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# Task design using a web-based proof learning system to support students' emerging strategic knowledge of how to construct alternative proofs to the same problem

Mikio Miyazaki (Shinshu University, Japan)

https://orcid.org/0000-0001-8475-8502

Taro Fujita (University of Exeter, UK)

https://orcid.org/0000-0002-3547-456X

Keith Jones (University of Southampton, UK)

https://orcid.org/0000-0003-3677-8802

Abstract: This study explores how a proving task with technology can be designed to develop students' strategic knowledge of how to construct alternative proofs to the same problem, and how the designed task enriched their strategic knowledge in proving in the context of geometrical proof. The designed task had three components; open problem with flow-chart proofs, learning environment with web-based proof learning support system, and process of expressing strategic knowledge of how to reconstruct proofs. By analyzing experimental lessons with a grade 8 class (students aged 13-14), we found that these task components, and their interactions, contributed to developing students' strategic knowledge. Using open problems with flow-chart proofs in a web-based proof learning support system enabled students to find alternative proofs to the same problem, and promoted the process of them expressing their strategic knowledge of how to reconstruct proofs.

Key words: task design, web-based proof learning support system, strategic knowledge

## 1. Improvement of proof construction by using technology

The teaching and learning of proof and proving is acknowledged globally as a crucial part of mathematics education (Hanna & de Villiers, 2012) not only for echoing the nature of mathematics, but also for cultivating generic competencies of authentic explorative thinking (Miyazaki & Fujita, 2015). Yet students at the secondary school level (and beyond) suffer serious difficulties related to constructing and evaluating proofs in mathematics in general, and in geometry in particular (e.g. McCrone & Martin, 2004).

In order to improve on these difficulties, learning environments with technology for proof learning has been developed in two directions. One direction relates to developments in Artificial Intelligence (AI) by offering environments that focus on what might be considered more formal aspects of proving and how learners might be guided to construct correct proofs by appropriate feedback on their proofs and proving (from early systems, e.g. Anderson et al. 1986, to current initiatives, e.g. Wang & Su, 2017). The second direction is characterized by dynamic geometry environments (DGEs, such as *Cabri Express*, *Sketchpad Explorer* and *GeoGebra*). This direction has contributed to stimulating the use of conjecturing and the dialectical relationship between proofs and refutations in mathematics classrooms (e.g. González & Herbst, 2009; Komatsu & Jones, in press). A slightly different approach to the two directions is to create a domain-specific learning space or environment for students (e.g. Cabri-Eulid (Luengo, 2005)). In our study we work on integrating technologies into daily mathematics lessons so teachers and learners use technologies effectively to advance their learning (for our technology, see section 3.2.3). We argue that this approach might improve the status quo of proof learning at the secondary school level, especially in terms of constructing proofs as this is not yet satisfactory.

One way to improve students' capabilities related to constructing proofs by means of technology is to focus on the strategic knowledge needed by learners during not only proof constructions (Weber, 2001) but also during proof 'reconstructions' where students need to consider and apply this knowledge in order to change their proofs into alternatives. As such, and given that proving tasks with technology may contribute to developing strategic knowledge required for constructing alternative proofs, the purpose of this chapter is to explore how a proving task with technology can be designed to encourage students' emerging strategic knowledge of how to construct alternative proofs to the same problem, and how the designed task might enrich learners' strategic knowledge in proving in the context of geometrical proof that is commonly used to teach deductive proofs and proving in lower secondary schools (Fujita and Jones, 2014; also see section 4.1).

## 2. Strategic knowledge of how to construct alternative proofs to the same problem

### 2.1 Constructing alternative proofs in the process of explorative proving

Luengo (2005, p. 13) views the construction of a mathematical proof as "a problem solving activity" . In line with this, we take proving in mathematics is a fallibilistic activity (e.g. Lakatos, 1976) that involves producing statements inductively/deductively/analogically, planning and constructing proofs, looking back over proving processes and overcoming global/local counter-examples or errors, as well as utilizing already-proved statements in the context of working on further proofs (see Figure 1). We define *explorative proving* as having the following three phases that inter-relate: 1) producing propositions, 2) producing proofs (planning and

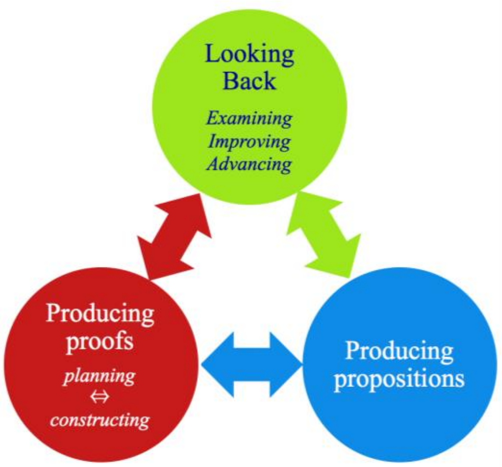


Fig. 1: Explorative proving

construction), and 3) looking back (examining, improving and advancing) (Miyazaki & Fujita, 2015).

In the process of explorative proving, students may need/want to be challenged to construct alternative proofs. When confronting mistakes and errors in their proving, they try to overcome them if they need to. When noticing more sophisticated ideas, they are willing to improve their previous proofs. At all these times, they may need to change the used theorems, assumptions or conclusions, rephrase their statements more clearly and so on. More globally, they sometimes need to switch from a direct proof to an indirect proof or vice versa, or overcome circular arguments (Fujita, Jones & Miyazaki, 2018), and so on. We call such activities of producing different proofs based on already constructed proofs, 'reconstructing (either correct/incorrect) proofs'.

2.2 Forms of strategic knowledge of how to reconstruct proofs

Strategic knowledge, the 'knowing-how' in problem solving (Greeno, 1978), enables students to "recall actions that are likely to be useful or to choose which action to apply among several alternatives" (Weber, 2001, p. 111). Such knowledge plays an important role in explorative proving.

Strategic knowledge of how to reconstruct proofs has a domain-specific aspect related to achieving successful resolutions in the concerned domain. Especially, proof problems usually request students to show the logical connection between premises and conclusions. In the solving process, students are engaged in the activity, including constructing and reconstructing proofs as appropriate to the proof problem. As such, we propose the following three types of strategic knowledge of how to reconstruct proofs; *strategic knowledge for constructing proofs*, *strategic knowledge for reconstructing proofs, strategic knowledge appropriate for solving the proof problem*. We consider each in turn.

Strategic knowledge for constructing proofs (SKC) is used to connect premises and conclusions in order to form an original proof even if the proof contains some errors. This can involve, at a minimum, distinguishing conclusions and premises, deciding which theorems and/or definitions can be applied, and arranging the ways of thinking backward from conclusions to premises and thinking forward in the opposite direction. Knowledge of thinking backward/forward, especially, can fundamentally inform students' proof construction (e.g. Heinz, et al., 2008). This can work successfully, particularly when supported by more general strategic knowledge such as 'planning' (Schoenfeld, 1985), 'working backward' (Anderson, 1995), metacognition, and so on. For example, in constructing a geometrical proof, students often need to choose an appropriate triangle congruent condition before choosing which singular propositions can be used as premises. In this case, they use SKC related to deciding which theorems and/or definitions can be applied to the particular proof problem.

