On reciprocal systems and controllability

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Abstract

In this paper, we extend classical results on (i) signature symmetric realizations, and (ii) signature symmetric and passive realizations, to systems which need not be controllable. These results are motivated in part by the existence of important electrical networks, such as the famous Bott-Duffin networks, which possess signature symmetric and passive realizations that are uncontrollable. In this regard, we provide necessary and sufficient algebraic conditions for a behavior to be realized as the driving-point behavior of an electrical network comprising resistors, inductors, capacitors and transformers.

Key words: Reciprocity; Passive system; Linear system; Controllability; Observability; Behaviors; Electrical networks.

1 Introduction

This paper is concerned with reciprocal systems (see, e.g., Casimir, 1963; Willems, 1972; Anderson and Vongpanitlerd, 2006; Newcomb, 1966; van der Schaft, 2011). Reciprocity is an important form of symmetry in physical systems which arises in acoustics (Rayleigh-Carson reciprocity); elasticity (the Maxwell-Betti reciprocal work theorem); electrostatics (Green's reciprocity); and electromagnetics (Lorentz reciprocity), where it follows as a result of Maxwell's laws (Newcomb, 1966, p. 43). Special cases of reciprocal systems include reversible systems, as arise in thermodynamics; and relaxation systems, such as viscoelastic materials (Willems, 1972). In addition, reciprocity is a property of important classes of electrical, mechanical and structural systems, such as lightly damped flexible structures (Ferrante and Ntogramatzidis, 2013). Our focus in this paper is on linear reciprocal systems. In contemporary systems and control theory, a linear reciprocal system is typically defined as a system with a symmetric transfer function. A fundamental result in systems and control theory states that if the transfer function is also proper, then the system possesses a so-called *signature symmetric realization* (see Willems, 1972; Anderson and Vongpanitlerd, 2006; Fuhrmann, 1983; Youla and Tissi, 1966). However, this result is subject to one notable caveat: the system is assumed to be controllable.

Practical motivation for developing a theory of reciprocity that does not assume controllability arises from electrical networks. Notably, the driving-point behavior of an electrical network comprising resistors, inductors, capacitors and transformers (an RLCT network) is necessarily reciprocal, and also passive, [2] but it need not be controllable (see Çamlibel et al., 2003; Willems, 2004; Hughes, 2017d). Indeed, as noted by Çamlibel et al. (2003), it is not known what (uncontrollable) behaviors can be realized as the driving-point behavior of an RLCT network. In addition, an RLCT network need not possess an impedance function, so the conventional definition of a reciprocal system as one with a symmetric transfer function is inappropriate for such networks.

The purpose of this paper is to address the aforementioned limitations with the theory of reciprocity. The paper is structured as follows. In Section 3, we review the classical theory of reciprocal systems, to highlight the limitations of the existing theory and the contributions of this paper. The main results follow in Sections 4–7, and are summarised in the next two paragraphs.

In Section 4, we provide a formal definition of reciprocity (Definition 5), which was first proposed by Newcomb (1966). The main advantage of this definition is that it does not assume the existence of a symmetric transfer function. This is particularly fitting for electrical networks as these need not possess an impedance function. We then provide a 2-part theorem which we call the *reciprocal behavior theorem*. In part 1 (Theorem 7), we provide necessary and sufficient conditions

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[2] A system is passive if the net energy that can be extracted from the system into the future is bounded above (this bound depending only on the past trajectory of the system).

for a system to be reciprocal in terms of the differential equations describing the system. We also prove that, for any given reciprocal system, a simple transformation of the system's variables yields a reciprocal system with a proper symmetric transfer function. Part 2 (Theorem 9) then proves the existence of a signature symmetric realization for any given system with a proper symmetric transfer function (irrespective of controllability).

Section 6 contains another 2-part theorem: the *passive and reciprocal behavior theorem*. Part 1 (Theorem 17) provides necessary and sufficient algebraic conditions for a system to be passive *and* reciprocal in terms of the differential equations describing the system. This theorem also answers the first open problem posed in Çamlibel et al. (2003) in the more general setting of multi-port networks: it shows that a behavior $\mathcal{B}$ is realizable as the driving-point behavior of an RLCT network if and only if $\mathcal{B}$ is passive and reciprocal. Part 2 (Theorem 18) then proves the existence of a passive and signature symmetric realization for any given passive system with a proper symmetric transfer function. The results in this section build on earlier results in (Hughes, 2017c,a) on systems which are passive but not necessarily reciprocal. The extension to consider passive *and* reciprocal systems is by no means trivial, and depends on a number of supplementary lemmas that are provided in Section 7 and Appendix B. Finally, the proofs in the paper, together with existing results in the literature, provide an algorithm for constructing an RLCT network realization of an arbitrary given reciprocal and passive behavior. This is illustrated by two examples in Section 8.

2 Notation and Preliminaries

We denote the real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, and the open and closed right-half plane by $\mathbb{C}_+$ and $\overline{\mathbb{C}}_+$. If $\lambda \in \mathbb{C}$, then $\bar{\lambda}$ denotes its complex conjugate. The polynomials, rational functions, and proper (i.e., bounded at infinity) rational functions in the indeterminate ξ with real coefficients are denoted $\mathbb{R}[\xi], \mathbb{R}(\xi)$, and $\mathbb{R}_p(\xi)$. The $m \times n$ matrices with entries from $\mathbb{R}$ (resp., $\mathbb{R}[\xi]$, $\mathbb{R}(\xi)$, $\mathbb{R}_p(\xi)$) are denoted $\mathbb{R}^{m\times n}$ (resp., $\mathbb{R}^{m\times n}[\xi]$, $\mathbb{R}^{m\times n}(\xi)$, $\mathbb{R}_p^{m\times n}(\xi)$), and n is omitted if $n = 1$. We denote the block column and block diagonal matrices with entries $H_1, \ldots, H_n$ by $\text{col}(H_1 \ \cdots \ H_n)$ and $\text{diag}(H_1 \ \cdots \ H_n)$, and we will use horizontal and vertical lines to indicate the partition in block matrix equations (e.g., see (B.5)). If $H \in \mathbb{R}^{m\times n}, \mathbb{R}^{m\times n}[\xi]$, or $\mathbb{R}^{m\times n}(\xi)$, then H^T denotes its transpose, and if H is nonsingular (i.e., $\det(H) \neq 0$) then H^{-1} denotes its inverse. If $H \in \mathbb{R}^{m\times n}$, then $\text{rank}(H)$ denotes its rank; and if $G \in \mathbb{R}^{m\times n}(\xi)$, then $\text{normalrank}(G) := \max_{\lambda\in\mathbb{C}}(\text{rank}(G(\lambda)))$. If $M \in \mathbb{R}^{m\times m}$, then $\text{spec}(M) := \{\lambda \in \mathbb{C} \mid \det(\lambda I - M) = 0\}$; and if, in addition, M is symmetric, then $M > 0$ ($M \geq 0$) indicates that M is positive (non-negative) definite. A matrix $\Sigma \in \mathbb{R}^{n\times n}$ is called a signature matrix if it is diagonal and all of its entries are either 1 or -1. A $V \in \mathbb{R}^{n\times n}[\xi]$ is called unimodular if $\det(V)$ is a non-zero constant (equivalently, V is nonsingular with $V^{-1} \in \mathbb{R}^{n\times n}[\xi]$). If $H \in \mathbb{R}^{n\times n}(\xi)$, then H is called positive-real if H is analytic in $\mathbb{C}_+$ and $H(\bar{\lambda})^T + H(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$.

The (k-vector-valued) locally integrable functions are denoted $\mathcal{L}_1^{\text{loc}}\left(\mathbb{R}, \mathbb{R}^k\right)$ (Polderman and Willems, 1998, Defs. 2.3.3, 2.3.4), and we equate any two locally integrable functions that differ only on a set of measure zero. The (k-vector-valued) infinitely differentiable functions with bounded support on the left (resp., bounded support on the right, bounded support) are denoted $\mathcal{D}_+\left(\mathbb{R}, \mathbb{R}^k\right)$ (resp., $\mathcal{D}_-\left(\mathbb{R}, \mathbb{R}^k\right)$, $\mathcal{D}\left(\mathbb{R}, \mathbb{R}^k\right)$). The convolution operator is denoted by $\star$; i.e., if $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{D}_+\left(\mathbb{R}, \mathbb{R}^k\right)$, then $(\mathbf{w}_1 \star \mathbf{w}_2)(t) = \int_{-\infty}^{\infty} \mathbf{w}_1(\tau)^T \mathbf{w}_2(t-\tau) d\tau$.

A main contribution of this paper is to develop a theory of reciprocal systems which doesn't assume controllability, observability, or the existence of a transfer function. This is relevant to electric networks which can possess uncontrollable or unobservable internal modes, and whose driving-point currents and voltages need not adhere to the conventional system theoretic input-output view. The natural framework to formalise these issues is the behavioral approach (Polderman and Willems, 1998). Accordingly, the remainder of this section contains relevant definitions and results on behaviors.

We consider behaviors (systems) defined as the set of weak solutions (see Polderman and Willems, 1998, Sec. 2.3.2) to a differential equation:

$$\mathcal{B} = \{\mathbf{w} \in \mathcal{L}_1^{\text{loc}}\left(\mathbb{R}, \mathbb{R}^q\right) \mid R(\tfrac{d}{dt})\mathbf{w} = 0\}, \ R \in \mathbb{R}^{p\times q}[\xi]. \quad (2.1)$$

The behavior $\mathcal{B}$ is called *controllable* if, for any two trajectories $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{B}$ and $t_0 \in \mathbb{R}$, there exists $\mathbf{w} \in \mathcal{B}$ and $t_1 \geq t_0$ such that $\mathbf{w}(t) = \mathbf{w}_1(t)$ for all $t \leq t_0$ and $\mathbf{w}(t) = \mathbf{w}_2(t)$ for all $t \geq t_1$ (Polderman and Willems, 1998, Def. 5.2.2). From (Polderman and Willems, 1998, Th. 5.2.10), $\mathcal{B}$ in (2.1) is controllable if and only if $\text{rank}(R(\lambda))$ is the same for all $\lambda \in \mathbb{C}$.

