2} \left(1 - \frac{\omega}{k}\right). \tag{157}
\]

Finally, plugging all above equations (153), (154), (155), (156) and (157) into the Hamilton-Jacobi-Bellman equation in (36) and performing some algebra yields,

\[
\rho a + \rho b \frac{(k - \omega)^{1 - \frac{1}{\eta}}}{1 - \frac{1}{\eta}} = \frac{b^{-\eta} (k - \omega)^{1 - \frac{1}{\eta}}}{1 - \frac{1}{\eta}} - \frac{1}{1 - \frac{1}{\eta}} +
\]

\[
+ b (k - \omega)^{-\frac{1}{\eta}} \left[ \eta \frac{(R - r_f)^2}{\sigma^2} \left(1 - \frac{\omega}{k}\right) k + r_f k - b^{-\eta} k - \chi + b^{-\eta} \omega \right] -
\]

\[
- \frac{1}{2\eta} \left( \eta \frac{R - r_f}{\sigma} \right)^2 b (k - \omega)^{1 - \frac{1}{\eta}}. \tag{158}
\]
Further, we get rid of the constant term by setting,

$$ a = -\frac{1}{\rho \left( 1 - \frac{1}{n} \right)} $$

(159)

and divide both sides of equation (158) by expression $b \left( k - \omega \right)^{1 - \frac{1}{n}}$, which after some algebraic steps give,

$$ \left[ \frac{\rho - b^{-\eta}}{1 - \frac{1}{\eta}} - \frac{\eta}{2} \left( \frac{R - r_f}{\sigma} \right)^2 \right] k = \left[ \frac{\rho - b^{-\eta}}{1 - \frac{1}{\eta}} - \frac{\eta}{2} \left( \frac{R - r_f}{\sigma} \right)^2 \right] \omega = (r_f - b^{-\eta}) k + b^{-\eta} \omega - \chi . $$

(160)

With the aid of the method of undetermined coefficients we can assure that the proposed value for $J(k)$ is operative. We equalize respectively the coefficients of $k$ and the constant terms from both sides of (160) in order to obtain values for $b$ and $\omega$. To be precise, the whole coefficient of $k$ in (160) is set to zero, namely,

$$ \frac{\rho - b^{-\eta}}{1 - \frac{1}{\eta}} - \frac{\eta}{2} \left( \frac{R - r_f}{\sigma} \right)^2 = r_f - b^{-\eta} , $$

(161)

which can be rearranged to,

$$ b^{-\eta} = \rho \eta + (1 - \eta) r_f - \frac{\eta (\eta - 1)}{2} \left( \frac{R - r_f}{\sigma} \right)^2 . $$

(162)

Furthermore, all constant terms in (160) are added up to zero, hence,

$$ \left[ \frac{\rho - b^{-\eta}}{1 - \frac{1}{\eta}} - \frac{\eta}{2} \left( \frac{R - r_f}{\sigma} \right)^2 \right] \omega = \chi - b^{-\eta} \omega . $$

(163)

Combining both equations (161) and (163) gives,

$$ (r_f - b^{-\eta}) \omega = \chi - b^{-\eta} \omega , $$

which can be simplified to,

$$ \omega = \frac{\chi}{r_f} . $$

(164)
Inserting (164) and (162) into equation (156) gives the following decision rule for consumption,

\[ c = \left[ \rho \eta + (1 - \eta) r_f - \frac{\eta (\eta - 1)}{2} \left( \frac{R - r_f}{\sigma} \right)^2 \right] k + \eta \chi \frac{r_f - \rho + \frac{\eta - 1}{2} \left( \frac{R - r_f}{\sigma} \right)^2}{r_f}, \]  

(165)

affirming the correctness of the statement made in Proposition 1. Now, it only remains to confirm that Assumption 2 ensures that $\xi > 0$. The condition $\xi > 0$ applied to equation (165) induces,

\[ \eta^2 - \left[ 1 - \frac{r_f - \rho}{\frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2} \right] \eta - \frac{r_f}{\frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2} < 0. \]  

(166)

In order to find roots of the second-order polynomial (166) we take the discriminant with respect to $\eta$,

\[ \left[ \frac{r_f - \rho}{\frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2} \right]^2 + 2 \frac{r_f + \rho}{\frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2} + 1 > 0. \]

Since the discriminant is positive, there exist two real distinct roots. Moreover, both roots are non-zero and have alternating signs as captured by the constant term of (166). Yet, parameter $\eta$ is restricted to be greater than zero and hence we only care about the positive root of the second-order polynomial (166), which constitutes an upper bound on parameter $\eta$ coinciding with the formula for $\bar{\eta}$ in Assumption 2. \(\square\)

**Proof of the statement in Proposition 2**

We plug the optimal consumption and portfolio decision rules, $C(k)$ and $\Phi(k)$, into the budget constraint in equation (35) and perform some algebra to obtain,

\[ dk = \eta \left[ \frac{\eta + 1}{2} \left( \frac{R - r_f}{\sigma} \right)^2 + r_f - \rho \right] \left( k - \frac{\chi}{r_f} \right) \, dt + \eta \left( \frac{R - r_f}{\sigma} \right) \left( k - \frac{\chi}{r_f} \right) \, dz. \]  

(167)
Next, we apply Itô’s lemma on stochastic differential equation (167), which gives,
\[
d \ln \left( k - \frac{X}{r_f} \right) = \eta \left[ \frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2 + r_f - \rho \right] dt + \eta \left( \frac{R - r_f}{\sigma} \right) dz.
\] (168)

Finally integrating both sides of equation (168) using Itô’s integral with respect to Brownian motion (notice that by convention \( z(0) = 0 \)) gives proof to the Proposition 2. \( \square \)

**Proof of the statement in Proposition 5**

Fix any point in time \( t > 0 \). The optimal consumption formula \( C(k^*(t)) = \xi k^*(t) + \psi \) in Proposition 1 combined with the result of Proposition 2 yields,
\[
E[C(k^*(t))] = \xi E[\nu(t)] \left( k_0 - \frac{X}{r_f} \right) + \xi \frac{X}{r_f} + \psi,
\] (169)
in which,
\[
\nu(t) \equiv e^{\eta \left[ r_f - \rho + \frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2 \right] t + \eta \left( \frac{R - r_f}{\sigma} \right) z(t)}.
\]

Combining equations (40) and (41) gives \( \psi = \chi \left( 1 - \xi/r_f \right) \) reducing equation (169) to,
\[
E[C(k^*(t))] = \xi E[\nu(t)] \left( k_0 - \frac{X}{r_f} \right) + \chi.
\] (170)

Subsequently, the variation coefficient of consumption is,
\[
CoeffVar(C(k^*(t))) = \frac{\xi \left\{ Var[\nu(t)] \right\}^{1/2}}{\xi E[\nu(t)] + \frac{\chi}{k_0 - \frac{X}{r_f}}}
\] (171)
and due to \( E[\nu(t)] \) and \( \left\{ Var[\nu(t)] \right\}^{1/2} \) being strictly greater than zero,\(^{64}\) the variation coefficient (171) implies that \( dCoeffVar(C(k^*(t)))/dk_0 > 0 \). Now we turn to the derivation

\(^{64}\)Notice that the formula for expected value of growth rate of discretionary resources is,
\[
E[\nu(t)] = e^{\eta \left[ r_f - \rho + \frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2 \right] t},
\]
while the formula for variance of growth rate is given by,
\[
Var[\nu(t)] = e^{2\eta \left[ r_f - \rho + \frac{1}{2} \left( \frac{R - r_f}{\sigma} \right)^2 \right] t} \left\{ e^{\eta \left( \frac{R - r_f}{\sigma} \right)^2 t} - 1 \right\}.
\]

159
of consumption growth volatility. First, we insert the consumption decision rule, \( C(k) \), and the portfolio allocation rule, \( \Phi(k) \), into the budget constraint in (35). After rearranging terms, we get,

\[
dk = \theta \left( k - \frac{\chi}{r_f} \right) dt + \eta \left( \frac{R - r_f}{\sigma} \right) \left( k - \frac{\chi}{r_f} \right) dz ,
\]  

(172)

in which

\[
\theta = \eta \left[ \frac{\eta + 1}{2} \left( \frac{R - r_f}{\sigma} \right)^2 + r_f - \rho \right].
\]

Next, we apply Itô’s lemma on equation (172) using the formula for optimal consumption \( C(k^*(t)) = \xi k^*(t) + \psi = \xi [k^*(t) - \chi/r_f] + \chi \) with the aim of obtaining the rate of growth for consumption. The result is,

\[
d\ln [C(k^*)] = \left\{ \xi \frac{\dot{k}^*}{\xi \dot{k}^* + \chi} - \frac{1}{2} \left[ \xi \eta \left( \frac{R - r_f}{\sigma} \right) \right]^2 \left( \frac{\dot{k}^*}{\xi \dot{k}^* + \chi} \right)^2 \right\} dt + \\
+ \xi \eta \left( \frac{R - r_f}{\sigma} \right) \frac{\dot{k}^*}{\xi \dot{k}^* + \chi} dz ,
\]  

(173)

in which \( \dot{k}^* = k^* - \chi/r_f \). Finally, the variance of consumption growth rate is given by,

\[
Var (d\ln [C(k^*)]) = \left[ \xi \eta \left( \frac{R - r_f}{\sigma} \right) \right]^2 \left( \frac{\dot{k}^*}{\xi \dot{k}^* + \chi} \right)^2 dt .
\]  

(174)

and thus is an increasing function of current wealth, \( k^* \).

\[ \square \]

**Proof of the statement in Proposition 6**

We presume that solutions to the problem with time-varying subsistence consumption are interior. Our proposed functional form for the value function is,

\[
J(k, \bar{K}) = a + b \frac{(k - \gamma \bar{K})^{1 - \frac{1}{n}}}{1 - \frac{1}{n}} .
\]  

(175)
From our guessed value function in (175) we obtain,

\[ J_k(k, \bar{K}) = b \left( k - \gamma \bar{K} \right)^{-\frac{1}{\eta}} , \]  
\( (176) \)

\[ J_{kk}(k, \bar{K}) = -\frac{1}{\eta} b \left( k - \gamma \bar{K} \right)^{-\frac{1}{\eta} - 1} , \]  
\( (177) \)

\[ J_R(k, \bar{K}) = -\gamma b \left( k - \gamma \bar{K} \right)^{-\frac{1}{\eta}} , \]  
\( (178) \)

\[ J_{\bar{K}R}(k, \bar{K}) = -\frac{\gamma^2}{\eta} b \left( k - \gamma \bar{K} \right)^{-\frac{1}{\eta} - 1} , \]  
\( (179) \)

\[ J_{kR}(k, \bar{K}) = \frac{\gamma}{\eta} b \left( k - \gamma \bar{K} \right)^{-\frac{1}{\eta} - 1} . \]  
\( (180) \)