Strategic knowledge for reconstructing proofs (SKRc) is used to evaluate the previous proofs and improve/advance them. This can involve checking premises (definitions, axioms, etc.) and the validity of the used theorems during reconstructing proofs.

Moreover, from a local point of view, it also involves organizing the elements of proofs to overcome any errors. For example, in the case of geometrical proofs, students often need to switch between three types of congruent triangle conditions in accordance with the problem conditions. In this case, they use SKRc related to validating the used theorems that can be applied.

Finally, strategic knowledge appropriate for solving the proof problem (SKSP) is used to supplement the parts of the logical connection in constructing and reconstructing proofs by using SKC and SKRc. This can involve considering the conditions of the problem to be solved and using theorems appropriate to them. For example, even if students can choose or switch an appropriate triangle congruent condition in constructing a geometrical proof, they also need to find/replace pairs of sides/angles, or triangles correctly. In this case, they use SKSP related to using theorems according to the problem conditions.

## 3. Task design to develop strategic knowledge of how to reconstruct proofs

In developing students' strategic knowledge of how to reconstruct proofs, it is necessary to organise activities for students that cultivate their strategic knowledge in proving. In this section we discuss key ideas for our task design and how technology is used to support students' learning.

### 3.1 Necessary components of a task for problem solving

A 'task' generally means an activity that needs to be undertaken and accomplished. In education, a task is expected to have distinct roles to guide learners' attention and interests towards desirable aspects of the concerned content (Doyle, 1983) that, in the context of mathematics education, encompasses thinking about, developing, using, and making sense of mathematics (Stein et al, 1996). For more on task design in mathematics education, see the various chapters in Watson and Ohtani (2015).

In order to undertake and accomplish a task as an intended activity in education, the task should be designed with essential components that lead learners toward the targets of the activity. Given that an intended activity is related to problem solving, the two questions 'What should be solved?' and 'How should it be solved' should be addressed. The first can be answered by the characteristics of a problem to be tackled. The second question has two aspects; a process to solve the problem and an environment that encourages the process. The former can be designed for managing and directing learners' solving processes toward appropriate ones. The latter can be designed for supporting learners' solving processes. Therefore, in this study, three components are adopted to design a task to develop strategic knowledge of how to reconstruct proofs: a *problem* to be tackled, a *learning environment* where to solve the problem, and a *process* to solve the problem.

## 3.2 Three components of a task designed to develop strategic knowledge of how to reconstruct proofs

Here we consider “what to design; which tools are necessary, or beneficial, for task design and under what conditions” (Jones and Pepin, 2016, p. 115). In our task design, we use ‘open problems with flow-chart proofs’ (what to design), web-based proof support system (hereinafter ‘the system’) that provides automatic translations of figural to symbolic objects and feedback in accordance with the type of errors (which tools are necessary, or beneficial) and specially-designed worksheets for students (under what conditions). We explain the detail below.

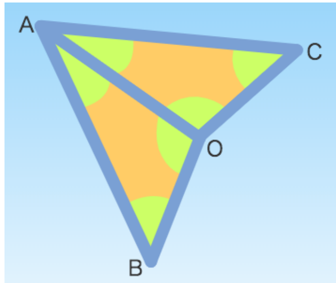
### 3.2.1 Open problems with flowchart proofs

To develop strategic knowledge of how to reconstruct proofs, domain-specific strategic knowledge is applied in the problem-solving process consisting of a series of propositions and their transformations. In order to actualize, and be conscious of, this domain-specific strategic knowledge, open problems with a flow-chart proof format can be adopted. Open problems encourage students to construct multiple solutions by deciding the assumptions and intermediate propositions necessary to deduce a given conclusion. A flow-chart proof format can demonstrate the logical chains of propositions, and support students to construct proofs systematically.

Based on a scaffolding analysis (Sherin et al., 2004), the use of open problems with flowchart proofs can enhance the structural understanding of proofs because it is expected that flow-chart format provides a visualization of the structural aspects of proofs in geometry, and encourages thinking backward/forward interactively because learners freely choose assumptions to prove the conclusion (Miyazaki, Fujita & Jones, 2015). Open problems can function successfully in developing strategic knowledge of how to reconstruct proofs because students are encouraged to switch from an existing proof to an alternative proof by appropriate use of the viewpoint of proof structure; that is, “the relational network via deductive reasoning that combines singular and universal propositions” (Miyazaki, Fujita & Jones, 2017, p. 226).

For example, the problem in Fig. 2 is intentionally designed so that students can freely choose which assumptions they use to draw the conclusion ∠ABO = ∠ACO. One solution is to show ∠ABO = ∠ACO by using SSS condition. Nevertheless, other solutions can be found. One alternative solution might be to use AO = AO as the same line and hence Δ ABO ≡ Δ ACO can be shown by assuming AB = AC and ∠ OAB = ∠ OAC using the SAS condition. During the time to find these solutions, by using flow-chart proofs format students can distinguish theorems and pairs of sides/angles/triangles, and then think backward/forward flexibly.

In the right-hand side diagram, you will prove ∠ABO = ∠ACO by showing that these triangles are congruent. What else do you need to add to prove this? What type of condition of congruence and what property of congruent figures do you use in there? Let's complete the flow-chart!



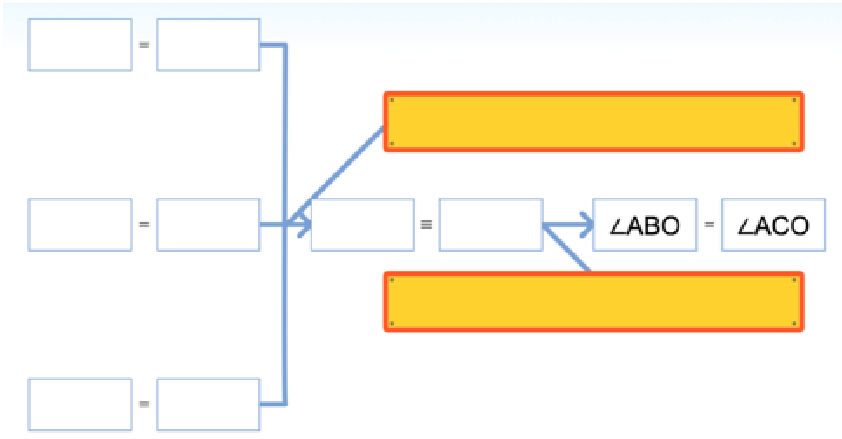




Fig. 2 Example of open problems with Flow-chart proofs

3.2.2 Learning environment with web-based proof learning support system

In order to support students to tackle open problems, our web-based system has three characteristics; translating automatically figural to symbolic objects, reviewing a learner's answer, providing systematic feedback during/after constructing proofs (Miyazaki, Fujita, Jones, & Iwanaga, 2017; Fujita, Jones & Miyazaki, 2018).