We pay particular attention to state-space systems:

$$\begin{aligned}\mathcal{B}_s = \{(\mathbf{u}, \mathbf{y}, \mathbf{x}) \in \mathcal{L}_1^{\text{loc}}\left(\mathbb{R}, \mathbb{R}^n\right) \times \mathcal{L}_1^{\text{loc}}\left(\mathbb{R}, \mathbb{R}^n\right) \times \mathcal{L}_1^{\text{loc}}\left(\mathbb{R}, \mathbb{R}^d\right) \mid \\ \tfrac{d\mathbf{x}}{dt} = A\mathbf{x} + B\mathbf{u} \text{ and } \mathbf{y} = C\mathbf{x} + D\mathbf{u}\}, \\ A \in \mathbb{R}^{d\times d}, B \in \mathbb{R}^{d\times n}, C \in \mathbb{R}^{n\times d} \text{ and } D \in \mathbb{R}^{n\times n}. \quad (2.2)\end{aligned}$$

Here, we call (A, B) *controllable* if $\mathcal{B}_s$ is controllable; and we call (C, A) *observable* if $(\mathbf{u}, \mathbf{y}, \mathbf{x}) \in \mathcal{B}_s$ and $(\mathbf{u}, \mathbf{y}, \hat{\mathbf{x}}) \in \mathcal{B}_s$ imply $\mathbf{x} = \hat{\mathbf{x}}$ (Polderman and Willems, 1998, Def. 5.3.2). These concepts are equivalent to the well known algebraic controllability/observability conditions (Polderman and Willems, 1998, Ch. 5).

We also consider behaviors obtained by transforming and/or eliminating variables in a behavior $\mathcal{B}$ as in (2.1). For example, associated with the state-space system $\mathcal{B}_s$ in (2.2) is the corresponding external behavior $\mathcal{B}_s^{(\mathbf{u},\mathbf{y})} = \{(\mathbf{u}, \mathbf{y}) \mid \exists \mathbf{x} \text{ with } (\mathbf{u}, \mathbf{y}, \mathbf{x}) \in \mathcal{B}_s\}$. More generally, if $T_1 \in \mathbb{R}^{p_1\times q}, \ldots, T_n \in \mathbb{R}^{p_n\times q}$ are such that $\text{col}(T_1 \ \cdots \ T_n) \in \mathbb{R}^{q\times q}$ is a nonsingular real matrix, and m is an integer satisfying $1 \leq m \leq n$, then we denote the projection of

$\mathcal{B}$ onto $T_1\mathbf{w}, \ldots, T_m\mathbf{w}$ by

$$\mathcal{B}^{(T_1\mathbf{w},\ldots,T_m\mathbf{w})} = \{(T_1\mathbf{w},\ldots,T_m\mathbf{w}) \mid \exists (T_{m+1}\mathbf{w},\ldots,T_n\mathbf{w}) \text{ such that } \mathbf{w} \in \mathcal{B}\}.$$

A representation for the behavior $\mathcal{B}^{(T_1\mathbf{w},\ldots,T_m\mathbf{w})}$ can be obtained by the so-called elimination theorem (see Appendix A). In particular, by eliminating the state variables $\mathbf{x}$ from $\mathcal{B}_s$, we obtain a behavior of the form

$$\hat{\mathcal{B}}=\{(\mathbf{u},\mathbf{y})\in\mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^n)\times\mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^n) \mid \hat{P}(\tfrac{d}{dt})\mathbf{u}=\hat{Q}(\tfrac{d}{dt})\mathbf{y}\},$$
$$\hat{P},\hat{Q}\in\mathbb{R}^{n\times n}[\xi], \hat{Q} \text{ nonsingular and } \hat{Q}^{-1}\hat{P} \text{ proper.} \quad (2.3)$$

More specifically, from (Hughes, 2016, Secs. 2 and 4) we have the following lemma on behavioral realizations.

Lemma 1 *Let $\mathcal{B}_s$ be as in (2.2) and $\mathcal{A}(\xi):=\xi I - A$. There exist polynomial matrices $\hat{P},\hat{Q},Y,Z,U,V,E,F,G$ where*

1. $\begin{bmatrix} Y & Z \\ U & V \end{bmatrix} \begin{bmatrix} -D & I & -C \\ -B & 0 & \mathcal{A} \end{bmatrix} = \begin{bmatrix} -\hat{P} & \hat{Q} & 0 \\ -E & -F & G \end{bmatrix}$;
2. $\begin{bmatrix} Y & Z \\ U & V \end{bmatrix}$ *is unimodular; and*
3. *G is nonsingular (i.e., $0 \not\equiv \det(G) \in \mathbb{R}[\xi]$).*

Furthermore, if conditions 1–3 hold and $\hat{\mathcal{B}}$ is as in (2.3), then $\mathcal{B}_s^{(\mathbf{u},\mathbf{y})} = \hat{\mathcal{B}}$, and we say that (A,B,C,D) is a realization of $(\hat{P},\hat{Q})$. Also, if $\hat{\mathcal{B}}$ is as in (2.3), then there exists $\mathcal{B}_s$ as in (2.2) and polynomial matrices Y,Z,U,V,E,F and G satisfying conditions 1–3.

Remark 2 For a given behavior $\hat{\mathcal{B}}$ as in (2.3), algorithms for computing a realization (A,B,C,D) for $(\hat{P},\hat{Q})$ (i.e., a state-space system $\mathcal{B}_s$ such that $\mathcal{B}_s^{(\mathbf{u},\mathbf{y})} = \hat{\mathcal{B}}$) are described in (Fuhrmann et al., 2007, Sec. 4.7) and (Hughes, 2016, Sec. 4). Such behavioral realizations are not unique. Indeed, it is easily shown from (Hughes, 2017a, Note A.3) that $(\hat{A},\hat{B},\hat{C},\hat{D})$ is another realization for $(\hat{P},\hat{Q})$ if and only if (i) $\hat{D}+\hat{C}(\xi I-\hat{A})^{-1}\hat{B} = D + C(\xi I - A)^{-1}B$; and (ii) there exist matrices $T_1 \in \mathbb{R}^{\hat{d}\times d}$ and $T_2 \in \mathbb{R}^{d\times\hat{d}}$ such that $CA^iT_1 = \hat{C}\hat{A}^i$ for $i = 0,1,2,\ldots$, and $\hat{C}\hat{A}^kT_2 = CA^k$ for $k = 0,1,2,\ldots$. Note that the equivalence of transfer functions (condition (i)) is necessary but not sufficient. E.g., let $B = 0$, $C = 1$ and $D = 1$, so $D + C(\xi I - A)^{-1}B = 1$ for all $A \in \mathbb{R}$. If $A = -1$, then $(u,y) \in \mathcal{B}_s^{(u,y)}$ if and only if there exists $k_1 \in \mathbb{R}$ such that $y(t) = u(t) + k_1e^{-t}$. But if $A = 0$, then $(u,y) \in \mathcal{B}_s^{(u,y)}$ if and only if there exists $k_2 \in \mathbb{R}$ such that $y(t) = u(t) + k_2$.

3 Signature symmetric realizations of symmetric transfer functions

The following fundamental result in systems and control theory states that any given controllable system with a proper symmetric transfer function has a so-called *signature symmetric realization.*

Lemma 3 *Let $\hat{\mathcal{B}}$ in (2.3) be controllable. Then the following are equivalent.*

1. *$\hat{Q}^{-1}\hat{P}$ is symmetric.*
2. *There exists $\mathcal{B}_s$ as in (2.2) and a signature matrix $\Sigma_i \in \mathbb{R}^{d\times d}$ such that (i) $\hat{\mathcal{B}} = \mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$; (ii) (A,B) is controllable; (iii) (C,A) is observable; and (iv) $A\Sigma_i = \Sigma_iA^T$, $\Sigma_iC^T = B$, and $D = D^T$.*

PROOF. If $\hat{\mathcal{B}}$ in (2.3) is controllable, then there exists $\mathcal{B}_s$ as in (2.2) which satisfies (i)–(iii) in condition 2 (see Hughes, 2017c, App. D). Furthermore, $D + C(\xi I - A)^{-1}B = (\hat{Q}^{-1}\hat{P})(\xi)$. Thus, if A,B,C and D are as in condition 2 then $((\hat{Q}^{-1}\hat{P})(\xi))^T = D^T + B^T(\xi I - A^T)^{-1}C^T = D + C\Sigma_i(\xi I - \Sigma_iA^T\Sigma_i)^{-1}\Sigma_iB = D + C(\xi I - A)^{-1}B = (\hat{Q}^{-1}\hat{P})(\xi)$, so $\hat{Q}^{-1}\hat{P}$ is symmetric. This proves that $2 \Rightarrow 1$. The proof of $1 \Rightarrow 2$ then follows from (Willems, 1972, Th. 6). This proof proceeds by first showing that, if $\hat{A} \in \mathbb{R}^{d\times d}, \hat{B} \in \mathbb{R}^{d\times n}, \hat{C} \in \mathbb{R}^{n\times d}$ and $\hat{D} \in \mathbb{R}^{n\times n}$ are such that $\hat{D} + \hat{C}(\xi I - \hat{A})^{-1}\hat{B}$ is symmetric, $(\hat{A},\hat{B})$ is controllable, and $(\hat{C},\hat{A})$ is observable, then there exists a nonsingular symmetric $P \in \mathbb{R}^{d\times d}$ such that $P\hat{A} = \hat{A}^TP$, $\hat{C}^T = P\hat{B}$ and $\hat{D} = \hat{D}^T$. Note that, with the notation $\hat{V}_o = \text{col}(\hat{C}\ \hat{C}\hat{A}\ \ldots\ \hat{C}\hat{A}^{d-1})$ and $\hat{V}_c = [\hat{B}\ \hat{A}\hat{B}\ \ldots\ \hat{A}^{d-1}\hat{B}]$, then $P\hat{V}_c = \hat{V}_o^T$, whereupon P can be computed from the explicit formula $P = \hat{V}_o^T\hat{V}_c^T(\hat{V}_c\hat{V}_c^T)^{-1}$. Since P is symmetric, then there exists a signature matrix $\Sigma_i \in \mathbb{R}^{d\times d}$ and a nonsingular $T \in \mathbb{R}^{d\times d}$ such that $P = T^T\Sigma_iT$. We then let $A := T\hat{A}T^{-1}, B := T\hat{B}, C := \hat{C}T^{-1}$ and $D := \hat{D}$.

Of particular interest are controllable systems with proper symmetric transfer functions that are positive-real. These arise as the impedances of electrical networks containing resistors, inductors, capacitors and transformers (RLCT networks). In fact, every known physical system with a non-symmetric positive-real impedance actually contains active components (see Ferrante et al., 2016). A second fundamental result in systems and control theory is that any controllable system with a proper symmetric positive-real transfer function has a so-called *passive and signature symmetric realization*, in accordance with the following lemma.

Lemma 4 *Let $\hat{\mathcal{B}}$ in (2.3) be controllable. Then the following are equivalent:*

1. *$\hat{Q}^{-1}\hat{P}$ is positive-real and symmetric.*
2. *There exists $\mathcal{B}_s$ as in (2.2) and a signature matrix $\Sigma_i \in \mathbb{R}^{d\times d}$ such that (i) $\hat{\mathcal{B}} = \mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$; (ii) (A,B) is controllable; (iii) (C,A) is observable;*

   (iv) $\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} + \begin{bmatrix} -A & -B \\ C & D \end{bmatrix}^T \geq 0$*; and*

   (v) $A\Sigma_i = \Sigma_iA^T$, $\Sigma_iC^T = B$, and $D = D^T$.