The equations (55) and (176) can be combined to give,

\[ c = b^{-\eta} \left( k - \gamma \bar{K} \right) + \gamma \bar{C} , \]  
\( (181) \)

whereas equation (56) together with (176), (177) and (180) leads to,

\[ \phi = \eta \frac{R - r_f}{\sigma^2} \left( 1 - \gamma \frac{\bar{K}}{k} \right) + \bar{\Phi} \gamma \frac{\bar{K}}{k} . \]  
\( (182) \)

Inserting equations (181) and (182) into the budget constraint in (52) yields,

\[ dk = \left\{ \left[ \eta \left( \frac{R - r_f}{\sigma} \right)^2 - b^{-\eta} + r_f \right] k + \gamma \left[ b^{-\eta} + (R - r_f) \bar{\Phi} - \eta \left( \frac{R - r_f}{\sigma} \right)^2 \right] \bar{K} - \gamma \bar{C} \right\} dt \]

\[ + \left[ \eta \frac{R - r_f}{\sigma} k + \gamma \left( \sigma \bar{\Phi} - \eta \frac{R - r_f}{\sigma} \right) \bar{K} \right] dz . \]  
\( (183) \)

The investor’s budget equation (183) permits exact linear aggregation of the law of motion of individual’s wealth across richer and poorer households. Hence, our guessed value function is consistent with the setup of the model. Next, we examine whether we can find a constant value of \( b \) that corroborates the solution. We linearly aggregate equation (182) and obtain the portfolio allocation of the average household with \( \bar{K} \) units of wealth, namely,

\[ \bar{\Phi} = \eta \frac{R - r_f}{\sigma^2} . \]  
\( (184) \)
Plugging the portfolio share (184) of the household with $\tilde{K}$ into individual’s portfolio decision (182) yields,

$$
\phi = \Phi = \eta \frac{R - r_f}{\sigma^2}, \quad \text{for } k > 0.
$$

(185)

Accordingly, we linearly aggregate equation (181) and find that,

$$
\tilde{C} = b^{-\eta} \tilde{K}, \quad \text{for } \tilde{K} > 0
$$

(186)

and inserting this expression for $\tilde{C}$ into (181) gives,

$$
c = b^{-\eta} k, \quad \text{for } k > 0.
$$

(187)

Finally, plugging all equations from (175) to (182) into the Hamilton-Jacobi-Bellman equation in (54), and setting $a = -1/[\rho (1 - 1/\eta)]$, results after some algebraic steps in,

$$
\frac{\rho - \frac{1}{\eta} b^{-\eta}}{1 - \frac{1}{\eta}} - r_f = (R - r_f) \frac{\phi k - \gamma \bar{\phi} \tilde{K}}{k - \gamma \tilde{K}} - \frac{\sigma^2}{2\eta} \left( \frac{\phi k - \gamma \bar{\phi} \tilde{K}}{k - \gamma \tilde{K}} \right)^2.
$$

(188)

Now we substitute $\phi = \Phi$ from equation (185) into (188) and obtain the expression for $b$ that is equals to $\xi^{-\Phi}$ validating the statement of Proposition 6. Based on this result, the role of condition in Assumption 3 is reaffirmed. □
Appendix C: Proof of Propositions

Proof that equation (74) results in time-separable additive preferences with HARA utility when setting \( \gamma = 1/\eta \)

We make an assumption beforehand: using the following transformation \( \tilde{c} = c - \chi \), we get \( f(c, J) = \tilde{f}\left(\tilde{c}, \tilde{J}\right) \), where \( \tilde{f}\left(\tilde{c}, \tilde{J}\right) \) is a normalized aggregator function as in the Epstein and Duffie (1992a,b), in which \( \tilde{J} \) is its continuation utility (notice the following \( \tilde{f}\left(\tilde{c}, \tilde{J}\right) = f(c, J)\mid_{\chi=0} \)); due to the identity \( f(c, J) = \tilde{f}\left(\tilde{c}, \tilde{J}\right) \) we can make use of the known result, namely: if \( \gamma = 1/\eta \), then \( \tilde{f}\left(\tilde{c}, \tilde{J}\right)\mid_{\gamma=1/\eta} \) leads to the continuation utility function of the form \( \tilde{J}(t) = \rho \int_{t}^{\infty} e^{-\rho(s-t)}c(s)^{1-\gamma} / (1 - \gamma) \, ds \); the assumption we need here is: \( \tilde{f}\left(\tilde{c}, \tilde{J}\right)\mid_{\gamma=1/\eta} \) leads to \( \tilde{J}(t) = \rho \int_{t}^{\infty} e^{-\rho(s-t)}\tilde{c}(s)^{1-\gamma} / (1 - \gamma) \, ds \), which is the result we aim to obtain. In order to check whether the intuition and assumption provided above may fail we carry out a formal proof.

Furthermore, throughout this proof we employ an expectation operator to investigate whether the result holds for a parameter \( \chi > 0 \), when consumption is uncertain.

Equation (74) leads to,

\[
f(c, J)\mid_{\gamma=1/\eta} = \rho \frac{(c - \chi)^{1-\gamma}}{1 - \gamma} - \rho J. \tag{189}
\]

The derivative of \( J \) with respect to time \( t \) and evaluated at \( t \) is given by \( J'(t) \). Equation (73) results in \( E_t[J'(t)] = -E_t[f(c(t), J(t))] \), and combined with equation (189) gives,

\[
-E_t[J'(t)] = \rho E_t \left\{ \frac{|c(t) - \chi|^{1-\gamma}}{1 - \gamma} \right\} - \rho E_t[J(t)].
\]

Next, we multiply both sides by expression \((1/\rho)e^{-\rho t}\), integrate the equation w.r.t. (with respect to) time and obtain,

\[
-\frac{1}{\rho} E_0 \left[ \int_0^T e^{-\rho t} J'(t) \, dt \right] = E_0 \left\{ \int_0^T e^{-\rho t} \frac{|c(t) - \chi|^{1-\gamma}}{1 - \gamma} \, dt \right\} - E_0 \left[ \int_0^T e^{-\rho t} J(t) \, dt \right], \tag{190}
\]
for any \( T \geq 0 \). Once we have applied integration by parts, we get,

\[
\int_0^T e^{-\rho t} J(t) \, dt = -\frac{1}{\rho} \left[ e^{-\rho T} J(T) - J(0) \right] + \frac{1}{\rho} \int_0^T e^{-\rho t} J'(t) \, dt . \tag{191}
\]

We substitute the expression (191) into equation (190) and obtain,

\[
J(0) = E_0 \left\{ \rho \int_0^T e^{-\rho t} \frac{c(t) - \chi}{1 - \gamma} \, dt \right\} + e^{-\rho T} E_T \left[ J(T) \right] . \tag{192}
\]

Since \( T \) is arbitrary, we expect equation (192) to hold for all \( T \geq 0 \). Here, we provide a requirement that guarantee that expected utility function is well defined for all \( T \geq 0 \), i.e.,

\[-\infty < E_T \left[ J(T) \right] < \infty \quad \text{for any } T \geq 0 ,\]

gives that \( \lim_{T \to \infty} e^{-\rho T} E_T \left[ J(T) \right] = 0 \). Hence, equation (192) leads to,

\[
J(T) = E_T \left\{ \rho \int_T^\infty e^{-\rho(t-T)} \frac{c(t) - \chi}{1 - \gamma} \, dt \right\} , \quad \text{for any } T \geq 0 , \tag{193}
\]

which is the proof of the statement that equation (74) with \( \gamma = 1/\eta \) implies time-separable additive preferences with HARA utility. \( \Box \)

**Proof that IES = \( \eta (1 - \chi/c) \)**

We consider two different time instants, namely \( t \) and \( t+\Delta t \), for \( t \geq 0, \Delta t > 0 \). According to the definition of \( J(t) \) in (73), the IES at \( t \) is given by,

\[
IES(t) = -\lim_{\Delta t \to 0} \frac{d \ln \left[ \frac{c(t+\Delta t)}{c(t)} \right]}{d \ln \left[ \frac{g(t+\Delta t)}{f(x(t), J(t))} \right]} . \tag{194}
\]

For \( \Delta t > 0 \) we obtain,

\[
J(t) = E_t \left[ \int_t^{t+\Delta t} f(c(\tau), J(\tau)) \, d\tau \right] + E_{t+\Delta t} \left[ \int_t^{\infty} f(c(\tau), J(\tau)) \, d\tau \right] , \tag{195}
\]

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where $\Delta t_-$ approaches $\Delta t$ from below. The equation (195) together with the definition of (73) gives,

$$
\lim_{\Delta t \to 0} \frac{\partial J (t)}{\partial c (t + \Delta t)} = \lim_{\Delta t \to 0} \left\{ E_t \left[ f_J (c (\tau), J (\tau)) d\tau \right] + 1 \right\} \cdot f_c (c (t + \Delta t), J (t + \Delta t)) ,
$$

(196)

where the integral in $\lim_{\Delta t \to 0} \left\{ E_t \left[ f_J (c (\tau), J (\tau)) d\tau \right] + 1 \right\}$ is an admissible approximation obtained from $E_t \left[ f_J (c (\tau), J (\tau)) \cdot \partial J (\tau) / \partial c (t + \Delta t) d\tau \right]$, for $\Delta t \to 0$.

Combining equations (196) and (194) results in,

$$
IES (t) = - \lim_{\Delta t \to 0} \frac{d \ln \left[ \frac{c(t + \Delta t)}{c(t)} \right]}{d \left\{ \lim_{\Delta t \to 0} \left\{ E_t \left[ f_J (c (\tau), J (\tau)) d\tau \right] + 1 \right\} + \lim_{\Delta t \to 0} \ln \left[ \frac{f_c (c(t + \Delta t), J(t + \Delta t))}{f_c (c(t), J(t))} \right] \right\} .
$$

(197)

Since $\lim_{\Delta t \to 0} \{ \ln [x (t + \Delta t)] - \ln [x (t)] \} = [\dot{x} (t) / x (t)] dt$ (where $\dot{x} (t) \equiv dx (t) / dt$), we can express equation (197) as,

$$
IES (t) = - \frac{d \left[ \frac{c(t)}{c(0)} \right]}{d \left\{ \lim_{\Delta t \to 0} \left\{ E_t \left[ f_J (c (\tau), J (\tau)) d\tau \right] + 1 \right\} + \frac{d \ln [f_c (c(t), J(t))]}{dt} \right\} .
$$

(198)

Notice that the association between a discrete-time rate of growth, $g_d$, and its continuous-time growth counterpart, $g_c$, is given as $g_c = \ln (1 + g_d)$. The transition from discrete time to continuous time occurs for $\Delta t \to 0$.