Our system has some similarities to the Cabri-Euclid system designed by Luengo (2005). The Cabri-Euclid system supports learners' proof construction process with workspaces dedicated to dynamic representations of geometrical figures, graphs to represent algorithms, and text spaces for exploiting direct manipulation and a set of operators to express statements and organise them under a precise ''proof'' format"" (p. 4). At the same time, Cabri-Euclid gives feedback to learners to support their proof construction process, e.g. "if the student attempts a deduction without giving a deduction rule, the software will ask the student for a theorem or a definition." (p. 11)

Cabri-Euclid is a powerful tool to support learners' proof construction process, but one issue is that students need to learn specific ways of inputting their texts as "The textual component of Cabri-Euclide has a limited linguistic capability." (ibid., p. 5). As we have described above, our system adopts a flow-chart format as we consider this is more intuitive for novice learners so that they do not have to learn specific commands and language to use the system.

Unlike Cabri-Euclid, the diagrams in our system for representing geometrical figures are static, but this is because of a technological reason -- the interface of the system translates automatically from figural to symbolic objects. For example, to insert the symbol Δ ABO into the target box of flow-chart proof, by clicking the triangle on the diagram, the character gives message "ΔABO closed", and red circles appear on the possible boxes (Fig. 3). Next, by clicking the proper box, the symbol ΔABO appears inside the box. The opposite way (firstly click the box, then click the triangle) can also be executed. This automatic transformation of symbols is intended to reduce the representational barrier that students encounter when proof writing.

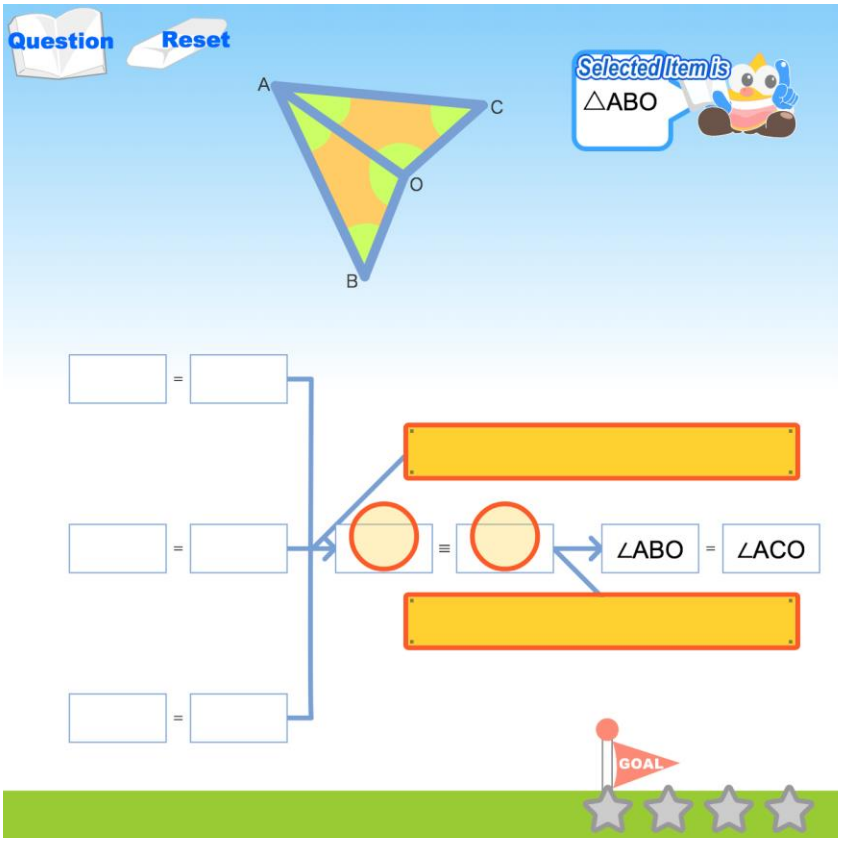



Fig. 3 Interface of web-based proof learning support system

In order to complete the open problem, students are expected to seek multiple solutions, and, in such processes, are expected to find it useful to review the solutions that they have already found. Similarly, in the case of open problems with flow-chart proofs, students might find difficulties in distinguishing whether the on-going proof is different from their previous proofs. Our system supports students to review their correct proofs. For example, the problem shown in Fig. 2 has four kinds of solutions indicated with the number of stars '☆'. If learners find correct solutions, the stars corresponding to these solutions are filled in yellow with an icon that shows the used congruent triangle condition. In the case illustrated (Fig. 4), the learners have already found two solutions with different conditions. As part of their search for the two remaining solutions, they can review their previous answers by clicking each yellow star.

In the process of proving, there can occur different kinds of errors. Students may notice some errors, but not others, and it is necessary to give support for this, e.g. by providing feedback. However, it is not an easy task to organize automated feedback from the computer. In fact, this was an issue for Cabri-Euclid -- Luengo wrote that "during experimentations, … the erroneous message is not understood by the students because there are several errors and the system give the first that it found." (p. 26). In order to deal with such situations, our system provides systematic, user-friendly feedback during/after constructing proofs and this is perhaps one of the most important technological features of our system (for the examples of learners' use of the system's feedback, see Fujita, Jones and Miyazaki, 2018). There are four categories of feedback; Category A: errors related to hypothetical syllogism, Category B: errors related to universal instantiation, Category C: errors related to singular propositions, Category D: errors related to proof-format. Category A, B, C are based on a viewpoint of proof structure. For example, Category A incudes a logical circulation that use conclusions as assumptions (Fig. 4), Category B includes an error of choosing theorems, and Category C includes an error of choosing pairs of sides/angles. Category D includes proof-format errors, usually rather trivial ones related to singular propositions. Particularly in geometrical proof, a singular proposition concerning angles or sides should correspond to a singular proposition concerning triangle congruency.