PROOF. See (Willems, 1972, Th. 7).

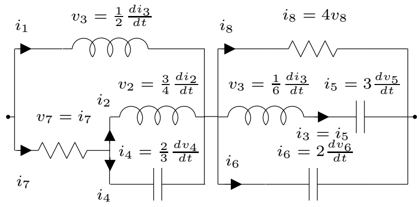

Fig. 1. Bott-Duffin realization of the driving-point behavior $(\frac{d}{dt}+1)(\frac{d^2}{dt^2}+\frac{d}{dt}+1)i = (\frac{d}{dt}+1)(\frac{d^2}{dt^2}+\frac{d}{dt}+4)v$

Using the reactance extraction approach, any realization of the form of Lemma 4 gives rise to an RLCT network whose impedance is equal to $\hat{Q}^{-1}\hat{P}$ (see Anderson and Vongpanitlerd, 2006). However, Lemma 4 contains several notable assumptions that are not satisfied by many RLCT networks. First, the theorem assumes the existence of a proper symmetric transfer function, yet not all RLCT networks possess a proper impedance (see Hughes, 2017c, Sec. 3). Second, the theorem assumes the system is controllable, but not all RLCT networks have controllable driving-point behaviors. Examples include the famous Bott-Duffin networks and their simplifications (see Hughes and Smith, 2014, 2017; Hughes, 2017d). One such network is provided in Fig. 1, whose behavior is described by the state-space realization

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -2 & 0 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & -\sqrt{3} \\ \sqrt{3} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -\sqrt{2} \\ 0 \\ 0 \\ \sqrt{\frac{3}{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} i$$

$$v = \begin{bmatrix} -\sqrt{2} & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\mathbf{x} + i, \text{ where}$$

$$\mathbf{x} = \mathrm{col}(\tfrac{v_1}{\sqrt{2}} \;\; \tfrac{\sqrt{3}}{2}v_2 \;\; \tfrac{v_3}{\sqrt{6}} \;\; \sqrt{\tfrac{2}{3}}i_4 \;\; \sqrt{3}i_5 \;\; \sqrt{2}i_6)$$

This realization satisfies conditions (iv) and (v) of Lemma 4, but is neither controllable nor observable.

The aforementioned RLCT networks indicate the importance of removing the assumptions of controllability, observability, and existence of a proper symmetric transfer function from Lemmas 3 and 4. This is the objective of this paper. Theorem 9 (resp., 18) generalizes Lemma 3 (resp., 4) to systems that need not be controllable. Also, Theorems 7 and 17 extend the results to systems that do not necessarily possess a proper symmetric transfer function. In particular, Theorem 17 provides necessary and sufficient conditions for a behavior to be realizable by an RLCT network.

To conclude this section, we discuss some recent developments in the literature on uncontrollable systems, and we contrast these with the results in the present paper. Motivation for developing a theory of reciprocity that does not assume controllability was provided in the behavioral literature in Çamlibel et al. (2003); Willems (2004). Indeed, Çamlibel et al. (2003) stated an open problem that we solve in this paper: *what behaviors are realizable as the port* (driving-point) *behavior of a circuit containing a finite number of passive resistors, capacitors, inductors and transformers*? This question concerns (not necessarily controllable) systems that are both passive and reciprocal. There have since been papers that have considered the question of uncontrollable passive systems (e.g., Hughes, 2017c), and uncontrollable (cyclo)-dissipative systems (e.g., Pal and Belur, 2008).[3] But none of these papers consider uncontrollable reciprocal systems. For example, consider the behavior $\tilde{\mathcal{B}} := \{(i,v) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}) \times \mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}) \mid (\frac{d}{dt}+1)i = (\frac{d}{dt}+1)(\frac{dv}{dt}+v)\}$. It has been shown in (Hughes, 2017c,b) that $\tilde{\mathcal{B}}$ can be realized as the driving-point behavior of an electrical network containing resistors, inductors, transformers *and gyrators* (an RLCTG network). The present paper proves that (i) this behavior has a signature symmetric realization; and (ii) it can be realized without gyrators (i.e., by an RLCT network).

In fact, as discussed by Willems (2004), the subject of uncontrollable reciprocal systems is related to a subtle yet significant question in the development of the theory of uncontrollable (cyclo)-dissipative systems: whether to allow unobservable storage functions. In particular, both Çamlibel et al. (2003) and Pal and Belur (2008) define (cyclo)-dissipativity in terms of the existence of an observable storage function. In the context of passive systems, this implies that the energy which can be extracted from a given system from the time t_0 onwards is bounded above by the value at t_0 of a differential form in the external variables.[4]

The restriction to systems with observable storage functions is not fitting for this paper, as unobservable storage functions frequently arise in electrical networks. In fact, if we consider an uncontrollable behavior with a state-space realization that satisfies the signature symmetry of condition (v) of Lemma 4, then it can be shown that this realization is both uncontrollable and unobservable (the energy that can be extracted from this system from time t_0 onwards is bounded above by $\frac{1}{2}\mathbf{x}(t_0)^T\mathbf{x}(t_0)$, yet $\mathbf{x}(t_0)$ cannot be inferred from measurements of $\mathbf{u}(t_0), \frac{d\mathbf{u}}{dt}(t_0), \frac{d\mathbf{u}}{dt}(t_0), \ldots$ and $\mathbf{y}(t_0), \frac{d\mathbf{y}}{dt}(t_0), \frac{d^2\mathbf{y}}{dt^2}(t_0), \ldots$). It can also be shown that any RLCT realization of an uncontrollable behavior necessarily has an unobservable storage function, corresponding to the energy stored in the network's inductors and capacitors. Accordingly, the approach in this paper

[3] A system is *cyclo-dissipative* if it has a (not necessarily non-negative) storage function with respect to some supply rate. It is shown in (Hughes, 2017a) that a system is passive (in the sense of Definition 13 of the present paper) if it is cyclo-dissipative with respect to the energy supplied to the system, and the associated storage function is non-negative.

[4] I.e., a function of the external variables and their derivatives of any given order at the time t_0. For an electric network, the external variables correspond to the driving-point currents and voltages, while for the state-space system $\mathcal{B}_s$ in (2.2) these variables correspond to the input $\mathbf{u}$ and output $\mathbf{y}$.

is aligned with Hughes (2017c). That paper provided a theory of passivity that does not assume controllability or observability (and removes other alternative assumptions prevalent in the literature). In contrast, this paper focuses on developing the theory of reciprocal systems.

4 Reciprocal behaviors

Following the motivation outlined in the previous sections, our focus in this paper is on systems of the form:

$$\mathcal{B}=\{(\mathbf{i},\mathbf{v})\in\mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^n)\times\mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^n) \mid P(\tfrac{d}{dt})\mathbf{i}=Q(\tfrac{d}{dt})\mathbf{v}\},$$
$$\text{with } P,Q\in\mathbb{R}^{n\times n}[\xi], \text{normalrank}([P \quad -Q])=n. \quad (4.1)$$

The driving-point behavior of any passive electrical circuit necessarily has the above form, where $\mathbf{i}$ denotes the driving-point currents and $\mathbf{v}$ the corresponding driving-point voltages (see Hughes, 2017b). We note that the partitioning $(\mathbf{i},\mathbf{v})$ need not be an input-output partition in the sense of (Polderman and Willems, 1998, Def. 3.3.1). Specifically, Q need not be nonsingular, and if Q is nonsingular then $Q^{-1}P$ need not be proper. In this more general setting, it is not possible to define a reciprocal system as a system whose transfer function is symmetric. Instead, we adopt the following definition from (Newcomb, 1966, Def. 2.7).

Definition 5 (Reciprocal system) *Let $\mathcal{B}$ be as in (4.1). $\mathcal{B}$ is called* reciprocal *if, for any given $(\mathbf{i}_a,\mathbf{v}_a),(\mathbf{i}_b,\mathbf{v}_b)\in\mathcal{B}\cap(\mathcal{D}_+(\mathbb{R},\mathbb{R}^n)\times\mathcal{D}_+(\mathbb{R},\mathbb{R}^n))$, then $\mathbf{v}_b\star\mathbf{i}_a=\mathbf{i}_b\star\mathbf{v}_a$.*

Remark 6 Our objective in this paper is to develop a concept of reciprocity that is consistent with the existence of signature symmetric realizations, and the driving-point behaviors of RLCT networks. In Lemma 11, we will define the so-called controllable and autonomous parts of a behavior ($\mathcal{B}_c$ and $\mathcal{B}_a$), and we note here that a behavior is reciprocal if and only if its controllable part ($\mathcal{B}_c$) is reciprocal. In particular, it will follow from Theorems 9 and 17 that whether a system has a signature symmetric realization depends only on its controllable part, and whether the driving-point behavior of an electric network can be realized without gyrators also depends only on its controllable part.

The next theorem shows that any given reciprocal system $\mathcal{B}$ can be transformed into a system of the form of (2.3) that is also reciprocal (condition 3 in Theorem 7). In addition, a necessary and sufficient condition for reciprocity is provided in terms of the polynomial matrices P and Q (condition 2 in Theorem 7).

Theorem 7 (Reciprocal behavior theorem, part 1) *Let $\mathcal{B}$ be as in (4.1). The following are equivalent:*

1. *$\mathcal{B}$ is reciprocal.*
2. *$PQ^T=QP^T$.*
3. *There exist real matrices $T_1\in\mathbb{R}^{r\times n}$ and $T_2\in\mathbb{R}^{(n-r)\times n}$ such that (i) $col(T_1 \; T_2)$ is a permutation matrix; and (ii) $\hat{\mathcal{B}} := \mathcal{B}^{(col(T_1\mathbf{i} \; -T_2\mathbf{v}),col(T_1\mathbf{v} \; T_2\mathbf{i}))}$ takes the form of (2.3) and $\hat{Q}^{-1}\hat{P}$ is symmetric.*

Remark 8 A well known result in behavioral theory is that any behavior $\mathcal{B}$ of the form of (4.1) necessarily has an input-output partition. However, condition 3 of Theorem 7 is not a trivial application of this result. Specifically, in the definition of a reciprocal system (Definition 5), the system's variables are partitioned into two halves (in the context of electrical networks, these two halves correspond to the driving-point currents and voltages). Condition 3 of Theorem 7 implies that if the system is reciprocal then it is possible to choose as input a subset of the variables from one of the sets together with the complementary variables from the other set. Note from the example in (Hughes, 2017c, Rem. 11) that this need not be true if the system is not reciprocal.

We will also show that the system $\hat{\mathcal{B}}$ in condition 3 of Theorem 7 has a state-space realization $\hat{\mathcal{B}}=\mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$ with the properties described in the next theorem.