Consequently, the term $\lim_{\Delta t \to 0} \left\{ E_t \left[ f_J (c (\tau), J (\tau)) d\tau \right] + 1 \right\} / \Delta t$ in (198) converges to $f_J (c (t), J (t))$, which implies that,

$$
IES (t) = - \left\{ \frac{d \left[ f_J (c (t), J (t)) + \frac{d \ln [f_c (c(t), J(t))]}{dt} \right]}{d \left[ \frac{c(t)}{c(0)} \right]} \right\}^{-1} .
$$

(199)

We get $f_c = \rho \left[ (1 - \gamma) J (1/n - \gamma) / (1 - \gamma) \cdot (c - \chi)^{-1/n} \right]$ from equation (74), which results in $d \ln (f_c) / dt = (1/\eta - \gamma) / (1 - \gamma) \cdot \left( J / J \right) - (1/\eta) [c / (c - \chi)] \cdot (\dot{c} / c)$ and becomes,

$$
\frac{d \ln [f_c (c (t), J (t))]}{dt} = \frac{1}{\eta} \left[ f_J (c (t), J (t)) \frac{c (t)}{J (t)} \right] \cdot \frac{1}{1 - \gamma} .
$$

(200)
once we have noticed that equation (73) implies \( \dot{J}(t) = -f(c(t), J(t)) \). Performing some algebra leads to,

\[
f_J(c(t), J(t)) = \frac{1}{\gamma} \cdot \frac{f(c(t), J(t))}{J(t)}.
\]  

(201)

Once we plug (200) and (201) into equation (199) we obtain,

\[
IES(t) = \eta \left\{ \frac{d \left[ \frac{1}{1-c(t)} \cdot \frac{c(t)}{c(t)} \right]}{d} \right\}^{-1}.
\]  

(202)

We assume that the current level of consumption, \( c(t) \), is constant over time, and we focus solely on the effect of the variation in consumption growth rate on the variation in the growth rate of marginal utility of consumption. When we do so, the equation (202) leads to \( IES(t) = \eta [1 - \chi/c(t)] \), which constitutes the proof of the statement above. □

**Proof of Proposition 1**

Our guess on the value function’s functional form is,

\[
V(a, y) = b \frac{(a + \psi y - \omega)^{1-\gamma}}{1-\gamma},
\]  

(203)

which gives,

\[
V_a(a, y) = b (a + \psi y - \omega)^{-\gamma},
\]  

(204)

and

\[
f_c(c, V(a, y)) = \rho b^{1-\frac{1}{1-\gamma}} (a + \psi y - \omega)^{\frac{1}{1-\gamma}} (c - \chi)^{-\frac{1}{\gamma}}.
\]  

(205)

Combining equations (204), (205) and (77) gives,

\[
c = \rho b^{-\frac{1}{1-\gamma}} (a + \psi y - \omega) + \chi.
\]  

(206)

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In a similar way, substituting the appropriate partial derivatives into (78) implies,

$$
\phi^T = \frac{1}{\gamma} (\sigma\sigma^T)^{-1} (\mathbf{R}^T - r_f \mathbf{1}^T) \left(1 + \psi \frac{y}{a} - \frac{\omega}{a}\right) - \sigma_y \psi \frac{y}{a} \left(\rho_y \sigma^{-1}\right)^T .
$$

(207)

Next, we plug (206), (203), (74), (207), and all derivatives coming from (203) into the Hamilton-Jacobi-Bellman equation (76) and obtain,

$$
\rho b \frac{(a + \psi y - \omega)^{1-\gamma}}{1 - \frac{1}{\eta}} = b \frac{\rho b^{1-\frac{1}{\eta}}}{1 - \frac{1}{\eta}} (a + \psi y - \omega)^{1-\gamma} +
$$

$$
+b (a + \psi y - \omega)^{-\gamma} \left\{ \frac{1}{\gamma} (\mathbf{R} - r_f \mathbf{1}) (\sigma\sigma^T)^{-1} (\mathbf{R}^T - r_f \mathbf{1}^T) (a + \psi y - \omega) - \right.
$$

$$
- \sigma_y \psi \frac{y}{a} \rho_y \sigma^{-1} (\mathbf{R}^T - r_f \mathbf{1}^T) + r_f a + y - \chi - \rho b^{1-\frac{1}{\eta}} (a + \psi y - \omega) \right\} -
$$

$$
- \gamma \frac{\alpha^2 b}{2} (a + \psi y - \omega)^{-\gamma-1} \left\{ \frac{1}{\gamma} (\mathbf{R} - r_f \mathbf{1}) (\sigma\sigma^T)^{-1} \left(1 + \psi \frac{y}{a} - \frac{\omega}{a}\right) - \sigma_y \psi \frac{y}{a} \rho_y \sigma^{-1} \right\} \times
$$

$$
\times \sigma\sigma^T \left\{ \frac{1}{\gamma} (\sigma\sigma^T)^{-1} (\mathbf{R}^T - r_f \mathbf{1}^T) \left(1 + \psi \frac{y}{a} - \frac{\omega}{a}\right) - \sigma_y \psi \frac{y}{a} \rho_y \sigma^{-1} \right\} +
$$

$$
+ \psi b (a + \psi y - \omega)^{-\gamma} \mu_y y - \gamma \frac{b \psi^2 (\sigma_y y)^2 (a + \psi y - \omega)^{-\gamma-1} - \gamma \sigma_y ayb \psi (a + \psi y - \omega)^{-\gamma-1}}{2} \times
$$

$$
\times \left\{ \frac{1}{\gamma} (\mathbf{R} - r_f \mathbf{1}) (\sigma\sigma^T)^{-1} \left(1 + \psi \frac{y}{a} - \frac{\omega}{a}\right) - \sigma_y \psi \frac{y}{a} \rho_y \sigma^{-1} \right\} \sigma \rho_y^T .
$$

(208)

Equation (208), after some algebra, results in,

$$
\rho - \frac{1}{\eta^2} \rho b^{1-\frac{1}{\eta}} = \frac{1}{2\gamma} (\mathbf{R} - r_f \mathbf{1}) (\sigma\sigma^T)^{-1} (\mathbf{R}^T - r_f \mathbf{1}^T) = r_f a + \psi \left[ \frac{\sigma_y \psi y}{r_f} (\sigma\sigma^T)^{-1} \right] \left( \frac{y}{a + \psi y - \omega} \right) +
$$

$$
\frac{1}{2\gamma} \left( \frac{\sigma_y \psi y}{a + \psi y - \omega} \right)^2 \rho_y \rho_y^T - 1 .
$$

(209)

Since we have made the following assumption \(\rho_{y,1}^2 + ... + \rho_{y,N}^2 = 1\), \(\rho_y \rho_y^T = 1\), the last term in equation (209) vanishes. In addition, we set

$$
\omega = \chi/r_f
$$

(210)
and

\[ \psi = 1 + \psi \left[ \mu_y - \sigma_y \left( R - r_f 1 \right) (\rho_y \sigma^{-1})^T \right] \frac{r_f}{r_f}, \]  

which implies

\[ \psi = 1/r_y. \]  

Once we plug (211) into (209) we receive the following expression,

\[ \frac{\rho - \frac{1}{n} \rho^n b^{-\eta} \eta^{-\frac{1}{\gamma}}} {1 - \frac{1}{n}} - \frac{1}{2\gamma} (R - r_f 1) (\sigma \sigma^T)^{-1} (R^T - r_f 1^T) = r_f. \]  

Solving equation (213) for \( \rho^n b^{-\eta(1-1/n)/(1-\gamma)} \) leads to,

\[ \rho^n b^{-\eta \frac{1-\frac{1}{\gamma}}{1-\frac{1}{n}}} = \xi, \]  

where \( \xi \) is defined in equation (81). Furthermore, plugging (212) and (210) into equation (207) implies (79). Plugging expressions (212) and (210) into (203) shows that Assumption 1 is necessary as well as sufficient to have a well-defined \( V(a, y) \). The requirement \( \xi > 0 \) in (81) coincides with the condition in Assumption 2 guaranteeing that \( c \geq \chi \) for all \( (a, y) \) under Assumption 1 and equation (80), which completes the proof. \( \square \)

**Proof of equations (94), (95) and (93)**

The decomposition of variance-covariance matrix \( \Sigma \) is:

\[ \Sigma = \sigma \sigma^T = \begin{bmatrix} \sigma_s & 0 \\ \rho_{s,b} \sigma_b & \sigma_b \sqrt{1 - \rho^2_{s,b}} \end{bmatrix}, \begin{bmatrix} \sigma_s & \rho_{s,b} \sigma_b \\ 0 & \sigma_b \sqrt{1 - \rho^2_{s,b}} \end{bmatrix}, \]  

with

\[ \sigma^{-1} = \frac{1}{\sigma_s \sigma_b \sqrt{1 - \rho^2_{s,b}}} \begin{bmatrix} \sigma_b \sqrt{1 - \rho^2_{s,b}} & 0 \\ -\rho_{s,b} \sigma_b & \sigma_s \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_s} & 0 \\ -\frac{\rho_{s,b}}{\sigma_s \sqrt{1 - \rho^2_{s,b}}} & \frac{1}{\sigma_s \sqrt{1 - \rho^2_{s,b}}} \end{bmatrix}, \]  

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hence,

$$\rho_y \sigma^{-1} = \begin{bmatrix} \rho_{y,s} & \rho_{y,b} \\ \rho_{y,s} & \rho_{y,b} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sigma_s} & 0 \\ -\frac{\rho_{s,b}}{\sigma_s \sqrt{1-\rho_{s,b}^2}} & \frac{1}{\sigma_b \sqrt{1-\rho_{s,b}^2}} \end{bmatrix}$$

or,

$$\rho_y \sigma^{-1} = \begin{bmatrix} \frac{\rho_{y,s}}{\sigma_s} - \frac{\rho_{y,b} \rho_{s,b}}{\sigma_s \sqrt{1-\rho_{s,b}^2}} & \frac{\rho_{y,b}}{\sigma_b \sqrt{1-\rho_{s,b}^2}} \\ -\frac{\rho_{s,b} \sigma_s \sigma_b}{\sigma_s \sqrt{1-\rho_{s,b}^2}} & \frac{\rho_{s,b}^2}{\sigma_s \sqrt{1-\rho_{s,b}^2}} \end{bmatrix}. \quad (217)$$