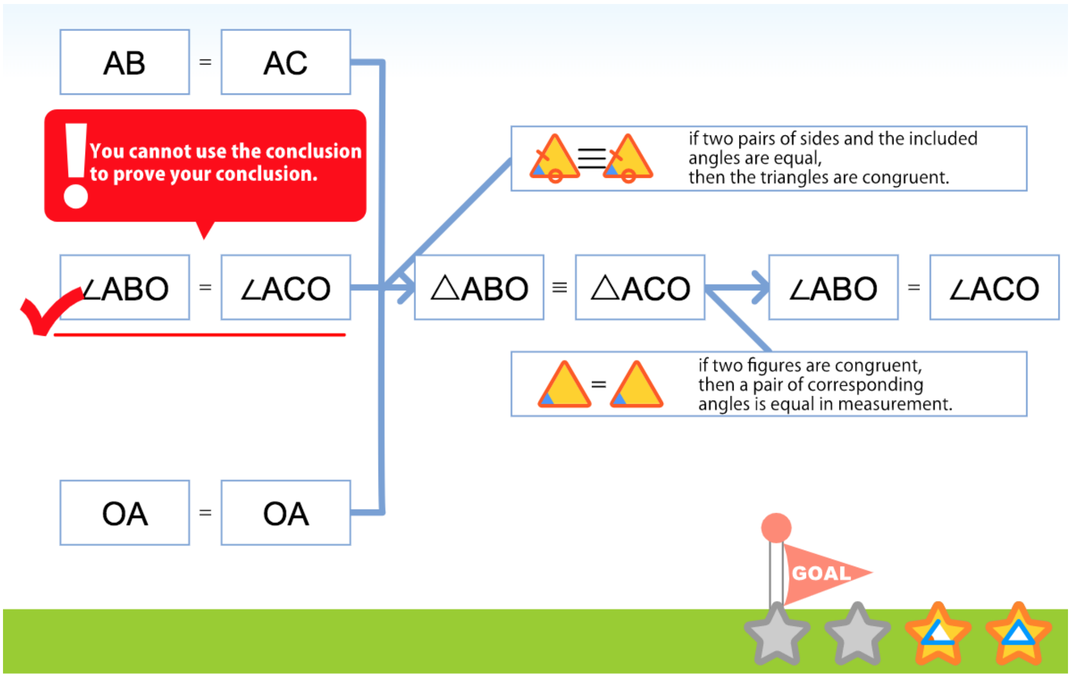



Fig. 4 Example of feedback of Category A

### 3.2.3 Process of expressing strategic knowledge of how to reconstruct proofs

In reconstructing proofs, the applied strategic knowledge is fundamentally implicit for students. Therefore, there needs an intervention that includes strategic knowledge of how to reconstruct proofs to enable them to express their ideas on changing their proofs into alternatives. Expressing the applied strategic knowledge encourages students to make explicit, and notice, their own ways of reconstructing proofs. This recognition enables them to apply their own strategic knowledge intentionally, and to expand the range of its application.

In order to realize the intervention, our approach is to devise a worksheet comprising some boxes that facilitate students expressing their ideas on changing their proofs into alternatives; see Fig. 5. Using the worksheet, students first write a proof checked by our system in ①. After finding an alternative proof by using our system, they write this proof in ②, and then express their ideas on changing their proof into an alternative in ③. The worksheet consists of a series of boxes. The number of boxes depends on how many correct proofs there are for the open problem. For example, the problem in Fig. 3 has four correct proofs so the worksheet for the problem consists of three boxes connected successively; as shown in Appendix 1.

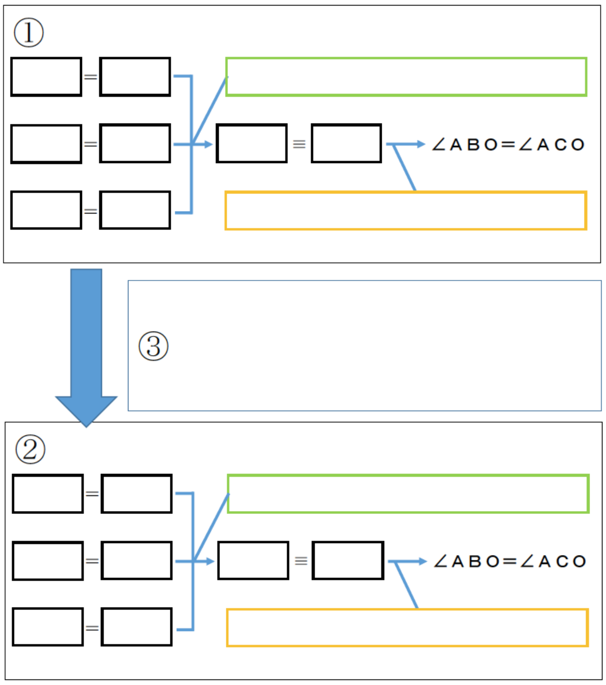



Fig. 5 worksheet used by students

## 4. Experimental lessons

### 4.1 Learning deductive proving in geometry in Japan

In Japan, deductive proof is explicitly taught in ‘Geometry’ in Grade 8. Although the Japanese national ‘Course of Study’ prescribes no official teaching sequence, some kinds of progressions can be found in the seven authorized textbooks (Fujita & Jones, 2014). Most schools follow the progressions in the textbooks. Building on geometrical reasoning in earlier grades, students in Grade 8 gradually access deductive proofs through studying properties of angles and lines, triangles, and quadrilaterals. After learning congruent triangle conditions, they reach deductive proofs through learning the structure of deductive proofs and how to construct the proofs, and then explore and prove properties of triangles and properties of quadrilaterals. Recently, explorative proving (see section 2.1) is also emphasized in teaching and assessment (Miyazaki & Fujita, 2015).

4.2 General information of classes and the plan of the experimental lessons

The experimental lessons were carried out in a grade 8 class of an attached junior high school of a national university in Japan. The lessons took place after the students had learnt properties of angles and lines, triangles, and quadrilaterals, and used these in related proofs. The class had forty students, and was taught by a teacher with more than 20 years of teaching experience. The class~~es~~ was relatively homogeneous with the most recent mathematics test scores being closely equal. Each lesson had the following instructional flow; understanding a problem (checking the conditions and the goal, reminding of what they learned, etc.), planning how to solve it (finding the ways of solving and guessing the solution), solving it based on the plan, discussing ways to solve this type of problem, summarizing ideas of how to solve this type of problem.

Four lessons (each 50 minutes long) were planned, as below. The 1st and 2nd lessons were conventional and aligned with the authorized textbooks. In contrast, the 3rd and 4th lesson were designed to develop strategic knowledge of how to reconstruct proof shown as set out above (section 3.2).

1st: Understanding the meaning of congruent figures and their properties

Students learn that congruent figures can put on top of each other, and then solve a problem to find the sides of corresponding sides/angles of congruent figures by using the properties of congruent figures.

2nd: Understanding three conditions of congruent triangles

Students explore the way of constructing congruent triangles, and then find three conditions of congruent triangles. Finally, students apply these conditions and to state them they used to find by solving a problem that requires to find congruent pairs among a lot of triangles by using.

3rd: Constructing flow-chart proof with conditions of congruent triangles

Students solve an open problem with flow-chart proofs (3.2.1) with conditions of congruent triangles that requires one step of deductive reasoning by using web-based proof learning support system (3.2.2) as they express strategic knowledge of how to reconstruct proofs on their worksheet (3.2.3).

4th: Constructing flow-chart proof with conditions of congruent triangles and properties of congruent figures

Students solve an open problem with flow-chart proofs with conditions of congruent triangles and properties of congruent figures that requires two steps of deductive reasoning by using the system as they express strategic knowledge of how to reconstruct proofs on their worksheet.

### 4.3 Data and analysis

Data from 3rd and 4th lessons that adopted our task design principles were collected by three video cameras. One recorded the whole class activity, the others recorded two students' activity individually. These students were chosen by the mathematics teacher based on daily performances and behaviors. For the purposes of this paper, we select one student, Tatsumi (all names are pseudonyms) , to exemplify emerging strategic knowledge of how to reconstruct proofs by using the web-based proof learning support system effectively. His strategic knowledge was captured by his interactions with the system and writings on his worksheet concerning his ideas of how he changed his proofs into alternatives (we refer to such ideas as 'tips').