Theorem 9 (Reciprocal behavior theorem, part 2) *Let $\hat{\mathcal{B}}$ be as in (2.3). Then the following are equivalent.*

1. *$\hat{\mathcal{B}}$ is reciprocal.*
2. *There exists $\mathcal{B}_s$ as in (2.2) and a signature matrix $\Sigma_i\in\mathbb{R}^{d\times d}$ such that (i) $\hat{\mathcal{B}}=\mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$; and (ii) $A\Sigma_i=\Sigma_i A^T$, $\Sigma_i C^T=B$, and $D=D^T$.*

The two-part reciprocal behavior theorem (Theorems 7 and 9) is proved in Section 5. Then, in Sections 6–7, we consider behaviors that are both reciprocal and passive.

Remark 10 We emphasise that Lemma 3 is concerned only with controllable systems, whereas Theorem 9 is applicable to any system of the form of (2.3), irrespective of controllability. Note that, if $\hat{\mathcal{B}}$ in (2.3) is not controllable, and $\mathcal{B}_s$ in (2.2) satisfies $\hat{\mathcal{B}}=\mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$, then (A,B) cannot be controllable, so Lemma 3 does not apply.

5 Reciprocity and signature symmetric realizations

The purpose of this section is to prove the reciprocal behavior theorem, parts 1 and 2 (Theorems 7 and 9). We first present the following lemma on the so-called *controllable* and *autonomous* parts of a behavior.

Lemma 11 *Let $\mathcal{B}$ be as in (4.1). The following hold:*

1. *There exist $F,\tilde{P},\tilde{Q},U,V\in\mathbb{R}^{n\times n}[\xi]$ such that*
   (i) $P=F\tilde{P}$ and $Q=F\tilde{Q}$; and
   (ii) $\begin{bmatrix}\tilde{P} & -\tilde{Q}\\ U & V\end{bmatrix}$ is unimodular.
   Also, if $F,\tilde{P},\tilde{Q},U,V\in\mathbb{R}^{n\times n}[\xi]$ satisfy (i)–(ii);
   $\mathcal{B}_c := \{(\mathbf{i},\mathbf{v})\in\mathcal{L}_1^{loc}(\mathbb{R},\mathbb{R}^n)\times\mathcal{L}_1^{loc}(\mathbb{R},\mathbb{R}^n) \mid \tilde{P}(\frac{d}{dt})\mathbf{i}=\tilde{Q}(\frac{d}{dt})\mathbf{v}\}$; and
   $\mathcal{B}_a := \{(\mathbf{i},\mathbf{v})\in\mathcal{L}_1^{loc}(\mathbb{R},\mathbb{R}^n)\times\mathcal{L}_1^{loc}(\mathbb{R},\mathbb{R}^n) \mid P(\frac{d}{dt})\mathbf{i}=Q(\frac{d}{dt})\mathbf{v}$ and $U(\frac{d}{dt})\mathbf{i}=-V(\frac{d}{dt})\mathbf{v}\}$,
   then $(\mathbf{i},\mathbf{v})\in\mathcal{B} \iff$ there exist $(\mathbf{i}_1,\mathbf{v}_1)\in\mathcal{B}_c$ and $(\mathbf{i}_2,\mathbf{v}_2)\in\mathcal{B}_a$ with $\mathbf{i}=\mathbf{i}_1+\mathbf{i}_2$ and $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$.
2. *There exist $M,N\in\mathbb{R}^{n\times n}[\xi]$ such that*
   (i) $PM=QN$; and
   (ii) $rank(col(M \; N)(\lambda))=n$ for all $\lambda\in\mathbb{C}$.
   Also, if $M,N\in\mathbb{R}^{n\times n}[\xi]$ satisfy (i)–(ii), then $(\mathbf{i},\mathbf{v})\in\mathcal{B}\cap(\mathcal{D}_+(\mathbb{R},\mathbb{R}^n)\times\mathcal{D}_+(\mathbb{R},\mathbb{R}^n))$ if and only if there

exists $\mathbf{z} \in \mathcal{D}_+(\mathbb{R}, \mathbb{R}^n)$ *such that* $\mathbf{i} = M(\frac{d}{dt})\mathbf{z}$ *and* $\mathbf{v} = N(\frac{d}{dt})\mathbf{z}$. *In particular,* $(\mathbf{i}, \mathbf{v}) \in \mathcal{B}_c$.

PROOF. This requires only minor modifications to the proof of Lemma 17 in (Hughes, 2017c).

PROOF OF THEOREM 7 (see p. 5). We let M and N be as in Lemma 11, and we will show the equivalence of conditions 1–3 to the additional condition:
4. $M^T N = N^T M$.
Specifically, we will prove $1 \iff 4 \iff 3 \iff 2$.

$1 \iff 4$. Let $(\mathbf{i}_a, \mathbf{v}_a), (\mathbf{i}_b, \mathbf{v}_b) \in \mathcal{B} \cap (\mathcal{D}_+(\mathbb{R}, \mathbb{R}^n) \times \mathcal{D}_+(\mathbb{R}, \mathbb{R}^n))$. Then, from Lemma 11, there exist $\mathbf{z}_a, \mathbf{z}_b \in \mathcal{D}_+(\mathbb{R}, \mathbb{R}^n)$ such that

$$\mathbf{i}_a = M(\tfrac{d}{dt})\mathbf{z}_a, \mathbf{i}_b = M(\tfrac{d}{dt})\mathbf{z}_b, \mathbf{v}_a = N(\tfrac{d}{dt})\mathbf{z}_a \text{ and } \mathbf{v}_b = N(\tfrac{d}{dt})\mathbf{z}_b.$$

Now, consider a fixed but arbitrary $t_0 \in \mathbb{R}$, and let

$$\hat{\mathbf{z}}_b(t) = \mathbf{z}_b(t_0 - t) \text{ for all } t \in \mathbb{R}.$$

Then $\hat{\mathbf{z}}_b \in \mathcal{D}_-(\mathbb{R}, \mathbb{R}^n)$, $\mathbf{i}_b(t_0 - \tau) = (M(-\frac{d}{dt})\hat{\mathbf{z}}_b)(\tau)$ and $\mathbf{v}_b(t_0 - \tau) = (N(-\frac{d}{dt})\hat{\mathbf{z}}_b)(\tau)$ for all $\tau \in \mathbb{R}$. Thus,

$$(\mathbf{v}_b \star \mathbf{i}_a)(t_0) = \textstyle\int_{-\infty}^{\infty} (N(-\frac{d}{dt})\hat{\mathbf{z}}_b)(\tau)^T (M(\frac{d}{dt})\mathbf{z}_a)(\tau) d\tau, \text{ and}$$
$$(\mathbf{i}_b \star \mathbf{v}_a)(t_0) = \textstyle\int_{-\infty}^{\infty} (M(-\frac{d}{dt})\hat{\mathbf{z}}_b)(\tau)^T (N(\frac{d}{dt})\mathbf{z}_a)(\tau) d\tau.$$

It follows from van der Schaft and Rapisarda (2011, Sec. 2.2) that

$$\begin{aligned}(\mathbf{v}_b \star \mathbf{i}_a - \mathbf{i}_b \star \mathbf{v}_a)(t_0) \\ = \textstyle\int_{-\infty}^{\infty} \hat{\mathbf{z}}_b(\tau)^T ((N^T M - M^T N)(\tfrac{d}{dt})\mathbf{z}_a)(\tau) d\tau.\end{aligned}$$

Since t_0 is arbitrary, then we conclude that $\mathcal{B}$ is reciprocal if and only if the above integral is zero for all $\mathbf{z}_a \in \mathcal{D}_+(\mathbb{R}, \mathbb{R}^n)$ and $\hat{\mathbf{z}}_b \in \mathcal{D}_-(\mathbb{R}, \mathbb{R}^n)$. In particular, if $N^T M = M^T N$, then $\mathcal{B}$ is reciprocal. Conversely, note that if the above integral is zero for all $\mathbf{z}_a \in \mathcal{D}_+(\mathbb{R}, \mathbb{R}^n)$ and $\hat{\mathbf{z}}_b \in \mathcal{D}_-(\mathbb{R}, \mathbb{R}^n)$, then $(N^T M - M^T N)(\frac{d}{dt})\mathbf{z}_a \equiv 0$ for all $\mathbf{z}_a \in \mathcal{D}(\mathbb{R}, \mathbb{R}^n)$ (since otherwise the integral is strictly positive with $\hat{\mathbf{z}}_b = (N^T M - M^T N)(\frac{d}{dt})\mathbf{z}_a$). It then follows from (Polderman and Willems, 1998, Secs. 2.5.6 and 3.3) that $N^T M = M^T N$.

$4 \Rightarrow 3$. First, bring $\text{col}(M \;\; N)$ into column proper form. In other words, let U be a unimodular matrix with $\text{col}[M \;\; N]U =: W$, in which the leading coefficient matrix W^L of W has full column rank (see Wolovich, 1974, Sec. 2.5). Next, partition W^L compatibly with $\text{col}(M \;\; N)$ as $W^L = \text{col}(W_1^L \;\; W_2^L)$, let r denote the rank of W_1^L, permute the columns of W_1^L so the first r columns are linearly independent, and then permute the rows so the first r rows are linearly independent. This gives permutation matrices $T = \text{col}(T_1 \;\; T_2) \in \mathbb{R}^{n \times n}$ and $S = [S_1 \;\; S_2] \in \mathbb{R}^{n \times n}$ and an $X \in \mathbb{R}^{r \times (n-r)}$ such that

$$\begin{bmatrix} \hat{M} \\ \hat{N} \end{bmatrix} = \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \\ \hline \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{bmatrix} := \begin{bmatrix} T_1 & 0 \\ 0 & -T_2 \\ \hline 0 & T_1 \\ T_2 & 0 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} U \begin{bmatrix} S_1 & S_2 \end{bmatrix}$$

is in column proper form, and its leading coefficient matrix $\text{col}(\hat{M}^L \;\; \hat{N}^L)$ takes the form

$$\begin{bmatrix} \hat{M}^L \\ \hat{N}^L \end{bmatrix} = \begin{bmatrix} \hat{M}_{11}^L & \hat{M}_{11}^L X \\ \hat{M}_{21}^L & \hat{M}_{22}^L \\ \hline \hat{N}_{11}^L & \hat{N}_{12}^L \\ \hat{N}_{21}^L & \hat{N}_{21}^L X \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & -T_2 \\ \hline 0 & T_1 \\ T_2 & 0 \end{bmatrix} \begin{bmatrix} W_1^L \\ W_2^L \end{bmatrix} \begin{bmatrix} S_1 & S_2 \end{bmatrix}$$

where $\hat{M}_{11}^L$ is nonsingular. It is then easily verified that $\hat{M}^T \hat{N} - \hat{N}^T \hat{M} = (US)^T (M^T N - N^T M)(US) = 0$. We will show that $\hat{M}^L$ is nonsingular, and it follows that $\hat{N}\hat{M}^{-1}$ is proper (see Rapisarda and Willems, 1997, Sec. 2). We then let $\hat{P} := [PT_1^T \;\; QT_2^T]$ and $\hat{Q} := [QT_1^T \;\; -PT_2^T]$, we recall that $PM = QN$, and we find that $\hat{P}\hat{M} = \hat{Q}\hat{N}$. This implies that $\hat{Q}$ is nonsingular with $\hat{Q}^{-1}\hat{P} = \hat{N}\hat{M}^{-1}$, which is symmetric since $\hat{N}\hat{M}^{-1} = (\hat{M}^{-1})^T \hat{M}^T \hat{N} \hat{M}^{-1} = (\hat{M}^{-1})^T \hat{N}^T \hat{M} \hat{M}^{-1} = (\hat{M}^{-1})^T \hat{N}^T$. Finally, with $\mathbf{i}_1 := T_1 \mathbf{i}, \mathbf{v}_1 := T_1 \mathbf{v}, \mathbf{i}_2 := T_2 \mathbf{i}$ and $\mathbf{v}_2 := T_2 \mathbf{v}$, then it is easily shown that $\hat{\mathcal{B}}$ takes the form indicated in the present theorem statement.