Since, \( \Sigma^{-1} = (\sigma \sigma^T)^{-1} = \frac{1}{\sigma_s^2 \sigma_b^2 (1-\rho_{s,b}^2)} \begin{bmatrix} \sigma_b^2 & -\rho_{s,b} \sigma_s \sigma_b \\ -\rho_{s,b} \sigma_s \sigma_b & \sigma_s^2 \end{bmatrix} \), the term \( \frac{1}{\gamma} (R - r_f \mathbf{1}) (\sigma \sigma^T)^{-1} \) in equation (79), after some algebra, becomes,

$$\frac{1}{\gamma} (R - r_f \mathbf{1}) (\sigma \sigma^T)^{-1} = \frac{1}{\gamma} \cdot \frac{1}{1-\rho_{s,b}^2} \begin{bmatrix} R_s - r_f & R_b - r_f \\ -\frac{R_s - r_f}{\sigma_s} & -\frac{R_b - r_f}{\sigma_b} \end{bmatrix} \cdot \begin{bmatrix} \frac{R_s - r_f}{\sigma_s} & \frac{R_b - r_f}{\sigma_b} \\ -\frac{R_s - r_f}{\sigma_s} & -\frac{R_b - r_f}{\sigma_b} \end{bmatrix}. \quad (218)$$

Combining equations (218) and (217) with (79) and imposing the restriction \( \rho_{y,s}^2 + \rho_{y,b}^2 = 1 \) leads to equations (94) and (95). Lastly, since \( r_y = r_f - \mu_y + \sigma_y (R - r_f \mathbf{1}) \rho_y \sigma^{-1} \), combining (217) with the restriction \( \rho_{y,s}^2 + \rho_{y,b}^2 = 1 \) implies equation (93). \( \Box \)
Data Appendix A

Construction of the series: Government Consumption (GC) and Transfers (T)

We retrieve government spending data from Government Finance Statistics (GFS) database. The data are available for the time period between 1972 and 2007. In order to construct the series on government consumption we use the following data categories: “Compensation of Employees”, “Use of Goods and Services”, “Other Expense” and “Grants to other governmental units”. The series for government consumption are expressed in current national currencies:

\[ GC_N = \text{Compensation\_Employees} + \text{Goods\_Services} + \text{Other\_Expense} + \text{Grants\_other\_government\_units} \]  

\[ GC_{N,i,t} : \text{government consumption in millions of current national currency} \]

The series on transfer payments are constructed using categories: “Subsidies” and “Social Benefits”. The series are also expressed in current national currencies.

\[ T_N = \text{Subsidies} + \text{Social\_Benefits} \]  

\[ T_{N,i,t} : \text{transfers in millions of current national currency (country} \ i, \ \text{year} \ t) \]

Per capita series are obtained through division by population. Here, we use population series from Penn World Table 6.3. The population data is reported in thousands, so:

\[ L_{i,t} = \frac{POP_{i,t} (1000s)}{1000} : \text{population in millions (country} \ i, \ \text{year} \ t) \]
As a result, per capita government expenditures for country $i$, in year $t$ are given as:

$$GC_{N,i,t} = \frac{GG_{N,i,t}}{L_{i,t}} : \text{per capita government consumption in current national currencies}$$

$$T_{N,i,t} = \frac{T_{N,i,t}}{L_{i,t}} : \text{per capita transfers in current national currencies}$$

**Per capita real GDP**

We retrieve GDP series from Penn World Table 6.3. Here, we use “Real Gross Domestic Income adjusted for Terms of Trade” (variable “rgdptt”). The data are per capita, in constant 2005 international dollars, adjusted for Purchasing Power Parity (PPP).

$$\hat{Y}_{R,i,t} : \text{per capita real GDP, PPP-adjusted (country } i, \text{ year } t)$$

**Construction of deflator**

We choose accordingly year 2005 as a base year and the United States as a base country for the conversion of government spending series.

First, we construct the deflator which will allow us to deflate the government expenditures,

$$Deflator_{US,t} = \frac{c_{gdp_{us,t}}}{Y_{N,us,t}} \cdot \frac{c_{gus,t}}{Y_{N,us,t}} \cdot \frac{c_{g_{i,t}}}{Y_{N,i,t}}$$

$$\approx \frac{c_{g_{i,t}}}{Y_{N,i,t}} \cdot \frac{c_{g_{i,t}}}{Y_{N,i,t}}$$

(221)

Deflator for the U.S. coincides with that for a country $i$, yet the adjustment for terms of trade in per capita real GDP (rgdptt) makes a slight difference.

Definitions of variables, which we use to construct the deflator, are:

$c_{g_{i,t}}$ : nominal per capita Gross Domestic Product of country $i$ (in year $t$), adjusted for PPP (base year and base currency: 2005 US dollars), the data source is Penn World Table 6.3.
$cg_{i,t}$: government consumption as a share of per capita nominal Gross Domestic Product $(cgdp_{i,t})$, adjusted for PPP (base year and base currency: 2005 US dollars), the data source is Penn World Table 6.3

$rgdp_{i,t}$: real per capita Gross Domestic Product of a country $i$ (year $t$), adjusted for PPP (base year and base currency: 2005 US dollars) from Penn World Table 6.3

$rgdptt_{i,t}$: $rgdp_{i,t}$ (as above) but adjusted for Terms of Trade, also adjusted for PPP (base year and base currency: 2005 US dollars), from Penn World Table 6.3

$kg_{i,t}$: government consumption as a share of per capita real GDP $(rgdp_{i,t})$, from Penn World Table 6.3

*Government spending series in constant international dollars*

Additional definitions of variables are:

$PPP_{i,t}$: Purchasing Power Parity over GDP (country $i$, year $t$), the number of respective currency units necessary to purchase goods that can be bought with 1 unit of the base currency. We retrieve the series $PPP_{i,t}$ from Penn World Table 6.3.65

$\hat{x}_{N,i,t}$: variable expressed in international dollars (country $i$, year $t$)

$x_{N,i,t}$: variable in current national currency (country $i$, year $t$),

hence $x_{N,i,t} = PPP_{i,t} \cdot \hat{x}_{N,i,t}$.

---

65 A precise definition of PPP is on http://pwt.econ.upenn.edu/Documentation/append61.pdf, page 2. Also, an older appendix of PWT on the same website gives a detailed derivation of PPP.
We convert government spending series into international dollars, namely:

\[ \hat{GC}_{N,i,t} = \frac{GC_{N,i,t}}{PPP_{i,t}} : \text{current price per capita government consumption, PPP-adjusted} \]

\[ \hat{T}_{N,i,t} = \frac{T_{N,i,t}}{PPP_{i,t}} : \text{current price per capita transfers, PPP-adjusted} \]

The conversion into real terms occurs as follows:

\[ \hat{GC}_{R,i,t} = \frac{\hat{GC}_{N,i,t}}{Deflator_{i,t}} : \text{constant per capita government consumption (in international dollars)} \]

\[ \hat{T}_{R,i,t} = \frac{\hat{T}_{N,i,t}}{Deflator_{i,t}} : \text{constant per capita transfers (in international dollars)} \]

**GDP shares of Government Consumption and Transfers**

The GDP shares of government expenditures are calculated using per capita real GDP (variable “rgdptt”):

\[ \frac{\hat{GC}_{R,i,t}}{\hat{Y}_{R,i,t}} : \text{GDP share of government consumption (country } i, \text{ time } t) \]

\[ \frac{\hat{T}_{R,i,t}}{\hat{Y}_{R,i,t}} : \text{GDP share of transfers (country } i, \text{ time } t) \]
Data Appendix B

The data in Table B1 are obtained from the Luxembourg Income Study (LIS) database. The summary statistics in Tables B3 through B6 are retrieved from the Survey of Consumer Finances (SCF) 2007.

Table B1  Average number of family members in the US.

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<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of family members</td>
<td>2.55</td>
<td>2.60</td>
<td>2.45</td>
<td>2.58</td>
<td>2.55</td>
<td>2.53</td>
<td>2.54</td>
</tr>
</tbody>
</table>

Source: Luxembourg Income Study.

Table B2  Derivation of the column (5), stocks/equity as a share of total resources, of Table 2.2.

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<th>Income Percentile</th>
<th>Financial Assets</th>
<th>Stocks</th>
<th>Other equity</th>
<th>Stocks &amp; other equity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>% of Net Worth&lt;sup&gt;a&lt;/sup&gt;</td>
<td>% of Net Worth&lt;sup&gt;b&lt;/sup&gt;</td>
<td>% of Net Worth&lt;sup&gt;c&lt;/sup&gt;</td>
<td>% of Net Worth&lt;sup&gt;d&lt;/sup&gt;</td>
</tr>
<tr>
<td>Less than 20</td>
<td>35.54</td>
<td>8.78</td>
<td>19.49</td>
<td>28.28</td>
</tr>
<tr>
<td>20-39.9</td>
<td>37.30</td>
<td>9.85</td>
<td>20.83</td>
<td>30.68</td>
</tr>
<tr>
<td>40-59.9</td>
<td>39.08</td>
<td>13.01</td>
<td>26.26</td>
<td>39.26</td>
</tr>
<tr>
<td>60-79.9</td>
<td>43.94</td>
<td>17.49</td>
<td>31.39</td>
<td>48.87</td>
</tr>
<tr>
<td>80-89.9</td>
<td>44.56</td>
<td>19.27</td>
<td>34.10</td>
<td>53.37</td>
</tr>
<tr>
<td>90-100</td>
<td>43.70</td>
<td>22.31</td>
<td>40.26</td>
<td>62.56</td>
</tr>
</tbody>
</table>

<sup>a</sup> To obtain this column we divide the average amounts given in Table B3 (see below) by the average amounts in Table B4.

<sup>b</sup> Here we multiply the percentages in column (1) of Table B2 with the respective average percentages of Table B5.

<sup>c</sup> Other equity is constructed from three SCF variables, namely “business equity”, “other residential property” and “equity in nonresidential property”. Yet, the data on other equity are not available for all income percentiles. Consequently, we
project the numbers in column (3) of Table B2 assuming the working hypothesis that stockholding pattern by income classes approximately resembles the pattern of other risky asset holdings. We normalize average percentages in the last column of Table B5 (so that the values sum up to 100%) and multiply them by 28.72% (namely by the percentage share of other equity in total assets). Specifically, the percentages of the last column of Table B6, which refer to "other" equity categories, are multiplied by 63.8% (namely by the average share of nonfinancial assets in total assets). The result gives 5.75% of "other residential property" in total assets, 5.29% of "equity in nonresidential property" in total assets, and 17.68% of "business equity" in total assets, summing up to 28.72%.

\( \) We add columns (2) and (3) of Table B2 and obtain the values of column (5) in Table 2.2.