In what follows, we analyze the fourth lesson as this aimed at students' reconstructing a flow-chart proof with conditions of congruent triangles and properties of congruent figures by using the designed task explained in section 3. The lesson devoted time to each phase of its instructional flow as follows (time in minutes); understanding a problem (checking the conditions and the goal, reminding of what they learned, etc., 1:25), planning how to solve it (finding the ways of solving and guessing the solution, 2:30), solving it individually based on the plan (finding flow-chart proof by using the system, and writing their proofs and strategic knowledge on their worksheet, 28:21), discussing in the class how to solve it (presenting some students' strategic knowledge and sharing/improving them, 10:42), summarizing ideas of how to solve this type of problem (organizing their strategic knowledge and applying the other problem, 7:33). In our analysis, we focus on the student activity related to the emergence of strategic knowledge of how to reconstruct proofs. As well as student Tsunami, we also refer to the work of other students during the 'discussion' phase of the lesson.

## 5. Analysis of the teaching experiment

### 5.1 Emerging strategic knowledge of how to reconstruct proofs

In the lesson, student Tatsumi easily constructed his proof with SSS condition by filling out all the cells of the flow-chart proof displayed in our proof learning support system, and wrote it down on his worksheet. Next, he changed "BO = CO" to "∠ABO = ∠ACO", and then changed the condition SSS to SAS. After checking this proof by the system, he also transcribed it to his worksheet, and after a while wrote his "Tips" for reconstructing proofs in ③ (see Fig. 6): "I noticed that a congruent triangle condition ought to be inserted in the green box, and also a property of congruent figures be inserted in the yellow box. So, I put a different condition for congruent

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triangles.”. In this “Tips”, he mentioned the order of the theorems used in the first proof, and the change of congruent triangle conditions. Thus, the former was triggered by the knowledge of keeping the order of theorems embedded in the previous proofs, and the latter by the knowledge of changing a theorem into another that can deduce the same property. These belong to the strategic knowledge for reconstructing proofs (SKRc).

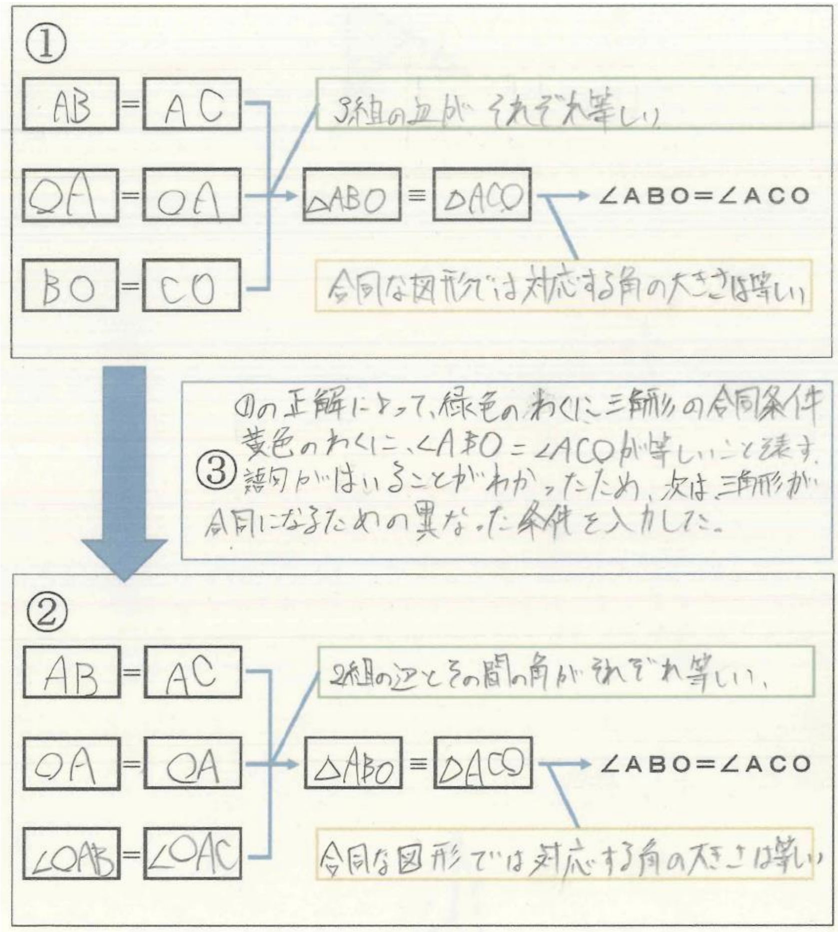


Fig. 6 Tatsumi’s description in the worksheet

In reconstructing the second proof to a third proof, and in keeping with SAS, Tatsumi changed “AB = AC” to “BO = CO” and then changed “∠ABO = ∠ACO” to “∠BOA = ∠COA”. After checking this proof using the web-based system, he also transcribed it to his worksheet, and after a while wrote his “Tips” as follows: “I made the second proof with SAS. In a pair of triangles, I noticed there were two way of using SAS, then I kept the shared side OA and chose the different sides.”. In this “Tips”, he mentioned how to change a pair of sides with keeping up SAS. Thus, this was triggered by the knowledge of changing the elements of proofs with remaining the theorem used in previous proofs. This belongs to the strategic knowledge proper to reconstructing proofs (SKRc). Notably, he pointed out there were two ways of using SAS. As he avoided using this problem’s conclusion, “∠ABO = ∠ACO”, in his proofs, this is likely to mean that he functioned with knowledge of avoiding logical circularity in his strategic knowledge for constructing proofs (SKC).

In reconstructing from his third to his fourth proof, he further changed “BO = CO” to “∠OAB = ∠OAC”, and checked his proof, then received from the system the feedback “Is this theorem OK? If you want to use it, what do you need to use?”. Based on this feedback, he realized his error, changed SAS to ASA, and then he completed all four correct proofs.