To complete the proof of the present implication, it remains to show that $\hat{M}^L$ is nonsingular, or equivalently that $\mathbf{z} \in \mathbb{R}^n$ and $\hat{M}^L \mathbf{z} = 0$ imply $\mathbf{z} = 0$. To see this, we denote the column degree of the jth column of $\text{col}(\hat{M} \;\; \hat{N})$ by $\hat{d}_j$, and we note that the entry in the ith row and jth column of $(\hat{M}^L)^T \hat{N}^L - (\hat{N}^L)^T \hat{M}^L$ is the coefficient of $\xi^{\hat{d}_i + \hat{d}_j}$ in the corresponding entry of $\hat{M}^T \hat{N} - \hat{N}^T \hat{M}$, which is necessarily zero. Now, let $\mathbf{z} \in \mathbb{R}^n$ satisfy $\hat{M}^L \mathbf{z} = 0$. Then $\hat{M}_{11}^L [I \;\; X]\mathbf{z} = 0$. Since $\hat{M}_{11}^L$ is nonsingular, it follows that $[I \;\; X]\mathbf{z} = 0$. Since, in addition, $(\hat{M}^L)^T \hat{N}^L - (\hat{N}^L)^T \hat{M}^L = 0$, then

$$\begin{aligned} 0 &= [I \;\; 0]((\hat{M}^L)^T \hat{N}^L - (\hat{N}^L)^T \hat{M}^L)\mathbf{z} \\ &= (\hat{M}_{11}^L)^T [\hat{N}_{11}^L \;\; \hat{N}_{12}^L]\mathbf{z} + (\hat{M}_{21}^L)^T \hat{N}_{21}^L [I \;\; X]\mathbf{z} \\ &= (\hat{M}_{11}^L)^T [\hat{N}_{11}^L \;\; \hat{N}_{12}^L]\mathbf{z}. \end{aligned}$$

As $\hat{M}_{11}^L$ is nonsingular, then $[\hat{N}_{11}^L \;\; \hat{N}_{12}^L]\mathbf{z} = 0$, and it follows that $\text{col}(\hat{M}^L \;\; \hat{N}^L)\mathbf{z} = 0$. But $\text{col}(\hat{M}^L \;\; \hat{N}^L)$ has full column rank as $\text{col}(\hat{M} \;\; \hat{N})$ is in column proper form, and we conclude that $\mathbf{z} = 0$.

$3 \Rightarrow 4$. Let $\hat{M}, \hat{N} \in \mathbb{R}^{n \times n}[\xi]$ be such that the columns of $\text{col}(\hat{M} \;\; \hat{N})$ are a basis for the right syzygy of $[\hat{P} \;\; -\hat{Q}]$ (see Willems, 2007, p. 85). Similar to before, we find that $\hat{M}$ is nonsingular and $\hat{N}\hat{M}^{-1} = \hat{Q}^{-1}\hat{P}$, which is symmetric. Also, there exists a unimodular U such that

$$\begin{bmatrix} M \\ N \end{bmatrix} U = \begin{bmatrix} T_1^T & 0 & 0 & T_2^T \\ 0 & -T_2^T & T_1^T & 0 \end{bmatrix} \begin{bmatrix} \hat{M} \\ \hat{N} \end{bmatrix}.$$

This follows from (Willems, 2007, pp. 84–85), noting from the definition of $\mathcal{B}$ and $\hat{\mathcal{B}}$ that the columns of the matrix on the right hand side of the above equation span the right syzygy of $[P \;\; -Q]$. It can then be verified that $U^T(M^TN - N^TM)U = \hat{M}^T\hat{N} - \hat{N}^T\hat{M} = 0$. Since U is nonsingular, this implies that $M^TN - N^TM = 0$.

3 $\iff$ 2. The proof is analogous to 4 $\iff$ 3. □

PROOF OF THEOREM 9 (see p. 5). That $2 \Rightarrow 1$ follows from Theorem 7, noting from the proof of Lemma 3 that $(\hat{Q}^{-1}\hat{P})(\xi) = D + C(\xi I - A)^{-1}B$, which is symmetric. To see that $1 \Rightarrow 2$, note initially that if $\hat{\mathcal{B}}$ is controllable then the result follows from Lemma 3 and Theorem 7. Otherwise, following (Hughes, 2017a, Notes A.1 and A.3) and (Polderman and Willems, 1998, Cor. 5.2.25), we can construct a realization $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ of $(\hat{P}, \hat{Q})$ such that $(\tilde{C}, \tilde{A})$ is observable; and

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix},\; \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix},\; \tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix},$$

where $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable, $(\tilde{C}_1, \tilde{A}_{11})$ is observable, and $\hat{Q}^{-1}\hat{P}(\xi) = \tilde{C}_1(\xi I - \tilde{A}_{11})^{-1}\tilde{B}_1 + D$, which is symmetric. It then follows from the proof of Lemma 3 that there exists a symmetric P such that $P\tilde{A}_{11} = \tilde{A}_{11}^TP$, $\tilde{C}_1^T = P\tilde{B}_1$, and $D = D^T$. Now, let

$$\hat{A} := \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & 0 \\ 0 & \tilde{A}_{22} & 0 \\ \tilde{A}_{12}^TP & 0 & \tilde{A}_{22}^T \end{bmatrix},\; \hat{B} := \begin{bmatrix} \tilde{B}_1 \\ 0 \\ \tilde{C}_2^T \end{bmatrix},\; \hat{C} := \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 & 0 \end{bmatrix},$$

and $S := \begin{bmatrix} P & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}$.

Since $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ is a realization for $(\hat{P}, \hat{Q})$, then so too is $(\hat{A}, \hat{B}, \hat{C}, D)$ (this follows from Remark 2, as $\hat{C}\hat{A}^k = [\tilde{C}\tilde{A}^k \;\; 0]$ for $k = 0, 1, 2, \ldots$). Also, $S\hat{A} = \hat{A}^TS$ and $S\hat{B} = \hat{C}^T$. Finally, as P is symmetric, there exists a real matrix R and a signature matrix $\tilde{\Sigma}_i$ such that $P = R^T\tilde{\Sigma}_iR$, and we define T and Σ_i (partitioned compatibly) as

$$T := \begin{bmatrix} R & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}I & -\frac{1}{\sqrt{2}}I \\ 0 & \frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \end{bmatrix}, \text{ and } \Sigma_i := \begin{bmatrix} \tilde{\Sigma}_i & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then $S = T^T\Sigma_iT$, and $A := T\hat{A}T^{-1}$, $B := T\hat{B}$, $C := \hat{C}T^{-1}$ satisfy the conditions of the present theorem. □

Remark 12 Note that the proofs of Lemma 3 and Theorem 9 provide an algorithm for the construction of the realization (A, B, C, D) and the signature matrix Σ_i in Theorem 9. Specifically, P in the proof of that theorem can be obtained from the explicit formula in the proof of Lemma 3, whereupon R and $\tilde{\Sigma}_i$ can be obtained from an eigenvalue decomposition for P.

6 Passive and reciprocal behaviors

In this section, we present our main results concerning passive and reciprocal systems. We define passivity in accordance with (Hughes, 2017c, Def. 5) as follows.

Definition 13 (Passive system) *$\mathcal{B}$ in (4.1) is called* passive *if, given any $(\mathbf{i}, \mathbf{v}) \in \mathcal{B}$ and any $t_0 \in \mathbb{R}$, there exists a $K \in \mathbb{R}$ (dependent on $(\mathbf{i}, \mathbf{v})$ and t_0) such that, if $t_1 \geq t_0$ and $(\tilde{\mathbf{i}}, \tilde{\mathbf{v}}) \in \mathcal{B}$ satisfies $(\tilde{\mathbf{i}}(t), \tilde{\mathbf{v}}(t)) = (\mathbf{i}(t), \mathbf{v}(t))$ for $t < t_0$, then $-\int_{t_0}^{t_1} \tilde{\mathbf{i}}^T(t)\tilde{\mathbf{v}}(t)dt < K$.*

Remark 14 Here, $-\int_{t_0}^{t_1} \tilde{\mathbf{i}}^T(t)\tilde{\mathbf{v}}(t)dt$ is the net energy extracted from the system between t_0 and t_1, and the bound K is necessarily non-negative (as the integral is zero when $t_1 = t_0$). Thus, Definition 13 formalises the concept that a system is passive if the net energy that can be extracted from the system into the future is bounded above (this bound depending only on the past trajectory of the system). It is shown in Hughes (2017a) that this definition is consistent with the existence of a non-negative quadratic state storage function with respect to the energy supplied to the system.

The next concept of a positive-real pair was introduced by Hughes (2017c), where it was shown that $\mathcal{B}$ in (4.1) is passive if and only if (P, Q) is a positive-real pair.

Definition 15 (Positive-real pair) *Let $P, Q \in \mathbb{R}^{n\times n}[\xi]$. We call (P, Q) a* positive-real pair *if:*

(a) $P(\lambda)Q(\bar{\lambda})^T + Q(\lambda)P(\bar{\lambda})^T \geq 0$ for all $\lambda \in \mathbb{C}_+$;
(b) rank$([P \;\; -Q](\lambda)) = n$ for all $\lambda \in \overline{\mathbb{C}}_+$; and
(c) if $\mathbf{p} \in \mathbb{R}^n[\xi]$ and $\lambda \in \mathbb{C}$ satisfy $\mathbf{p}(\xi)^T(P(\xi)Q(-\xi)^T + Q(\xi)P(-\xi)^T) = 0$ and $\mathbf{p}(\lambda)^T[P \;\; -Q](\lambda) = 0$, then $\mathbf{p}(\lambda) = 0$.