Table B3  Average values of financial assets for stockholding households (1000s of 2007 US dollars).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 20</td>
<td>19.30</td>
<td>14.90</td>
<td>21.50</td>
<td>23.30</td>
<td>28.20</td>
<td>25.40</td>
<td>29.60</td>
<td>23.26</td>
</tr>
<tr>
<td>20-39.9</td>
<td>37.90</td>
<td>32.10</td>
<td>43.90</td>
<td>52.80</td>
<td>52.10</td>
<td>46.80</td>
<td>49.30</td>
<td>44.99</td>
</tr>
<tr>
<td>40-59.9</td>
<td>50.90</td>
<td>50.70</td>
<td>57.70</td>
<td>66.80</td>
<td>84.90</td>
<td>79.70</td>
<td>78.80</td>
<td>68.43</td>
</tr>
<tr>
<td>60-79.9</td>
<td>78.80</td>
<td>74.60</td>
<td>93.20</td>
<td>124.20</td>
<td>175.70</td>
<td>161.60</td>
<td>165.50</td>
<td>124.80</td>
</tr>
<tr>
<td>80-89.9</td>
<td>114.60</td>
<td>127.20</td>
<td>164.40</td>
<td>215.50</td>
<td>275.00</td>
<td>266.90</td>
<td>237.60</td>
<td>200.17</td>
</tr>
<tr>
<td>90-100</td>
<td>584.00</td>
<td>554.50</td>
<td>664.90</td>
<td>984.30</td>
<td>1283.30</td>
<td>1209.40</td>
<td>1368.80</td>
<td>949.89</td>
</tr>
</tbody>
</table>

Source: Survey of Consumer Finances (2007)
### Table B4  Family net worth (1000s of 2007 dollars).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 20</td>
<td>39.8</td>
<td>48.2</td>
<td>60.3</td>
<td>63.1</td>
<td>62.3</td>
<td>78.5</td>
<td>105.9</td>
<td>65.5</td>
</tr>
<tr>
<td>20-39.9</td>
<td>109.6</td>
<td>93.8</td>
<td>108.5</td>
<td>124.4</td>
<td>138.2</td>
<td>135.3</td>
<td>134.5</td>
<td>120.6</td>
</tr>
<tr>
<td>40-59.9</td>
<td>165.2</td>
<td>146.8</td>
<td>137.2</td>
<td>160.1</td>
<td>191.0</td>
<td>215.0</td>
<td>210.6</td>
<td>175.1</td>
</tr>
<tr>
<td>60-79.9</td>
<td>219.9</td>
<td>202.2</td>
<td>216.0</td>
<td>259.9</td>
<td>345.2</td>
<td>372.5</td>
<td>372.6</td>
<td>284.0</td>
</tr>
<tr>
<td>80-89.9</td>
<td>360.0</td>
<td>329.9</td>
<td>349.6</td>
<td>419.7</td>
<td>529.9</td>
<td>541.5</td>
<td>614.2</td>
<td>449.2</td>
</tr>
<tr>
<td>90-100</td>
<td>1619.3</td>
<td>1394.5</td>
<td>1487.0</td>
<td>1976.3</td>
<td>2644.3</td>
<td>2788.0</td>
<td>3305.6</td>
<td>2173.6</td>
</tr>
</tbody>
</table>


### Table B5  Share of stockholdings in financial assets (%).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 20</td>
<td>13.08</td>
<td>13.92</td>
<td>13.51</td>
<td>22.55</td>
<td>39.01</td>
<td>32.15</td>
<td>38.78</td>
<td>24.71</td>
</tr>
<tr>
<td>40-59.9</td>
<td>16.67</td>
<td>20.99</td>
<td>27.75</td>
<td>38.29</td>
<td>47.34</td>
<td>43.38</td>
<td>38.61</td>
<td>33.29</td>
</tr>
<tr>
<td>60-79.9</td>
<td>21.60</td>
<td>28.19</td>
<td>35.61</td>
<td>47.04</td>
<td>51.62</td>
<td>41.83</td>
<td>52.67</td>
<td>39.80</td>
</tr>
<tr>
<td>80-89.9</td>
<td>26.04</td>
<td>32.90</td>
<td>40.90</td>
<td>49.26</td>
<td>57.62</td>
<td>48.04</td>
<td>47.92</td>
<td>43.24</td>
</tr>
<tr>
<td>90-100</td>
<td>35.13</td>
<td>40.18</td>
<td>45.83</td>
<td>62.23</td>
<td>59.89</td>
<td>57.19</td>
<td>56.82</td>
<td>51.04</td>
</tr>
</tbody>
</table>

Source: Survey of Consumer Finances (2007)
Table B6  Value of nonfinancial assets of all households (%).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Vehicles</td>
<td>5.6</td>
<td>5.7</td>
<td>7.1</td>
<td>6.5</td>
<td>6.0</td>
<td>5.1</td>
<td>4.5</td>
<td>5.8</td>
</tr>
<tr>
<td>Primary residence</td>
<td>45.7</td>
<td>47.0</td>
<td>47.3</td>
<td>47.0</td>
<td>47.3</td>
<td>50.3</td>
<td>48.3</td>
<td>47.6</td>
</tr>
<tr>
<td>Other residential property</td>
<td>8.1</td>
<td>8.5</td>
<td>8.2</td>
<td>8.7</td>
<td>8.2</td>
<td>10.2</td>
<td>11.2</td>
<td>9.0</td>
</tr>
<tr>
<td>Equity in nonresidential property</td>
<td>11.3</td>
<td>10.9</td>
<td>7.6</td>
<td>7.6</td>
<td>8.2</td>
<td>7.1</td>
<td>5.4</td>
<td>8.3</td>
</tr>
<tr>
<td>Business equity</td>
<td>27.1</td>
<td>26.4</td>
<td>27.7</td>
<td>28.6</td>
<td>28.8</td>
<td>25.8</td>
<td>29.6</td>
<td>27.7</td>
</tr>
<tr>
<td>Other</td>
<td>2.2</td>
<td>1.5</td>
<td>2.2</td>
<td>1.6</td>
<td>1.5</td>
<td>1.5</td>
<td>1.0</td>
<td>1.6</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Nonfinancial assets as a share of total assets</td>
<td>68.9</td>
<td>68.3</td>
<td>63.1</td>
<td>59.1</td>
<td>57.4</td>
<td>64.2</td>
<td>65.7</td>
<td>63.8</td>
</tr>
</tbody>
</table>

Source: Survey of Consumer Finances (2007)
Data Appendix C

In this appendix we provide detailed definition and construction of variables. All our data come from the Survey of Consumer Finances (SCF) 2007.

Direct and indirect holdings of stocks:

Direct holdings

Stocks of public companies

Indirect holdings through mutual funds

Savings accounts and money market accounts
Mutual fund investment
Trusts, annuities and managed investment accounts

Indirect holdings through retirement accounts

IRA/KEOGH accounts
Pension accounts
Present and future benefits from pensions.

Business equity: business actively managed and not actively managed

Total assets: all asset categories in the database of SCF 2007; namely: stocks, bonds, business, savings, checking accounts, retirement accounts, life insurance, main residence, other residential equity, nonresidential real estate, vehicles etc.

Total income: salaries, gains from interests and dividends, transfers, compensations etc.

Weights: used to normalize the sample in order to obtain a representative sample$^{66}$

Equivalence scale: $\sqrt{n}$ with $n$ representing the number of family members

$^{66}$For more details see the “Codebook for 2007 Survey of Consumer Finances”, more precisely the section “Analysis Weights”. 
Comparison of our data statistics with those in the SCF 2007

We show that our statistics match the median values in the SCF 2007 database. Instead of mean values we compare median values, as we want to see whether the distributions of variables are captured. Also, the means can be notably sensitive to outliers.

The following tables show the comparison between our data and the data from SCF 2007 database.

Table C1  Main variables (median values)

<table>
<thead>
<tr>
<th>Variables</th>
<th>SCF 2007</th>
<th>Our data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total assets</td>
<td>221.5</td>
<td>221.9</td>
</tr>
<tr>
<td>Total income</td>
<td>47.3</td>
<td>46.5</td>
</tr>
<tr>
<td>Stock equity</td>
<td>35.0</td>
<td>34.8</td>
</tr>
<tr>
<td>Business equity</td>
<td>100.5</td>
<td>80.6</td>
</tr>
</tbody>
</table>

Full sample (1000s of 2007 US dollars).

Table C2  Before-tax family income (median values)

<table>
<thead>
<tr>
<th>Income percentile</th>
<th>SCF 2007</th>
<th>our data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 20</td>
<td>12.30</td>
<td>12.30</td>
</tr>
<tr>
<td>20-39.9</td>
<td>28.8</td>
<td>28.8</td>
</tr>
<tr>
<td>40-59.9</td>
<td>47.3</td>
<td>47.1</td>
</tr>
<tr>
<td>60-79.9</td>
<td>75.1</td>
<td>74.9</td>
</tr>
<tr>
<td>80-89.9</td>
<td>114.0</td>
<td>114.8</td>
</tr>
<tr>
<td>90-100</td>
<td>206.9</td>
<td>209.0</td>
</tr>
</tbody>
</table>

Full sample (1000s of 2007 US dollars).
Table C3  Total family assets (families with positive asset holdings, median values)

<table>
<thead>
<tr>
<th>Percentile of income</th>
<th>SCF 2007</th>
<th>our data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 20</td>
<td>23.5</td>
<td>20.1</td>
</tr>
<tr>
<td>20-39.9</td>
<td>84.9</td>
<td>90.1</td>
</tr>
<tr>
<td>40-59.9</td>
<td>183.5</td>
<td>182.2</td>
</tr>
<tr>
<td>60-79.9</td>
<td>343.1</td>
<td>345.6</td>
</tr>
<tr>
<td>80-89.9</td>
<td>567.5</td>
<td>561.2</td>
</tr>
<tr>
<td>90-100</td>
<td>1358.4</td>
<td>1355.5</td>
</tr>
</tbody>
</table>

Full sample (1000s of 2007 US dollars).

Table C4  Holdings of different Assets (families with positive asset holdings, median values)

<table>
<thead>
<tr>
<th>Percentile of income</th>
<th>SCF 2007</th>
<th>our data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Stocks</td>
<td>Business</td>
</tr>
<tr>
<td>Less than 20</td>
<td>6.5</td>
<td>50.0</td>
</tr>
<tr>
<td>20-39.9</td>
<td>8.4</td>
<td>19.5</td>
</tr>
<tr>
<td>40-59.9</td>
<td>17.7</td>
<td>30.8</td>
</tr>
<tr>
<td>60-79.9</td>
<td>34.2</td>
<td>55.1</td>
</tr>
<tr>
<td>80-89.9</td>
<td>62.0</td>
<td>72.1</td>
</tr>
<tr>
<td>90-100</td>
<td>219.6</td>
<td>379.5</td>
</tr>
</tbody>
</table>

Full sample (1000s of 2007 US dollars).
Portfolio Shares of Risky Assets

Here we show how we calculate portfolio shares of risky assets by income percentiles. We use the following formula,

\[ SHARE_i = \frac{\sum_k \sum^N SHARE_{obs(n)} \text{ K}}{K}, \]

where \( n \) is the number of observations, \( k \) stands for the imputation number and \( i \) stands for the type of a risky asset. The tables below justify why we chose the full sample (and not a particular age group). As shown in Table C5 and Table C6 demographic and life-cycle biases do not seem to play a significant role, so we can utilize the entire SCF 2007 dataset for our calibration.