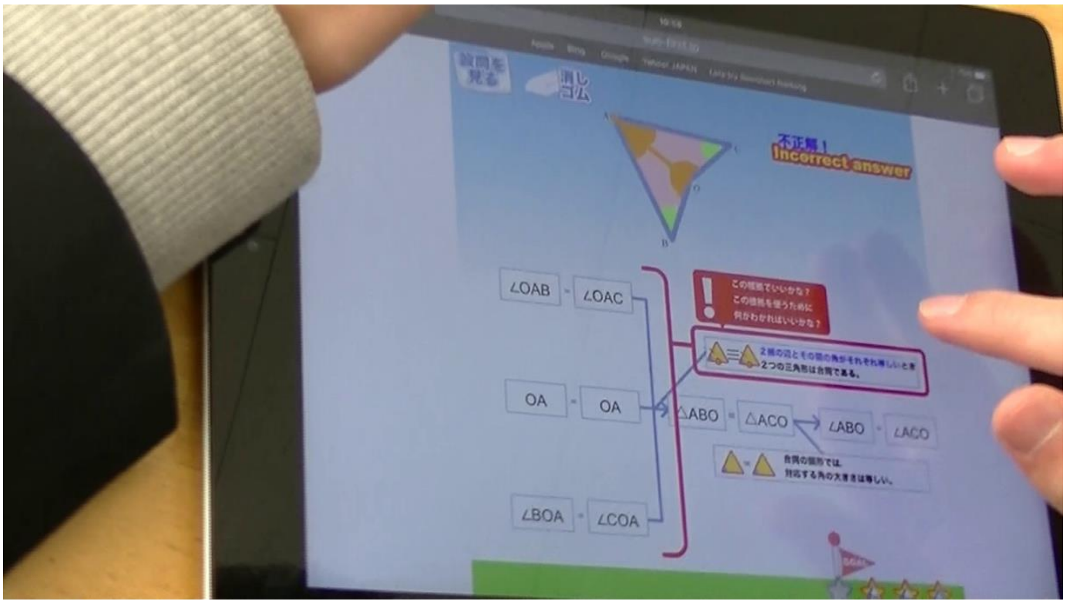


Fig. 7 System’s feedback for using congruent triangle conditions appropriately

After writing his fourth proof on his worksheet, he wrote his “Tips”: “I chose the condition that I didn’t use yet. The shared side OA is necessary to make proofs, then I chose the pairs of angles that included the side OA.”. In this “Tips”, he mentioned changing congruent triangle conditions and changing elements of proofs based on the givens of the problem. Thus, the former was triggered by the knowledge of changing a theorem into another that can deduce the same conclusion. This belongs to the strategic knowledge for reconstructing proofs (SKRc). Moreover, the latter was triggered by the knowledge of using theorems according to the conditions of the problem to be solved. This belongs to the strategic knowledge for solving the proof problem (SKSP).

## 5.2 Sharing strategic knowledge of how to reconstruct proofs in class

At the start of the phase of the lesson ‘discussing ways to solve this type of problem’, the class teacher asked four students to write on the blackboard their proof to the flow-chart problem. The teacher explained that these four proofs, shown in Table 1, were typical among the students in the class, and encouraged the class to focus on the differences between the proofs (minutes 35:01 - 36:14).

Table 1 Proofs on the blackboard chose by Teacher

| Proof | Assumption | | | | Conclusion |
|---|---|---|---|---|---|
| 1st | OA = OA<br>AB = AC<br>BO = CO | SSS | ΔABO ≡ ΔACO | CPCTC (Angle) | ∠ABO = ∠ACO |
| 2nd | OA = OA<br>AB = AC<br>∠OAB = ∠OAC | SAS | ΔABO ≡ ΔACO | CPCTC (Angle) | ∠ABO = ∠ACO |
| 3rd | OA = OA<br>BO = CO<br>∠BOA = ∠COA | SAS | ΔABO ≡ ΔACO | CPCTC (Angle) | ∠ABO = ∠ACO |
| 4th | OA = OA<br>∠OAB = ∠OAC<br>∠BOA = ∠COA | ASA | ΔABO ≡ ΔACO | CPCTC (Angle) | ∠ABO = ∠ACO |

For example, concerning the change from the first to the second proof, the teacher pointed out the mysterious picture drawn by student Shinnosuke on the blackboard to explain his idea of changing proofs (see Fig. 8), and guided him to explain the meaning of this picture as follows (minutes 36:34 - 37:32).

T61: Shinnosuke, so I asked you to draw the 'mystery' pictures below but what are these pictures about? What were you thinking when you drew this?

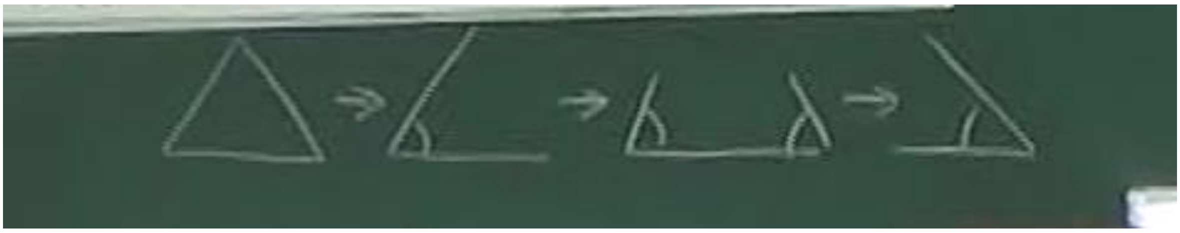

Fig. 8 Shinnosuke's mysterious picture

S105: Ah, yes, uh, I thought [my proofs] in that order.

T62: Ah, OK, that order (the four pictures).

S106: When I was working with the iPad, um, I thought this would be the least number of ways…

T63: The least number of ways… What do you mean, the least number of ways?

S107: Ah, well, that [the left SSS triangle], if I change one of the sides to angles, then I can have the second proof. Then change the side to angle, um, the third proof, and then angle to side, so the fourth one.

T64: That angles or that sides… Can you see? [saying to the other students in the class] Can you see? If you follow that [order], then we can complete with the least number of ways. The most efficient method, I would say. This way of thinking, it is interesting, isn't it? Yes? Three small claps!

As can be seen from this extract, Shinnosuke explained not only how to change the congruent triangle conditions, but also the reason why this order of change was very efficient. This suggests Shinnosuke applied his strategic knowledge of switching between theorems that can deduce the same property and of judging its efficiency. These belong to the strategic knowledge for reconstructing proofs (SKRc). After his explanation, the teacher wrote Shinnosuke's ideas on blackboard in order to record his strategic thinking, and praised him and his classmate.

Next, concerning the change from the second to the third proof, both of which use the SAS condition, the teacher asked student Yuki to explain the way and the reason of this change. Yuki explained that he chose the alternative pairs of side/angle different from the second proof because the common side OA had not been changed.

S118: OK, I have done ②, so, um, the sides OA are shared so I thought I cannot change. So I exchanged the rest of the side and angles.

This suggests that Yuki applied his strategic knowledge of changing the elements of proofs while retaining the theorem used in the previous proof. This belongs to the strategic knowledge for reconstructing proofs (SKRc). The teacher also wrote his idea on blackboard to share it in class.

In the last section of the lesson, on the change from the third to the fourth proof, the teacher asked student Narumi to explain the change, and the reason of this change. She showed her ideas as follows.

S119: Yes, um, um, as usual, OA, OA are always equal, so OA will be there. And the last one we need to prove angles ABO = ACO, so I used the other two angles.

T77: OK, I think you understood well. Do you see this (saying to the other students in the class)? Can you say it again? OK? He is going to say very important thing so please listen carefully!

S120: Um…

T78: These (OA=OA). We really need these (OA=OA).

S121: And, uh, the last thing I want to say is, uh, angles ABO=ACO, so, uh, of these three pairs of angles, I thought the other pairs than ABO=ACO.