Remark 16 Note that, in contrast with reciprocity, it is possible for the controllable part of a system to be passive yet for the system itself to not be passive. E.g., let $\mathcal{B}_s$ be as in (2.2) with $B = 0$, $C = 1$ and $D = 1$, so $D + C(\xi I - A)^{-1}B = 1$ for all $A \in \mathbb{R}$ (i.e., the controllable part of the system is independent of A). From Remark 2, if $A = -1$, then for any given $(u, y) \in \mathcal{B}_s^{(u,y)}$, there exists $k_1 \in \mathbb{R}$ such that $-\int_{t_0}^{t_1}(uy)(t)dt = \int_{t_0}^{t_1} -(u(t) + \frac{k_1}{2}e^{-t})^2 + \frac{k_1^2}{4}e^{-2t}dt \leq \frac{k_1^2}{8}e^{-2t_0}$, so this system is passive. But if $A = 0$, then there exists $(u, y) \in \mathcal{B}_s^{(u,y)}$ with $u = -\frac{k_2}{2} = -y$ for all $t \in \mathbb{R}$ and for some $0 \neq k_2 \in \mathbb{R}$, in which case $-\int_{t_0}^{t_1}(uy)(t)dt = \frac{k_2^2}{4}(t_1 - t_0)$. Thus, for any given $K \in \mathbb{R}$, there exists $t_1 > t_0$ such that $-\int_{t_0}^{t_1}(uy)(t)dt > K$, so this system is not passive.

In the next theorem, we state necessary and sufficient algebraic conditions for $\mathcal{B}$ in (4.1) to be passive and reciprocal. We also show that these conditions are equivalent to $\mathcal{B}$ being realizable by an RLCT network, thus solving the first open problem in Çamlibel et al. (2003).

Theorem 17 (Passive and reciprocal behavior theorem, part 1) *Let $\mathcal{B}$ be as in (4.1). The following are equivalent:*

1. $\mathcal{B}$ is passive and reciprocal.
2. (P, Q) is a positive-real pair and $PQ^T = QP^T$.
3. $\mathcal{B}$ is the driving-point behavior of an RLCT network.

In our final theorem, we generalize Lemma 4 to systems that need not be controllable.

Theorem 18 (Passive and reciprocal behavior

theorem, part 2) *Let $\hat{\mathcal{B}}$ be as in (2.3). Then the following are equivalent.*

1. *$\hat{\mathcal{B}}$ is passive and reciprocal.*
2. *There exists $\mathcal{B}_s$ as in (2.2) and a signature matrix $\Sigma_i \in \mathbb{R}^{d\times d}$ such that*
   (i) $\hat{\mathcal{B}} = \mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$;
   (ii) $\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} + \begin{bmatrix} -A & -B \\ C & D \end{bmatrix}^T \geq 0$; *and*
   (iii) $\Sigma_i A = A^T \Sigma_i$, $\Sigma_i B = C^T$, *and* $D = D^T$.

The two-part passive and reciprocal behavior theorem (Theorems 17 and 18) is proved in Section 7. The proofs can be combined with existing results in the literature to obtain a passive and reciprocal realization for any given passive and reciprocal system of the form of (2.3), and to obtain an RLCT realization for an arbitrary given passive and reciprocal system of the form of (4.1). This will be illustrated by two examples in Section 8.

7 Proof of the passive and reciprocal behavior theorem

The purpose of this section is to prove Theorems 17 and 18. These two theorems will be proved in reverse order. First, we prove the following result, which uses the supplementary lemmas in Appendix B.

Lemma 19 *Let $\hat{\mathcal{B}}$ in (2.3) be passive and reciprocal. Then there exists $\mathcal{B}_s$ as in (2.2) such that $\hat{\mathcal{B}} = \mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$ and the following properties both hold:*

1. *there exists $X \in \mathbb{R}^{d\times d}$ such that $X > 0$ and* $\begin{bmatrix} -XA - A^TX & C^T - XB \\ C - B^TX & D + D^T \end{bmatrix} \geq 0$; *and*
2. *there exists a symmetric nonsingular $S \in \mathbb{R}^{d\times d}$ such that $SA = A^TS$, $SB = C^T$, and $D = D^T$.*

PROOF. We will prove this first for the case in which $D + D^T > 0$, and then for the general case.

Case (i): $D + D^T > 0$. We let $\hat{A}, \hat{B}, \hat{C}, D$ and S be as in the proof of Theorem 9, and we let $A = \hat{A}, B = \hat{B}$ and $C = \hat{C}$. From that proof, condition 2 of the present theorem statement holds. Also, $\hat{\mathcal{B}} = \mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$ is passive and (C, A) is detectable. Thus, from Lemma B.2, there exists $K \in \mathbb{R}^{d\times d}$ such that $K > 0$ and $\Upsilon(K) \geq 0$, where $\Upsilon(K)$ is as in (B.1). It can then be verified that $X := K^{-1}$ satisfies condition 1 of the present theorem statement.

Case (ii): general case. Let $P_1 := \hat{P}$ and $Q_1 := \hat{Q}$, and consider the following three statements (c.f., Hughes, 2017a, proof of Th. 13):

(R1) $P_i, Q_i \in \mathbb{R}^{n_i\times n_i}[\xi]$ where (P_i, Q_i) is a positive-real pair and $Q_i^{-1}P_i$ is proper and symmetric.

(R2) P_i, Q_i are as in (R1), P_i is nonsingular, and $\lim_{\xi\to\infty}((Q_i^{-1}P_i)(\xi)) = \operatorname{diag}(I_{r_i}\ \ 0)$.

(R3) P_i, Q_i are as in (R1), and either $n_i = 0$ or $\lim_{\xi\to\infty}((Q_i^{-1}P_i)(\xi)) = I$.

By (Hughes, 2017c, Th. 7) and Theorem 7 of this paper, P_1, Q_1 satisfy condition (R1). Then, using Lemmas B.4 and B.6, we construct $P_2, \ldots, P_m, Q_2, \ldots, Q_m$ such that condition (R1) is satisfied, $n_i \leq n_{i-1}$, and $\deg(\det(Q_i)) \leq \deg(\det(Q_{i-1}))$, for $i = 2, \ldots, m$; and

(1) if, for $i = k-1$, (R2) is not satisfied, then (R2) is satisfied for $i = k$, and if P_{k-1} is singular then $n_k < n_{k-1}$ (Lemma B.4);

(2) if, for $i = k-1$, (R2) is satisfied but (R3) is not, then $\deg(\det(Q_k)) < \deg(\det(Q_{k-1}))$ (Lemma B.6).

This inductive procedure terminates in a finite number of steps with polynomial matrices P_m and Q_m that satisfy conditions (R1)–(R3). An example is given in Section 8.

Next, we consider the following three statements:

(S1) There exist polynomial matrices $\mathcal{A}_i(\xi) := \xi I - A_i$, $Y_i, Z_i, U_i, V_i, E_i, F_i$, and G_i, with G_i nonsingular (i.e., $0 \neq \det(G_i) \in \mathbb{R}[\xi]$), and

$$\begin{bmatrix} Y_i & Z_i \\ U_i & V_i \end{bmatrix} \begin{bmatrix} -D_i & I & -C_i \\ -B_i & 0 & \mathcal{A}_i \end{bmatrix} = \begin{bmatrix} -P_i & Q_i & 0 \\ -E_i & -F_i & G_i \end{bmatrix},$$

where the leftmost matrix is unimodular.

(S2) The matrix $X_i \in \mathbb{R}^{d_i\times d_i}$ satisfies $X_i > 0$ and

$$\Omega_i(X_i) := \begin{bmatrix} -A_i^TX_i - X_iA_i & C_i^T - X_iB_i \\ C_i - B_i^TX_i & D_i + D_i^T \end{bmatrix} \geq 0.$$

(S3) There exists a symmetric $S_i \in \mathbb{R}^{d_i\times d_i}$ such that $S_iA_i = A_i^TS_i$, $S_iB_i = C_i^T$ and $D_i = D_i^T$.

From case (i) and Lemma 1, there exist real matrices A_m, B_m, C_m, D_m, X_m and S_m such that (S1)–(S3) hold for $i = m$. Then, using Lemmas B.4 and B.6, we find that there exist real matrices A_i, B_i, C_i, D_i, X_i and S_i such that (S1)–(S3) hold for $i = m-1, \ldots, 1$. Since $P = P_1$ and $Q = Q_1$, then letting $A = A_1, B = B_1, C = C_1, D = D_1, S = S_1$ and $X = X_1$, we obtain a state-space realization $\hat{\mathcal{B}} = \mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$ with the required properties. □

PROOF OF THEOREM 18 (see p. 8). That 2 ⇒ 1 follows from Theorem 9 and (Hughes, 2017c, Th. 13), noting that condition 3 of that theorem holds with $X = I$. To see that 1 ⇒ 2, consider the realization in Lemma 19. From that theorem, $(\hat{P}, \hat{Q})$ has a realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with the following properties:

1. there exists $\tilde{X} \in \mathbb{R}^{d\times d}$ such that $\tilde{X} > 0$ and $\begin{bmatrix} -\tilde{X}\tilde{A} - \tilde{A}^T\tilde{X} & \tilde{C}^T - \tilde{X}\tilde{B} \\ \tilde{C} - \tilde{B}^T\tilde{X} & \tilde{D} + \tilde{D}^T \end{bmatrix} \geq 0$; and
2. there exists a symmetric nonsingular $\tilde{S} \in \mathbb{R}^{d\times d}$ such that $\tilde{S}\tilde{A} = \tilde{A}^T\tilde{S}$, $\tilde{S}\tilde{B} = \tilde{C}^T$, and $\tilde{D} = \tilde{D}^T$.