Table C5  Generated Portfolio Shares (full sample, 1000s of 2007 US dollars)

<table>
<thead>
<tr>
<th>Percentile of income</th>
<th>Risky assets (%)</th>
<th>Data information</th>
<th>Tax and after-tax income</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Stocks</td>
<td>Business</td>
<td>Total income</td>
</tr>
<tr>
<td>Less than 20</td>
<td>2.44</td>
<td>3.24</td>
<td>9.03</td>
</tr>
<tr>
<td>20-39.9</td>
<td>5.84</td>
<td>1.84</td>
<td>19.42</td>
</tr>
<tr>
<td>40-59.9</td>
<td>7.72</td>
<td>3.97</td>
<td>32.20</td>
</tr>
<tr>
<td>60-79.9</td>
<td>12.44</td>
<td>4.51</td>
<td>49.84</td>
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<tr>
<td>80-89.9</td>
<td>15.96</td>
<td>6.14</td>
<td>74.61</td>
</tr>
<tr>
<td>90-100</td>
<td>20.53</td>
<td>24.55</td>
<td>252.12</td>
</tr>
</tbody>
</table>
Table C6  Generated Portfolio Shares (age group 25-59, 1000s of 2007 US dollars)

<table>
<thead>
<tr>
<th>Percentile of Income</th>
<th>Risky Assets (%)</th>
<th>Data Information</th>
<th>Tax and After-tax Income</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Stocks</td>
<td>Business</td>
<td>Total Income</td>
</tr>
<tr>
<td>Less than 20</td>
<td>2.78</td>
<td>3.35</td>
<td>10.15</td>
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<tr>
<td>20-39.9</td>
<td>5.05</td>
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<td>22.93</td>
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<td>40-59.9</td>
<td>8.32</td>
<td>3.73</td>
<td>37.60</td>
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<tr>
<td>60-79.9</td>
<td>12.90</td>
<td>4.74</td>
<td>54.49</td>
</tr>
<tr>
<td>80-89.9</td>
<td>14.89</td>
<td>7.26</td>
<td>78.69</td>
</tr>
<tr>
<td>90-100</td>
<td>18.52</td>
<td>25.31</td>
<td>242.57</td>
</tr>
</tbody>
</table>

**Calibration**

We use equation (79) and perform brute force grid search to find the best combination of parameter values that minimize the difference between data and implied risky asset shares. We impose two restrictions on the parameter space Parameter:

1. Parameter should give a positive definite variance-covariance matrix for risky asset returns.

2. Parameter should produce positive fractions $\phi_{ij}^* \in [0, 1]$, in which $i$ indicates asset type and $j$ indicates income group of the income distribution.

Given $[\text{Parameter} \quad \mathbf{a}_{\text{data}} \quad \mathbf{y}_{\text{data}} \quad \chi]$, the risky asset fractions $\Phi^* = [\phi^*_a \quad \phi^*_y]$ are generated.⁶⁷

⁶⁷Notice that $a$ stands for wealth, $y$ stands for labor income and $\chi$ is the subsistence consumption.
We define the distance between implied and data risky asset shares ($\phi^*$ and $\phi^{\text{data}}$ respectively), namely:

$$D = 0.5 \cdot \Sigma_{j=1}^{6} \left( \phi^*_{s,j} - \phi^{\text{data}}_{s,j} \right)^2 + 0.5 \cdot \Sigma_{j=1}^{6} \left( \phi^*_{b,j} - \phi^{\text{data}}_{b,j} \right)^2$$

where [0.5 0.5] are weights assigned to the distances. Since we want to match both risky asset fractions, we assign equal weights. Here $j$ indicates the income group, $s$ stands for stocks and $b$ for business.

Algorithm

Given the enlarged parameter space $\widehat{\text{Parameter}}$ and data on assets and income, we perform an initial search and check:

- whether variance-covariance matrix is positive definite: if yes, continue;
  
  if not, reject the parameters space $\widehat{\text{Parameter}}$ and break the loop;

- whether risky asset shares are between 0 and 1, namely $\phi^*_{ij} \in [0,1]$: if yes, continue;
  
  if not, reject the parameter space $\widehat{\text{Parameter}}$ and break the loop.

Next the distance $D$ is calculated. We consider $D >= 0.2$ too large, so:

- if $D >= 0.2$, break the loop;

- otherwise, store the parameter combinations for the next search.
Second search uses parameter combinations from the first search. Also, we impose additional restrictions in the second search.

- load results from the first search,
  
  1. if $R_f \geq R_s$ or $R_f \geq 4\%$, drop the parameter combination,
  
  2. if $\phi_{s,j} > 30\%$ or $\phi_{b,j} > 30\%$, drop the parameter combination.
  
  3. store the remaining combinations for the third search.

Third search chooses the combination for which the distance $D$ is minimized.

- pick up the parameter combinations with minimal $D$. 

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Tables

Table I  Gini Coefficient of selected advanced economies.

<table>
<thead>
<tr>
<th>Country</th>
<th>Year</th>
<th>Gini Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>United States</td>
<td>2004</td>
<td>46.4</td>
</tr>
<tr>
<td>Portugal</td>
<td>2004</td>
<td>38.0</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>2005</td>
<td>34.0</td>
</tr>
</tbody>
</table>

Source: UNU-WIDER

Table II  Gini Coefficient of selected developing economies

<table>
<thead>
<tr>
<th>Country</th>
<th>Year</th>
<th>Gini Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Armenia</td>
<td>2004</td>
<td>45.5</td>
</tr>
<tr>
<td>Cambodia</td>
<td>2004</td>
<td>41.7</td>
</tr>
<tr>
<td>China</td>
<td>2004</td>
<td>46.9</td>
</tr>
<tr>
<td>Honduras</td>
<td>2004</td>
<td>54.5</td>
</tr>
<tr>
<td>India</td>
<td>2004</td>
<td>36.8</td>
</tr>
<tr>
<td>Malawi</td>
<td>2004</td>
<td>39.0</td>
</tr>
<tr>
<td>Mexico</td>
<td>2004</td>
<td>49.9</td>
</tr>
<tr>
<td>Nepal</td>
<td>2004</td>
<td>47.2</td>
</tr>
<tr>
<td>Pakistan</td>
<td>2004</td>
<td>31.2</td>
</tr>
<tr>
<td>Peru</td>
<td>2004</td>
<td>47.5</td>
</tr>
<tr>
<td>Ukraine</td>
<td>2004</td>
<td>41.0</td>
</tr>
<tr>
<td>Vietnam</td>
<td>2004</td>
<td>34.4</td>
</tr>
<tr>
<td>Zambia</td>
<td>2004</td>
<td>50.8</td>
</tr>
</tbody>
</table>

Source: UNU-WIDER United Nations University World Institute for Development Economics Research, World Income Inequality Database.

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### Table 1.1 Preference and technology parameters.

<table>
<thead>
<tr>
<th>Simulation</th>
<th>Preference parameters</th>
<th>Technology parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>1</td>
<td>0.96</td>
<td>0.45</td>
</tr>
<tr>
<td>2</td>
<td>0.96</td>
<td>0.475</td>
</tr>
<tr>
<td>3</td>
<td>0.96</td>
<td>0.45</td>
</tr>
<tr>
<td>4</td>
<td>0.96</td>
<td>0.45</td>
</tr>
</tbody>
</table>

### Table 1.2 Shock parameters.

<table>
<thead>
<tr>
<th>Simulation</th>
<th>TFP</th>
<th>Investment</th>
<th>Government</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\rho_1$</td>
<td>$\sigma_1$</td>
<td>$\rho_2$</td>
</tr>
<tr>
<td>1</td>
<td>0.9</td>
<td>0.0046</td>
<td>0.64</td>
</tr>
<tr>
<td>2</td>
<td>0.9</td>
<td>0.0038</td>
<td>0.64</td>
</tr>
<tr>
<td>3</td>
<td>0.9</td>
<td>0.0047</td>
<td>0.64</td>
</tr>
<tr>
<td>4</td>
<td>0.9</td>
<td>0.0044</td>
<td>0.64</td>
</tr>
</tbody>
</table>

186
Table 1.3  Distributional statistics sorted by wealth.

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0-49</th>
<th>49-51</th>
<th>51-100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fraction</td>
<td>0.49</td>
<td>0.02</td>
<td>0.49</td>
</tr>
<tr>
<td>Wealth</td>
<td>4.97</td>
<td>1</td>
<td>0.34</td>
</tr>
<tr>
<td>Earnings</td>
<td>2.15</td>
<td>1</td>
<td>0.60</td>
</tr>
</tbody>
</table>

Table 1.4  First moments.

<table>
<thead>
<tr>
<th></th>
<th>$\theta = 0.45, \chi = 1.2$</th>
<th>$\theta = 0.475, \chi = 1.5$</th>
<th>$\theta = 0.45, \chi = 1.2$</th>
<th>$\theta = 0.45, \chi = 1.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g = 0.19, \eta = 0.5$</td>
<td>$g = 0.19, \eta = 0.5$</td>
<td>$g = 0.14, \eta = 0.5$</td>
<td>$g = 0.14, \eta = 0.8$</td>
<td>$g = 0.14, \eta = 0.8$</td>
</tr>
<tr>
<td>$\ln(l)/\ln(w) = 2.71$</td>
<td>$\ln(l)/\ln(w) = 3.47$</td>
<td>$\ln(l)/\ln(w) = 2.65$</td>
<td>$\ln(l)/\ln(w) = 2.59$</td>
<td>$\ln(l)/\ln(w) = 2.59$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Equality</th>
<th>Inequality</th>
<th>Equality</th>
<th>Inequality</th>
<th>Equality</th>
<th>Inequality</th>
<th>Equality</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C/Y$</td>
<td>0.60</td>
<td>0.62</td>
<td>0.60</td>
<td>0.61</td>
<td>0.65</td>
<td>0.66</td>
<td>0.65</td>
<td>0.66</td>
</tr>
<tr>
<td>$T/Y$</td>
<td>0.00</td>
<td>0.13</td>
<td>0.00</td>
<td>0.13</td>
<td>0.00</td>
<td>0.16</td>
<td>0.00</td>
<td>0.15</td>
</tr>
<tr>
<td>$I/Y$</td>
<td>0.20</td>
<td>0.19</td>
<td>0.20</td>
<td>0.20</td>
<td>0.21</td>
<td>0.20</td>
<td>0.21</td>
<td>0.20</td>
</tr>
<tr>
<td>$L$</td>
<td>0.38</td>
<td>0.31</td>
<td>0.38</td>
<td>0.30</td>
<td>0.39</td>
<td>0.31</td>
<td>0.38</td>
<td>0.32</td>
</tr>
<tr>
<td>$\tau$</td>
<td>0.241</td>
<td>0.417</td>
<td>0.238</td>
<td>0.395</td>
<td>0.179</td>
<td>0.370</td>
<td>0.181</td>
<td>0.360</td>
</tr>
</tbody>
</table>
Table 1.5  Business cycle moments. Benchmark versus simulation 2.