T79: You thought about. That sounds very good (the others started clapping their hands). OK, angles ABO and ACO, and then… And then what? I forgot (laugh). What was it? What was the last thing you wanted to say?

S122: Equal.

T80: Yes, we want to say they are equal.

As above, Naomi pointed out that she did not choose "∠ABO = ∠ACO" because this is a conclusion to be proven and cannot be used as an assumption. This suggests that Naomi applied her strategic knowledge of avoiding logical circularity that belongs to strategic knowledge for constructing proofs (SKC). The teacher wrote Naomi's ideas that "We want to conclude angles ABO = ACO, so we cannot use this. Therefore, we have to find the other pairs of angles." on the blackboard, and praised her strategic thinking. Fig. 9 shows how the blackboard was used during the 4th lesson.

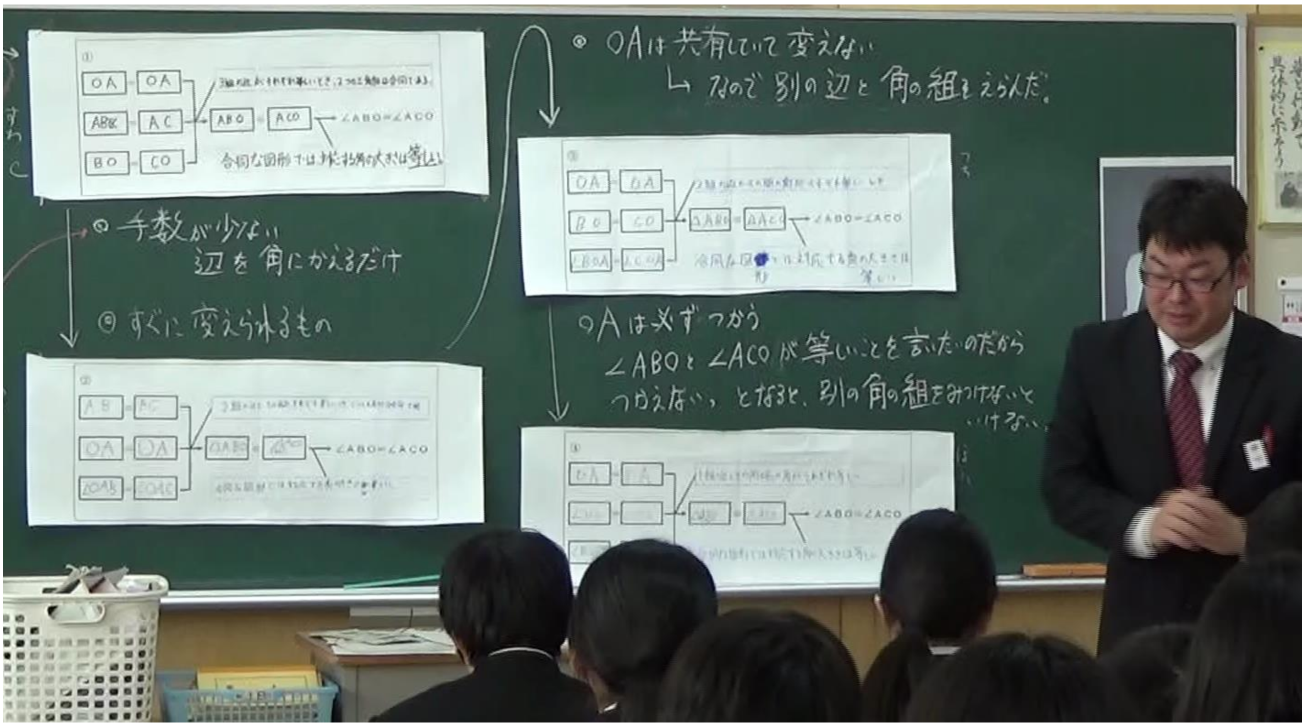


Fig. 9 Use of the blackboard in the 4th lesson (photo used with permission)

By analyzing the data from our teaching experiment, we identified the students' strategic knowledge of how to construct alternative proofs to the same problem. In the case of student Tatsumi, as in the analysis above, we identified the strategic knowledge that emerged during the classroom activity.

In doing so, we found that we could exemplify the three categories of strategic knowledge that we set out in section 2.2, viz:

- Strategic knowledge for constructing proofs (SKC)
  - ➢ Avoiding logical circularity (Tatsumi, Naomi)

- Strategic knowledge for reconstructing proofs (SKRc)
  - ➢ Keeping the order of theorems embedded in the previous proofs (Tatsumi)
  - ➢ Changing a theorem into another that can deduce the same property (Tatsumi)
  - ➢ Changing the elements of proofs with remaining the theorem used in previous proofs (Tatsumi, Yuki)
  - ➢ Switching between theorems that can deduce the same property (Shinnosuke)
  - ➢ Judging the efficiency of switching between theorems (Shinnosuke)
- Strategic knowledge appropriate for solving the proof problem (SKSP)
  - ➢ Using theorems according to the conditions of the problem to be solved (Tatsumi)

## 6. Discussion

Most of the examples of strategic knowledge listed immediately above echo the characteristics of geometrical proof based on triangle congruency. For example, switching between theorems that can be used to deduce the same property strongly relates to the three conditions of congruent triangles. This type of knowledge belongs to the "knowledge of the domain's proof techniques" that Weber (2001, p. 111) hypothesized as the type of strategic knowledge that undergraduates appear to lack.

In order for this strategic knowledge to emerge, it is necessary for students to notice it, and apply it, in the context of proof problem solving. Additionally, by expressing the knowledge students clarify and enrich their own thinking as well as noticing and applying it objectively. Moreover, the designed task to develop strategic knowledge of how to reconstruct proofs consists of three components described in section 3. These components and their interactions could contribute to students noticing, applying, and expressing their strategic knowledge.

In fact, Tatsumi firstly reconstructed his proof, and then could use his strategic knowledge that he had applied to reconstructing it as shown in 5.1. This activity was especially encouraged by open problem with flow-chart proofs and technologies of web-based proof learning support system. As shown above (section 3.2.1), open problem with flow-chart proofs encourage students to construct multiple solutions by supporting them when deciding the assumptions and intermediate propositions. Similar to the activities illustrated in our previous studies (e.g. Miyazaki, Fujita & Jones, 2015), the open problems with flow-chart proofs encouraged Tatsumi, on the one hand, to construct his proof, and reconstruct another based on the previous by switching the congruent triangle conditions and the properties of congruent figures, and changing the pairs of sides/angles. On the other hand, in every trail of reconstructing his proofs, he checked his proof by using the system and gained confidence to progress his proof solving. Additionally, he could take into account suggestions of how to resolve his error informed by the feedback of the system (e.g. "Is this theorem OK? If you want to use it, what do you need to use?"(see 5.2). Hence, the technologies of our system also encouraged him to reconstruct his proofs by providing systematic feedback according to his errors or mistakes, accompanied by offering a useful interface to switch between the elements of proof and suggesting whether his proof was correct or not.