Since $\tilde{X} > 0$, then there exists a nonsingular $\tilde{R} \in \mathbb{R}^{d\times d}$ such that $\tilde{X} = \tilde{R}^T\tilde{R}$. As $\tilde{S}$ is symmetric and nonsingular, then so too is $(\tilde{R}^{-1})^T\tilde{S}\tilde{R}^{-1}$. By considering an eigenvalue decomposition, we conclude that there exists a signature matrix $\Sigma_i = \operatorname{diag}(I\ \ -I) \in \mathbb{R}^{d\times d}$, a diagonal matrix $0 < W \in \mathbb{R}^{d\times d}$, and an orthogonal matrix $V \in \mathbb{R}^{d\times d}$ (i.e., $V^T = V^{-1}$), such that $(\tilde{R}^{-1})^T\tilde{S}\tilde{R}^{-1} = V\Sigma_iWV^T$. Here, $\Sigma_iW = W\Sigma_i$ is a diagonal matrix containing the eigenvalues of $(\tilde{R}^{-1})^T\tilde{S}\tilde{R}^{-1}$, which are real as $(\tilde{R}^{-1})^T\tilde{S}\tilde{R}^{-1}$ is symmetric. Now, let

$$\hat{Y} = \begin{bmatrix} -\hat{A} & -\hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} := \begin{bmatrix} V^T\tilde{R} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} -\tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}\begin{bmatrix} V^T\tilde{R} & 0 \\ 0 & I \end{bmatrix}^{-1}.$$

Then $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is a realization for $(\hat{P}, \hat{Q})$, and

$$\hat{Y} = \begin{bmatrix} \tilde{R}^{-1}V & 0 \\ 0 & I \end{bmatrix}^T\begin{bmatrix} -\tilde{X}\tilde{A} & -\tilde{X}\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}\begin{bmatrix} \tilde{R}^{-1}V & 0 \\ 0 & I \end{bmatrix},$$

which implies that $\hat{Y} + \hat{Y}^T \geq 0$. Next, let

$$Y = \begin{bmatrix} -A & -B \\ C & D \end{bmatrix} := \begin{bmatrix} W^{1/2} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} -\hat{A} & -\hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}\begin{bmatrix} W^{1/2} & 0 \\ 0 & I \end{bmatrix}^{-1}.$$

Then (A, B, C, D) is also a realization for $(\hat{P}, \hat{Q})$. Also, with the notation $G := \tilde{R}^{-1}VW^{-1/2}$, then

$$\begin{aligned} A = G^{-1}\tilde{A}G, B = G^{-1}\tilde{B}, C = \tilde{C}G, \text{ and} \\ G^T\tilde{S}G = W^{-1/2}V^T(\tilde{R}^{-1})^T\tilde{S}\tilde{R}^{-1}VW^{-1/2} \\ = W^{-1/2}\Sigma_i WW^{-1/2} \\ = \Sigma_i. \end{aligned}$$

Since, in addition, $\tilde{S}\tilde{A} = \tilde{A}^T\tilde{S}$ and $\tilde{S}\tilde{B} = \tilde{C}^T$, then $\Sigma_i A = G^T\tilde{S}\tilde{A}G = G^T\tilde{A}^T\tilde{S}G = A^T\Sigma_i$, $\Sigma_i B = G^T\tilde{S}\tilde{B} = G^T\tilde{C}^T = C^T$, and $D = D^T$, so $\text{diag}(-\Sigma_i \;\; I)Y$ is symmetric. Now, note that $W^{1/2}$ is diagonal since W is, and partition $W^{1/2}$ compatibly with Σ_i as $W^{1/2} = \text{diag}(F_1 \;\; F_2)$. Also, partition Y and $\hat{Y}$ compatibly with $\text{diag}(-\Sigma_i \;\; I) = \text{diag}(-I \;\; I \;\; I)$ as follows:

$$Y = \begin{bmatrix} F_1 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} -\hat{A}_{11} & -\hat{A}_{12} & -\hat{B}_1 \\ -\hat{A}_{21} & -\hat{A}_{22} & -\hat{B}_2 \\ \hat{C}_1 & \hat{C}_2 & \hat{D} \end{bmatrix}\begin{bmatrix} F_1^{-1} & 0 & 0 \\ 0 & F_2^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then, let

$$Z_{11} := -F_1\hat{A}_{11}F_1^{-1}, \;\; Z_{22} := \begin{bmatrix} F_2 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} -\hat{A}_{22} & -\hat{B}_2 \\ \hat{C}_2 & \hat{D} \end{bmatrix}\begin{bmatrix} F_2^{-1} & 0 \\ 0 & I \end{bmatrix}$$
$$Z_{12} := -F_1\begin{bmatrix} \hat{A}_{12} & \hat{B}_1 \end{bmatrix}\begin{bmatrix} F_2^{-1} & 0 \\ 0 & I \end{bmatrix}, Z_{21} := \begin{bmatrix} F_2 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} -\hat{A}_{21} \\ \hat{C}_1 \end{bmatrix}F_1^{-1}.$$

Since $\text{diag}(-\Sigma_i \;\; I)Y = \text{diag}(-I \;\; I \;\; I)Y$ is symmetric, then we conclude that Z_{11} and Z_{22} are both symmetric, and $Z_{12} = -Z_{21}^T$. Thus, $Y + Y^T = \text{diag}(2Z_{11} \;\; 2Z_{22})$, and to complete the proof it remains to show that $Z_{11} \geq 0$ and $Z_{22} \geq 0$. To prove this, we recall that $\hat{Y} + \hat{Y}^T \geq 0$, so

$$-\hat{A}_{11} - \hat{A}_{11}^T \geq 0, \text{ and } \begin{bmatrix} -\hat{A}_{22} & -\hat{B}_2 \\ \hat{C}_2 & \hat{D} \end{bmatrix} + \begin{bmatrix} -\hat{A}_{22} & -\hat{B}_2 \\ \hat{C}_2 & \hat{D} \end{bmatrix}^T \geq 0.$$

Since Z_{11} and Z_{22} are both symmetric, then their eigenvalues are all real. Now, let $\lambda < 0$, and let $\mathbf{z}$ be a real vector with $Z_{11}\mathbf{z} = \lambda\mathbf{z}$. Then $\hat{\mathbf{z}} := F_1^{-1}\mathbf{z}$ satisfies $-\hat{A}_{11}\hat{\mathbf{z}} = F_1^{-1}Z_{11}\mathbf{z} = \lambda F_1^{-1}\mathbf{z} = \lambda\hat{\mathbf{z}}$. Thus, $\hat{\mathbf{z}}^T(-\hat{A}_{11} - \hat{A}_{11}^T)\hat{\mathbf{z}} = 2\lambda\hat{\mathbf{z}}^T\hat{\mathbf{z}} \leq 0$. Since $(-\hat{A}_{11} - \hat{A}_{11}^T) \geq 0$, then we conclude that $\hat{\mathbf{z}} = 0$. It follows that the eigenvalues of Z_{11} are all real and non-negative, so $Z_{11} \geq 0$. A similar argument shows that $Z_{22} \geq 0$, and completes the proof. □

PROOF OF THEOREM 17 (see p. 7). That 1 $\iff$ 2 follows from (Hughes, 2017c, Th. 9) and Theorem 7 of the present paper.

If $\mathcal{B}$ takes the form of $\hat{\mathcal{B}}$ in (2.3), then from Theorem 18 it follows that $\mathcal{B}$ is passive and reciprocal if and only if $\mathcal{B}$ has a state-space realization with the properties outlined in condition 2 of that theorem. From (Anderson and Vongpanitlerd, 2006, Secs. 4.4 and 9.4), this holds if and only if $\mathcal{B}$ is the driving-point behavior of an RLCT network. It remains to consider the case in which $\mathcal{B}$ does not take the form of $\hat{\mathcal{B}}$ in (2.3), i.e., P, Q in (4.1) are such that Q is singular or $Q^{-1}P$ is not proper.

3 ⇒ 1. That $\mathcal{B}$ is passive follows from (Hughes, 2017b, Th. 6). It remains to show that $\mathcal{B}$ is reciprocal. As explained in (Hughes, 2017b, Sec. 2), any given RLCT network corresponds to a cascade loading of two networks: (i) N_1, in which all of the elements (resistors, inductors, capacitors and transformers) are removed and every single element port is replaced with an external port; and (ii) N_2, which contains each of the elements in the original circuit (disconnected from each other). Furthermore, the driving-point behaviors of N_1 and N_2 are both reciprocal.[5] Now, let $\mathcal{B}$ and $\tilde{\mathcal{B}}$ be fixed but arbitrary reciprocal behaviors, and let (i) $(\text{col}(\mathbf{i}_{a,1} \;\; \mathbf{i}_{a,2}), \text{col}(\mathbf{v}_{a,1} \;\; \mathbf{v}_{a,2})) \in \mathcal{B}$, $(\mathbf{i}_{a,3}, \mathbf{v}_{a,3}) \in \tilde{\mathcal{B}}$, $(\text{col}(\mathbf{i}_{b,1} \;\; \mathbf{i}_{b,2}), \text{col}(\mathbf{v}_{b,1} \;\; \mathbf{v}_{b,2})) \in \mathcal{B}$, and $(\mathbf{i}_{b,3}, \mathbf{v}_{b,3}) \in \tilde{\mathcal{B}}$; (ii) $\mathbf{i}_{a,3} = -\mathbf{i}_{a,2}$, $\mathbf{v}_{a,3} = \mathbf{v}_{a,2}$, $\mathbf{i}_{b,3} = -\mathbf{i}_{b,2}$, and $\mathbf{v}_{b,3} = \mathbf{v}_{b,2}$; and (iii) $\mathbf{i}_{a,1}, \mathbf{i}_{a,2}, \mathbf{v}_{a,1}, \mathbf{v}_{a,2}, \mathbf{i}_{b,1}, \mathbf{i}_{b,2}, \mathbf{v}_{b,1}$, and $\mathbf{v}_{b,2}$ have compact support on the left. Then it suffices to show that $\mathbf{v}_{b,1} \star \mathbf{i}_{a,1} = \mathbf{i}_{b,1} \star \mathbf{v}_{a,1}$. To prove this, note that, since $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are reciprocal, then

$$\begin{aligned} \mathbf{v}_{b,1} \star \mathbf{i}_{a,1} + \mathbf{v}_{b,2} \star \mathbf{i}_{a,2} - \mathbf{i}_{b,1} \star \mathbf{v}_{a,1} - \mathbf{i}_{b,2} \star \mathbf{v}_{a,2} = 0, \\ \text{and } \mathbf{v}_{b,3} \star \mathbf{i}_{a,3} - \mathbf{i}_{b,3} \star \mathbf{v}_{a,3} = 0, \end{aligned}$$

whence $\mathbf{v}_{b,1} \star \mathbf{i}_{a,1} - \mathbf{i}_{b,1} \star \mathbf{v}_{a,1} = 0$.

2 ⇒ 3. We will show the following:

(a) If $\lambda_0 \in \mathbb{C}_+$, $\mathbf{z} \in \mathbb{C}^n$ and $Q(\lambda_0)\mathbf{z} = 0$, then $Q\mathbf{z} = 0$.

(b) There exists a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ and a unimodular matrix $\hat{Y}$ such that

$$\begin{bmatrix} P & -Q \end{bmatrix} = \hat{Y}\begin{bmatrix} \hat{P} & -\hat{Q} \end{bmatrix}\begin{bmatrix} T & 0 \\ 0 & (T^T)^{-1} \end{bmatrix},$$

where $\hat{P}$ and $\hat{Q}$ have the compatible partitions

$$\hat{P} = \begin{bmatrix} \hat{P}_{11} & 0 \\ 0 & I \end{bmatrix} \text{ and } \hat{Q} = \begin{bmatrix} \hat{Q}_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

[5] To see this, note initially that the behavior of the network N_1 has the same form as the driving-point behavior of a transformer (Anderson and Newcomb, 1966). It is then easily verified that the driving-point behaviors of resistors, inductors, capacitors and transformers are all reciprocal, and so too are the driving-point behaviors of N_1 and N_2.

and where $\hat{Q}_{11}$ is nonsingular, $\hat{Q}_{11}^{-1}\hat{P}_{11}$ is symmetric, and $(\hat{P}_{11}, \hat{Q}_{11})$ is a positive-real pair.