\[ g = 0.19, \eta = 0.5 \]

\[ \theta = 0.45, \chi = 1.2 \ \text{(ELS=2.71) versus } \theta = 0.475, \chi = 1.5 \ \text{(ELS=3.47)} \]

<table>
<thead>
<tr>
<th></th>
<th>Standard Deviation</th>
<th>First Order Autocorrelation</th>
<th>Contemporaneous correlation with output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data ELS=2.71 ELS=3.47</td>
<td>Data ELS=2.71 ELS=3.47</td>
<td>Data ELS=2.71 ELS=3.47</td>
</tr>
<tr>
<td>Y</td>
<td>1.81 1.80 1.81</td>
<td>0.84 0.85 0.87</td>
<td>1.00 1.00 1.00</td>
</tr>
<tr>
<td>C</td>
<td>1.35 1.47 1.53</td>
<td>0.80 0.95 0.95</td>
<td>0.88 0.85 0.85</td>
</tr>
<tr>
<td>I</td>
<td>5.30 4.34 4.22</td>
<td>0.87 0.66 0.66</td>
<td>0.80 0.85 0.83</td>
</tr>
<tr>
<td>T</td>
<td>3.18 3.26</td>
<td>0.89 0.89 -0.37</td>
<td>0.92 0.78</td>
</tr>
<tr>
<td>L</td>
<td>1.79 0.64 0.78</td>
<td>0.88 0.36 0.54</td>
<td>0.88 0.57 0.79</td>
</tr>
<tr>
<td>T</td>
<td>0.31 0.34</td>
<td>0.87 0.89</td>
<td>0.50 0.26</td>
</tr>
<tr>
<td>T/Y</td>
<td>0.25 0.27</td>
<td>0.76 0.79 -0.89</td>
<td>0.67 0.40</td>
</tr>
<tr>
<td>w</td>
<td>0.68 1.54 1.29</td>
<td>0.66 0.95 0.96</td>
<td>0.12 0.94 0.93</td>
</tr>
<tr>
<td>r</td>
<td>0.30 0.19 0.17</td>
<td>0.60 0.78 0.76</td>
<td>-0.35 0.12 0.08</td>
</tr>
</tbody>
</table>


The cyclical pattern of T and T/Y are our calculated measures for U.S. Economy.
Table 1.6 Business cycle moments. Simulations 3 versus 4.

\[ g = 0.14, \theta = 0.45, \chi = 1.2 \]

\[ \eta = 0.5 \text{ versus } \eta = 0.8 \]

<table>
<thead>
<tr>
<th>Deviation</th>
<th>Autocorrelation</th>
<th>Contemporaneous correlation with output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard</td>
<td>First Order</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Data</th>
<th>( \eta = 0.5 )</th>
<th>( \eta = 0.8 )</th>
<th>Data</th>
<th>( \eta = 0.5 )</th>
<th>( \eta = 0.8 )</th>
<th>Data</th>
<th>( \eta = 0.5 )</th>
<th>( \eta = 0.8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>ELS=2.65</td>
<td>ELS=2.59</td>
<td>Data</td>
<td>ELS=2.65</td>
<td>ELS=2.59</td>
<td>Data</td>
<td>ELS=2.65</td>
<td>ELS=2.59</td>
</tr>
<tr>
<td>( Y )</td>
<td>1.81</td>
<td>1.80</td>
<td>1.81</td>
<td>0.84</td>
<td>0.85</td>
<td>0.85</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>( C )</td>
<td>1.35</td>
<td>1.44</td>
<td>1.49</td>
<td>0.80</td>
<td>0.95</td>
<td>0.94</td>
<td>0.88</td>
<td>0.87</td>
</tr>
<tr>
<td>( I )</td>
<td>5.30</td>
<td>4.34</td>
<td>4.77</td>
<td>0.87</td>
<td>0.66</td>
<td>0.60</td>
<td>0.80</td>
<td>0.85</td>
</tr>
<tr>
<td>( T )</td>
<td>3.20</td>
<td>3.10</td>
<td>0.92</td>
<td>0.91</td>
<td>-0.37</td>
<td>0.94</td>
<td>0.84</td>
<td></td>
</tr>
<tr>
<td>( L )</td>
<td>1.79</td>
<td>0.59</td>
<td>0.75</td>
<td>0.88</td>
<td>0.33</td>
<td>0.44</td>
<td>0.88</td>
<td>0.52</td>
</tr>
<tr>
<td>( T/Y )</td>
<td>0.33</td>
<td>0.34</td>
<td>0.87</td>
<td>0.89</td>
<td></td>
<td>0.67</td>
<td>0.39</td>
<td></td>
</tr>
<tr>
<td>( w )</td>
<td>0.68</td>
<td>1.58</td>
<td>1.50</td>
<td>0.66</td>
<td>0.95</td>
<td>0.95</td>
<td>0.12</td>
<td>0.95</td>
</tr>
<tr>
<td>( r )</td>
<td>0.30</td>
<td>0.17</td>
<td>0.17</td>
<td>0.60</td>
<td>0.76</td>
<td>0.72</td>
<td>-0.35</td>
<td>0.14</td>
</tr>
</tbody>
</table>


The cyclical pattern of \( T \) and \( T/Y \) are our calculated measures for U.S. Economy.
Table 1.7  Calculated shares and cyclical patterns of transfers T and transfers to output ratio T/Y for developing countries.

<table>
<thead>
<tr>
<th>Developing countries</th>
<th>T/Y</th>
<th>Corr(T,Y)</th>
<th>Corr(T/Y,Y)</th>
<th>(T+G)/Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bolivia</td>
<td>0.04</td>
<td>-0.18</td>
<td>-0.24</td>
<td>0.18</td>
</tr>
<tr>
<td>Colombia</td>
<td>0.06</td>
<td>0.32</td>
<td>0.09</td>
<td>0.14</td>
</tr>
<tr>
<td>India</td>
<td>0.03</td>
<td>0.58</td>
<td>0.24</td>
<td>0.11</td>
</tr>
<tr>
<td>Indonesia</td>
<td>0.01</td>
<td>0.27</td>
<td>0.23</td>
<td>0.08</td>
</tr>
<tr>
<td>Malaysia</td>
<td>0.04</td>
<td>0.35</td>
<td>0.06</td>
<td>0.16</td>
</tr>
<tr>
<td>Costa Rica</td>
<td>0.05</td>
<td>0.06</td>
<td>-0.09</td>
<td>0.16</td>
</tr>
<tr>
<td>Kuwait</td>
<td>0.07</td>
<td>0.21</td>
<td>-0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>Panama</td>
<td>0.05</td>
<td>0.47</td>
<td>-0.19</td>
<td>0.18</td>
</tr>
<tr>
<td>Thailand</td>
<td>0.01</td>
<td>0.03</td>
<td>-0.21</td>
<td>0.14</td>
</tr>
<tr>
<td>Tunisia</td>
<td>0.08</td>
<td>-0.07</td>
<td>-0.26</td>
<td>0.25</td>
</tr>
<tr>
<td>South Africa</td>
<td>0.03</td>
<td>0.16</td>
<td>0.03</td>
<td>0.23</td>
</tr>
<tr>
<td>Mauritius</td>
<td>0.06</td>
<td>0.30</td>
<td>0.07</td>
<td>0.18</td>
</tr>
<tr>
<td>Guatemala</td>
<td>0.01</td>
<td>-0.32</td>
<td>-0.39</td>
<td>0.08</td>
</tr>
<tr>
<td>Average</td>
<td>0.04</td>
<td>0.17</td>
<td>-0.07</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Cyclical components are estimated by means of Hodrick–Prescott Filter. Source: Social benefits and Subsidies were retrieved from GFS and summed up to constitute transfer payments. GDP series were retrieved from Penn World Tables 6.3.
Table 1.8  Calculated shares and cyclical patterns of T and T/Y for OECD countries.

<table>
<thead>
<tr>
<th>OECD countries</th>
<th>T/Y</th>
<th>Corr(T,Y)</th>
<th>Corr(T/Y,Y)</th>
<th>(T+G)/Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>AUS</td>
<td>0.08</td>
<td>-0.25</td>
<td>-0.46</td>
</tr>
<tr>
<td>Austria</td>
<td>AUT</td>
<td>0.19</td>
<td>-0.21</td>
<td>-0.54</td>
</tr>
<tr>
<td>Belgium</td>
<td>BEL</td>
<td>0.22</td>
<td>0.17</td>
<td>-0.36</td>
</tr>
<tr>
<td>Canada</td>
<td>CAN</td>
<td>0.09</td>
<td>-0.56</td>
<td>-0.78</td>
</tr>
<tr>
<td>Denmark</td>
<td>DNK</td>
<td>0.07</td>
<td>-0.61</td>
<td>-0.78</td>
</tr>
<tr>
<td>France</td>
<td>FRA</td>
<td>0.22</td>
<td>0.03</td>
<td>-0.41</td>
</tr>
<tr>
<td>Hungary</td>
<td>HUN</td>
<td>0.23</td>
<td>0.65</td>
<td>0.17</td>
</tr>
<tr>
<td>Iceland</td>
<td>ISL</td>
<td>0.09</td>
<td>-0.48</td>
<td>-0.70</td>
</tr>
<tr>
<td>Ireland</td>
<td>IRL</td>
<td>0.14</td>
<td>-0.26</td>
<td>-0.31</td>
</tr>
<tr>
<td>Korea, Republic of</td>
<td>KOR</td>
<td>0.03</td>
<td>0.70</td>
<td>0.44</td>
</tr>
<tr>
<td>Luxembourg</td>
<td>LUX</td>
<td>0.19</td>
<td>0.49</td>
<td>-0.66</td>
</tr>
<tr>
<td>Netherlands</td>
<td>NLD</td>
<td>0.20</td>
<td>-0.27</td>
<td>-0.56</td>
</tr>
<tr>
<td>Norway</td>
<td>NOR</td>
<td>0.16</td>
<td>0.40</td>
<td>-0.48</td>
</tr>
<tr>
<td>Portugal</td>
<td>PRT</td>
<td>0.15</td>
<td>-0.31</td>
<td>-0.64</td>
</tr>
<tr>
<td>Spain</td>
<td>ESP</td>
<td>0.12</td>
<td>-0.16</td>
<td>-0.62</td>
</tr>
<tr>
<td>Sweden</td>
<td>SWE</td>
<td>0.19</td>
<td>-0.48</td>
<td>-0.70</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>GBR</td>
<td>0.13</td>
<td>-0.76</td>
<td>-0.89</td>
</tr>
<tr>
<td>United States</td>
<td>USA</td>
<td>0.09</td>
<td>-0.37</td>
<td>-0.76</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td>0.14</td>
<td>-0.13</td>
<td>-0.50</td>
</tr>
</tbody>
</table>

Cyclical components are estimated by means of Hodrick-Prescott filter. Source: Social Benefits and Subsidies were retrieved from GFS and summed up to constitute transfer payments. GDP series were retrieved from Penn World Tables 6.3.
Table 2.1 Calibration of parameter values.