Moreover, Tatsumi was encouraged to express his strategic knowledge by writing his ideas on the worksheet composed of three boxes (see Fig. 6) as shown in section 5.1. Similarly, in analysis of the part of the 4th lesson on discussing how to solve the problem in the class (as shown in section 5.2), students were also inspired to pay attention to the ideas of how to reconstruct proofs, rather than the four proofs whose correctness was certificated by the system. Thus, the process of expressing strategic knowledge of how to reconstruct proofs contributed to formulating these students' strategic knowledge.

Concerning using technology, the system enables students to tackle open problem with flow-chart proofs efficiently and enhance their understanding of proof structure (Miyazaki, Fujita & Jones, 2017), informed by the systematic feedback (Fujita, Jones & Miyazaki, 2018) that is more user-friendly than earlier systems such as Cabri-Euclid (Luengo, 2005). As a result, students could begin to build their strategic knowledge of 'changing the elements of proofs whilst retaining the theorem used in previous proofs' by understanding universal instantiation as well as the knowledge related to the elements of proof. Also in terms of proof structure, students encountered the strategic knowledge 'avoiding logical circularity' on the whole structure of proof. Moreover, the system enabled students to recognize a proof as a group of 'modules' (Weber & Mejia-Ramos, 2011; Mejia-Ramos, et al., 2012). In the case of geometrical proofs by using triangle congruency, most proofs have the same groups of 'modules' that combines a congruent triangle condition with a property of congruent figures. Thus, the strategic knowledge of 'keeping the order of theorems embedded in the previous proofs' is certainly a fundamental key to reconstructing proofs, and most students are implicitly getting to apply this knowledge by their experience of proof construction. By using these technological advantages, open problems with flow-chart proofs came to be more effective for students to reconstruct proofs.

However, a teacher's instruction is another important factor (Parero and Aldon, 2016). In the 4th lesson, the teacher's instructions for students on how to find their proofs by using the system, and how to write their ideas clearly, took a role of organizing the process of expressing the strategic knowledge. Particularly, the instruction about looking back on how to think in changing their proofs was essential to expressing this knowledge. Moreover, the teacher praised Shinnosuke's idea concerning judging the efficiency of switching between theorems and Naomi's idea concerning avoiding logical circularity as shown in 5.2. These teacher's actions suggest what to be expressed as the strategic knowledge to reconstruct proofs.

## 7. Conclusion

This study explored how a proving task with technology can be designed to develop strategic knowledge of how to construct alternative proofs to the same problem, and how the designed task enriched learners' strategic knowledge in proving in the context of geometrical proof. For the former, the task we designed had three components; open problem with flow-chart proofs, learning environment with the web-based support system, and process of expressing strategic knowledge of how to reconstruct proofs. For the latter, through the observation of individual student activity, and the part of the 4th lesson on 'discussing ways to solve this type of problem', our analysis shows how these components, and their interactions encouraged students to notice, apply, and express this strategic knowledge.

The adoption of this designed task in the proof lessons was found to encourage students to carry out the process of explorative proving accompanied with reconstructing proofs as necessary. What is more, the way that the tasks were designed can contribute to advancing research on task design. Especially, concerning the role of technology, the support system can make open problems with flow-chart proofs more effective for students to reconstruct proofs, and promote the process of expressing strategic knowledge of how to reconstruct proofs. Nevertheless, a teacher's appropriate involvement and participation can help to ensure that students acquire this strategic knowledge. It shows the necessity of refining this design task in cooperation with teachers (Jones and Pepin, 2016) and designing teaching to organize students' activities in the most productive way.

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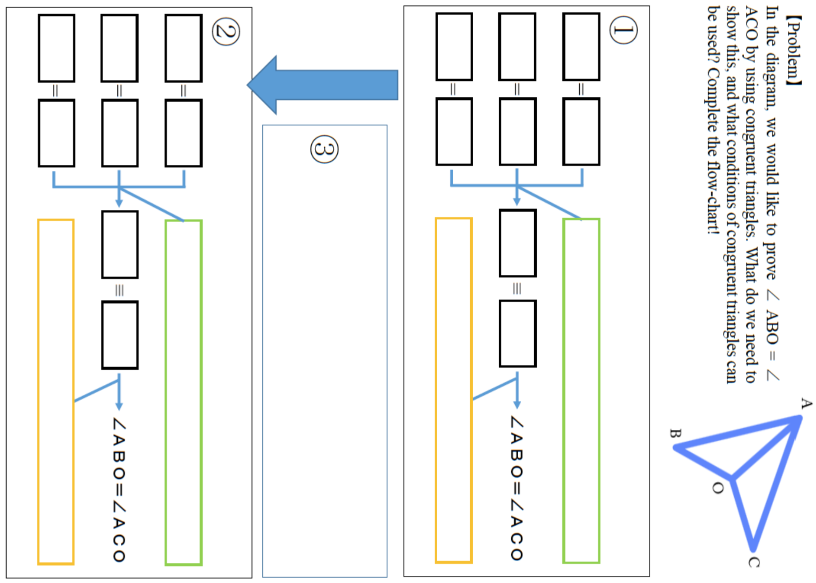
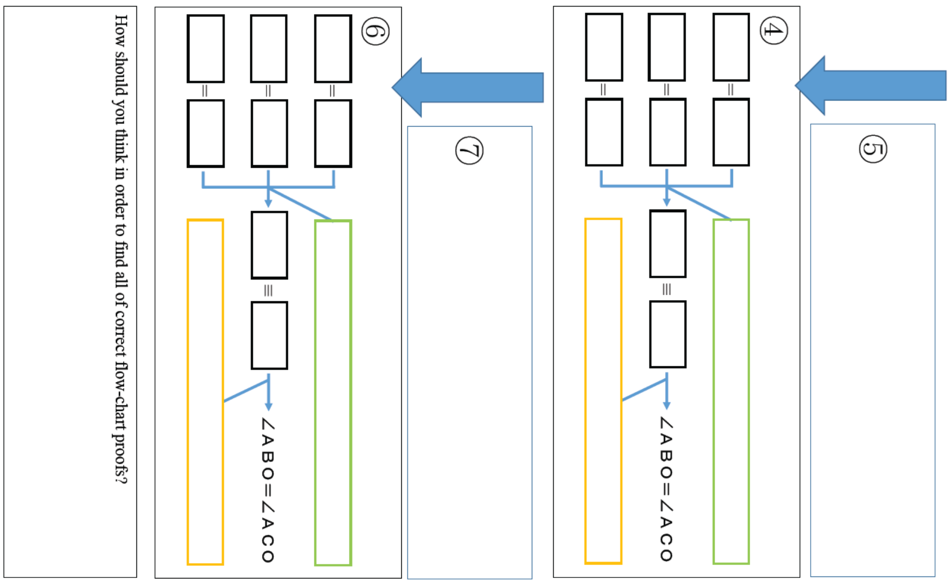
Appendix 1: Worksheet of Lesson 4 to express ideas on reconstructing proofs



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# Appendix 2: Post-test after the 4th lesson