<table>
<thead>
<tr>
<th>η</th>
<th>R</th>
<th>r_f</th>
<th>σ</th>
<th>ρ</th>
<th>χ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.23</td>
<td>7%</td>
<td>3%</td>
<td>20%</td>
<td>2.5%</td>
<td>6900&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
</tbody>
</table>

<sup>a</sup> Yearly subsistence consumption cost per capita in 2007 US Dollars. We follow Luxembourg Income Study (LIS) data in the choice of household size. The average US family size equals to 2.5 persons over years 1986-2004 (see for details Table B1 in the Data Appendix B).

Table 2.2 Data on wealth and income distributions, and calibration results for the benchmark χ = 6000 USD.

<table>
<thead>
<tr>
<th></th>
<th>(1)&lt;sup&gt;a&lt;/sup&gt;</th>
<th>(2)&lt;sup&gt;a&lt;/sup&gt;</th>
<th>(3)&lt;sup&gt;b&lt;/sup&gt;</th>
<th>(4)&lt;sup&gt;c&lt;/sup&gt;</th>
<th>(5)&lt;sup&gt;d&lt;/sup&gt;</th>
<th>(6)&lt;sup&gt;e&lt;/sup&gt;</th>
<th>(7)&lt;sup&gt;f&lt;/sup&gt;</th>
<th>(8)&lt;sup&gt;g&lt;/sup&gt;</th>
<th>(9)&lt;sup&gt;g&lt;/sup&gt;</th>
<th>(10)&lt;sup&gt;g&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 20</td>
<td>65.45</td>
<td>10.37</td>
<td>-1.83%</td>
<td>10.77</td>
<td>28.28</td>
<td>314.97</td>
<td>5.88</td>
<td>6.2</td>
<td>5.65</td>
<td>0.07</td>
</tr>
<tr>
<td>20-39.9</td>
<td>120.62</td>
<td>25.90</td>
<td>2.78%</td>
<td>25.18</td>
<td>30.68</td>
<td>692.67</td>
<td>5.34</td>
<td>15.36</td>
<td>12.58</td>
<td>0.16</td>
</tr>
<tr>
<td>40-59.9</td>
<td>175.12</td>
<td>43.88</td>
<td>6.47%</td>
<td>41.04</td>
<td>39.26</td>
<td>1045.62</td>
<td>6.58</td>
<td>17.94</td>
<td>14.28</td>
<td>0.18</td>
</tr>
<tr>
<td>60-79.9</td>
<td>284.03</td>
<td>69.86</td>
<td>14.28%</td>
<td>59.88</td>
<td>48.87</td>
<td>1464.84</td>
<td>9.48</td>
<td>19.39</td>
<td>15.20</td>
<td>0.20</td>
</tr>
<tr>
<td>80-89.9</td>
<td>449.24</td>
<td>104.97</td>
<td>22.63%</td>
<td>81.22</td>
<td>53.37</td>
<td>1999.15</td>
<td>11.99</td>
<td>20.35</td>
<td>15.80</td>
<td>0.21</td>
</tr>
<tr>
<td>90-100</td>
<td>2173.55</td>
<td>301.60</td>
<td>29.27%</td>
<td>212.90</td>
<td>62.56</td>
<td>5983.38</td>
<td>22.73</td>
<td>22.12</td>
<td>16.85</td>
<td>0.22</td>
</tr>
</tbody>
</table>

<sup>1</sup> Data family net worth (in 000s); <sup>2</sup> Data before tax family income (in 000s)

<sup>3</sup> Effective marginal tax rate (%), data, <sup>4</sup> Data after tax family income (in 000s)

<sup>5</sup> Fraction (%) of net worth held in stocks/equity (data)

<sup>6</sup> Total lifetime resources (in 000s)

<sup>7</sup> Fraction (%) of total resources held in stocks/equity (data)

<sup>8</sup> Fraction (%) of total resources held in stocks (model)

<sup>9</sup> Saving Rate (%) (model), <sup>10</sup> IES (Model)

<sup>a</sup> Survey of Consumer Finances (SCF, 2007) data on family net worth and before tax income are average values of years 1989 to 2007 (thousands, in 2007 US Dollars).
\(^b\) Tax rate values are calculated based on the data from the Federation of Tax Administrators (address: 444 N. Capital Street, Washington DC), and relate to year 2003. See for more details Grant et al. (2010, Table 2).

\(^c\) Amounts of after-tax family income are calculated by deducting respectively the effective marginal tax rates (column (3)) from before-tax family incomes (column (2)) (thousands, in 2007 US Dollars).

\(^d\) Values of stock/equity shares are projected based on the SCF data from year 2007. Projected fractions are averages of years 1989 to 2007 (see for more details Table B2 with its notes in Data Appendix B).

\(^e\) Entries of column (6) correspond to resource variable \(k\) in our model (values are reported in thousands of 2007 US Dollars). We construct the resource variable by summing up column (1) with the expected present discounted values of lifetime after-tax incomes (EPVY), which are calculated, using entries of column (4), namely after tax family earnings. The discount rate that we use to derive EPVY corresponds to the effective annual interest rate, \(\phi R + (1 - \phi)r_f\), of each income group, in which \(\phi\) is the projected risky fraction of net worth in column (5), whereas \(R\) and \(r_f\) are the parameterized interest rate values in Table 2.1. We compute family EPVY adopting 78 years as an average life duration.

\(^f\) Amounts in column (7) are calculated by multiplying columns (1) and (5) and dividing by column (6). These entries correspond to the fraction \(\Phi(k)\) in the model.

\(^g\) Calculations are made for the values of total lifetime resources in column (6), which proxy variable “\(k\)” in the model.
Table 2.3 Model-calculated relative risk aversion coefficients for varying levels of subsistence needs.

<table>
<thead>
<tr>
<th>Income Percentile</th>
<th>$\chi = \text{USD 6000 (benchmark)}$</th>
<th>$\chi = \text{USD 9000}$</th>
<th>$\chi = \text{USD 4500}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 20%</td>
<td>14.35</td>
<td>57.85</td>
<td>7.85</td>
</tr>
<tr>
<td>20%–39.9%</td>
<td>6.26</td>
<td>7.27</td>
<td>5.42</td>
</tr>
<tr>
<td>40%–59.9%</td>
<td>5.43</td>
<td>5.88</td>
<td>4.99</td>
</tr>
<tr>
<td>60%–79.9%</td>
<td>5.05</td>
<td>5.32</td>
<td>4.78</td>
</tr>
<tr>
<td>80%–89.9%</td>
<td>4.84</td>
<td>5.01</td>
<td>4.65</td>
</tr>
<tr>
<td>90%–100%</td>
<td>4.50</td>
<td>4.55</td>
<td>4.45</td>
</tr>
</tbody>
</table>

Table 3.1 Parameter values.

<table>
<thead>
<tr>
<th>Preference parameters</th>
<th>$\rho$</th>
<th>$\eta$</th>
<th>$\gamma$</th>
<th>$\chi$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2.5%</td>
<td>0.08</td>
<td>4.78</td>
<td>2940$^a$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mean returns</th>
<th>$r_f$</th>
<th>$\mu_y$</th>
<th>$R_s$</th>
<th>$R_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3.56%</td>
<td>1.15%</td>
<td>7.56%</td>
<td>18.0%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Standard deviations and correlations</th>
<th>$\sigma_y$</th>
<th>$\sigma_s$</th>
<th>$\sigma_b$</th>
<th>$\rho_{gs}$</th>
<th>$\rho_{gb}$</th>
<th>$\rho_{sb}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8.21%</td>
<td>21.0%</td>
<td>42.07%</td>
<td>48.93%</td>
<td>87.21%</td>
<td>1.74%</td>
</tr>
</tbody>
</table>

$^a$ Annual subsistence cost per person in 2007 US Dollars.
Figures

Figure 1.1 Impulse responses of output, consumption and investment to 1% TFP, investment, government shocks.
Figure 1.2 Impulse responses of labor, transfers, taxes and T/Y to 1% TFP, investment, government shocks.
Figure 1.3 Impulse responses of capacity utilization, capital, wages and interest rates to 1% TFP, investment, government shocks.
Figure 1.4 Impulse responses of each class’ income, consumption, investment and labor to 1% TFP shock.
Figure 1.5 Impulse responses of each class’ income, consumption, investment and labor to 1% investment shock.
Figure 1.6 Impulse responses of each class’ income, consumption, investment and labor to 1% government shock.
Figure 1.7 Net contribution of each income class.
Figure 2.1 Share (%) of total resources allocated to stocks.

Percentage share of total wealth resources allocated to stocks, Φ(k), is depicted against the percentiles of income. The data is retrieved from the Survey of Consumer Finances 2007 (see column (7) in Table 2.2). The model’s benchmark share Φ(k) is calculated using χ = 6900 USD (column (8) in Table 2.2 – subsistence consumption cost per head and month amounts to 230 USD, 2007 is the base year), the model’s sensitivity with respect to lower subsistence consumption costs, namely χ = 4500 USD (subsistence cost per head and month amounts to 150 USD), and higher subsistence consumption costs, namely χ = 9000 USD (subsistence cost per head and month 300 USD), gives respectively higher and lower implied Φ(k).
Figure 2.2 Implied saving rates (%) for various levels of subsistence costs.
Figure 3.1 Stocks and business as shares of total wealth depicted against income percentiles (2007 USD).
Figure 3.2 Model’s benchmark fraction $\Phi(a, y)$ versus the data depicted against income percentiles.
Figure 3.3 Model’s benchmark saving rate depicted against income percentiles.
Figure 3.4 Sensitivity analysis of $\Phi(a,y)$ with respect to different subsistence costs.
Figure 3.5 Sensitivity analysis of saving rate with respect to different subsistence costs.
Figure 3.6 Observed ratio of income to assets in the data depicted against income percentiles (assets and income are per equivalent adult (000s of 2007 USD)).
Figure 3.7 Sensitivity analysis of $\Phi(a, y)$ with respect to different $\rho_{ab}$. 

![Graph showing the share of stocks and business equity versus after-tax income categories per equivalent adult.](image)
Figure 3.8 Sensitivity analysis of saving rate with respect to different $\rho_{sb}$. 
Figure 3.9 Sensitivity analysis of saving rate with respect to different IES.
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