(c) With $\hat{P}_{11}$ and $\hat{Q}_{11}$ as in (b), then the limit $\lim_{\xi\to\infty}((1/\xi)(\hat{Q}_{11}^{-1}\hat{P}_{11})(\xi))$ exists and is nonnegative definite. Also, with the notation $K := \lim_{\xi\to\infty}((1/\xi)(\hat{Q}_{11}^{-1}\hat{P}_{11})(\xi))$, $\tilde{P}(\xi) := \hat{P}_{11}(\xi) - \hat{Q}_{11}(\xi)K\xi$, and $\tilde{Q}(\xi) := \hat{Q}_{11}(\xi)$, then $\tilde{Q}^{-1}\tilde{P}$ is proper and symmetric and $(\tilde{P}, \tilde{Q})$ is a positive-real pair.

Now, let $\tilde{P}, \tilde{Q}$ and K be as defined in (c). It follows from Theorem 18 and (Anderson and Vongpanitlerd, 2006, Secs. 4.4 and 9.4) that there exist RLCT networks N_1 and N_2 whose driving-point behaviors take the form $\{(\mathbf{i}, \mathbf{v}) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\tilde{n}}) \times \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\tilde{n}}) \mid \tilde{P}(\frac{d}{dt})\mathbf{i} = \tilde{Q}(\frac{d}{dt})\mathbf{v}\}$ and $\{(\mathbf{i}, \mathbf{v}) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\tilde{n}}) \times \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\tilde{n}}) \mid K\frac{d\mathbf{i}}{dt} = \mathbf{v}\}$, respectively. Next, let T be as in (b), partition T and T^{-1} compatibly with $\hat{P}$ as $T = \text{col}(T_1\ T_2)$ and $T^{-1} = [\hat{T}_1\ \hat{T}_2]$, and consider the behavior corresponding to the set of locally integrable solutions to

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \tilde{P}(\frac{d}{dt}) & 0 & 0 & -\tilde{Q}(\frac{d}{dt}) \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & I & -T_1^T & -T_2^T & 0 & 0 & 0 & 0 \\ T_1 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ T_2 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & K\frac{d}{dt} & 0 & -I & 0 \\ 0 & 0 & I & 0 & 0 & 0 & -I & -I \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{v} \\ \tilde{\mathbf{v}}_a \\ \tilde{\mathbf{v}}_b \\ \tilde{\mathbf{i}}_a \\ \tilde{\mathbf{i}}_b \\ \tilde{\mathbf{v}}_{a1} \\ \tilde{\mathbf{v}}_{a2} \end{bmatrix} = 0, \quad (7.1)$$

which is the driving-point behavior of the RLCT network in Fig. 2. We then let

$$U := \begin{bmatrix} \hat{Y} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix}, \text{ with}$$
$$Z(\xi) := \begin{bmatrix} -\tilde{Q}(\xi)\hat{T}_1^T & \tilde{P}(\xi) + \tilde{Q}(\xi)K\xi & 0 & \tilde{Q}(\xi) & -\tilde{Q}(\xi) \\ 0 & 0 & I & 0 & 0 \end{bmatrix},$$

and it is clear that U is unimodular since $\hat{Y}$ is unimodular. Then, following Appendix A, we pre-multiply both sides in (7.1) by $U(\frac{d}{dt})$, we note that $\hat{T}_1^T T_1^T = I$ and $\hat{T}_1^T T_2^T = 0$, and we find that the driving-point behavior of N is the set of locally integrable solutions to the differential equation

$$\hat{Y}\begin{bmatrix} (\tilde{P}(\frac{d}{dt}) + \tilde{Q}(\frac{d}{dt})K\frac{d}{dt})T_1 & -\tilde{Q}(\frac{d}{dt})\hat{T}_1^T \\ T_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{v} \end{bmatrix} = 0.$$

From (b) and (c), it follows that $[P\ \ -Q](\frac{d}{dt})\text{col}(\mathbf{i}\ \mathbf{v}) = 0$, so $\mathcal{B}$ is the driving-point behavior of N.

It remains to show conditions (a)–(c). To show condition (a), we let $\hat{P} := P - Q$ and $\hat{Q} := P + Q$. Since (P, Q) is a positive-real pair, then $\hat{Q}(\lambda)\hat{Q}(\bar{\lambda})^T - \hat{P}(\lambda)\hat{P}(\bar{\lambda})^T = 2(P(\lambda)Q(\bar{\lambda})^T + Q(\lambda)P(\bar{\lambda})^T) \geq 0$ for all $\lambda \in \mathbb{C}_+$. Now, suppose $\lambda_0 \in \mathbb{C}_+$ and $\mathbf{w} \in \mathbb{C}^n$ satisfy $\mathbf{w}^T\hat{Q}(\lambda_0) = 0$. Then $-\mathbf{w}^T\hat{P}(\lambda_0)\hat{P}(\bar{\lambda}_0)^T\bar{\mathbf{w}} \geq 0$, so $\mathbf{w}^T\hat{P}(\lambda_0) = 0$. But this implies that $\mathbf{w}^T[P\ \ -Q](\lambda_0) = 0$, whence $\mathbf{w} = 0$ since (P, Q) is a positive-real pair. We conclude that $\hat{Q}(\lambda)$ is nonsingular for all $\lambda \in \mathbb{C}_+$, and so $I - (\hat{Q}^{-1}\hat{P})(\lambda)((\hat{Q}^{-1}\hat{P})(\bar{\lambda}))^T \geq 0$ for all $\lambda \in \mathbb{C}_+$. This implies that $(\hat{Q}^{-1}\hat{P})^T$ is bounded-real in accordance with (Youla et al., 1959, Def. 16), and so $\hat{Q}^{-1}\hat{P}$ is bounded-real by (Youla et al., 1959, Cor. 7(c)). It then follows from (Youla et al., 1959, proof of Th. 7) that, if $\lambda_0 \in \mathbb{C}_+$ and $\mathbf{w} \in \mathbb{C}^n$ satisfy $(I - (\hat{Q}^{-1}\hat{P})(\lambda_0))\mathbf{w} = 0$, then $(I - \hat{Q}^{-1}\hat{P})\mathbf{w} = 0$. Now, let $\lambda_0 \in \mathbb{C}_+$ and $\mathbf{z} \in \mathbb{C}^n$ satisfy $Q(\lambda_0)\mathbf{z} = 0$. Then $(P + Q)^{-1}(\lambda_0)Q(\lambda_0)\mathbf{z} = \frac{1}{2}(I - (P + Q)^{-1}(\lambda_0)(P - Q)(\lambda_0))\mathbf{z} = 0$, whence $(I - (P + Q)^{-1}(P - Q))\mathbf{z} = 0$, and so $Q\mathbf{z} = \frac{1}{2}(P + Q)(I - (P + Q)^{-1}(P - Q))\mathbf{z} = 0$.

To show condition (b), we let $r := \text{normalrank}(Q)$, and we first show that there exists a nonsingular matrix $T = \text{col}(T_1\ T_2) \in \mathbb{R}^{n\times n}$ with $T_1 \in \mathbb{R}^{r\times n}$ such that $QT_2^T = 0$. Accordingly, let the columns of $W \in \mathbb{R}^{n\times(n-r)}[\xi]$ be a basis for the right syzygy of Q (see Willems, 2007, p. 85). For any given $\lambda_0 > 0$, $W(\lambda_0) \in \mathbb{R}^{n\times(n-r)}$ has full column rank and $Q(\lambda_0)W(\lambda_0) = 0$, so $QW(\lambda_0) = 0$ by condition (a). We then let T be a nonsingular matrix whose final $n-r$ rows are $W(\lambda_0)^T$.

Next, note that $QT_1^T \in \mathbb{R}^{n\times r}$ and $\text{normalrank}(QT_1^T) = \text{normalrank}(QT^T) = r$. Then, by considering the upper echelon form for QT_1^T (see Hughes, 2017c, Note A4), we obtain a unimodular $Y \in \mathbb{R}^{n\times n}$ such that $YQT^T =: \hat{Q}$ takes the form indicated in condition (b), where $\hat{Q}_{11} \in \mathbb{R}^{r\times r}[\xi]$ is nonsingular. Now, let $\tilde{P} := YPT^{-1}$. It is then easily shown that $(\tilde{P}, \hat{Q})$ is a positive-real pair since (P, Q) is. Accordingly, we consider a fixed but arbitrary $\lambda \in \mathbb{C}_+$, we partition $\tilde{P}$ compatibly with $\hat{Q}$ as

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix},$$

and it follows that $\tilde{P}(\lambda)(\hat{Q}(\bar{\lambda}))^T + \hat{Q}(\lambda)(\tilde{P}(\bar{\lambda}))^T \geq 0$, i.e.,

$$\begin{bmatrix} \tilde{P}_{11}(\lambda)(\hat{Q}_{11}(\bar{\lambda}))^T + \hat{Q}_{11}(\lambda)(\tilde{P}_{11}(\bar{\lambda}))^T & \hat{Q}_{11}(\lambda)(\tilde{P}_{21}(\bar{\lambda}))^T \\ \tilde{P}_{21}(\lambda)(\hat{Q}_{11}(\bar{\lambda}))^T & 0 \end{bmatrix} \geq 0.$$

This implies that $\tilde{P}_{21}(\lambda)(\hat{Q}_{11}(\bar{\lambda}))^T = 0$. Since this relationship holds for all $\lambda \in \mathbb{C}_+$, and $\hat{Q}_{11}$ is nonsingular, then we conclude that $\tilde{P}_{21} = 0$.

Next, it follows from (Hughes, 2017a, proof of Lem. D.3 condition 1) that $\tilde{P}_{22}$ is unimodular since $(\tilde{P}, \hat{Q})$ is a positive-real pair. Accordingly, we let

$$\hat{Y} = Y^{-1}\begin{bmatrix} I & \tilde{P}_{12} \\ 0 & \tilde{P}_{22} \end{bmatrix},$$

and $\hat{P}_{11} = \tilde{P}_{11}$, and we find that $\hat{Y}$ is unimodular and $[P\ \ -Q]$ has the form indicated in condition (b). Finally, it is easily shown that $(\hat{P}_{11}, \hat{Q}_{11})$ is a positive-real pair, and $\hat{P}\hat{Q}^T - \hat{Q}\hat{P}^T = \hat{Y}^{-1}(PQ^T - QP^T)(\hat{Y}^{-1})^T = 0$ so $\hat{Q}_{11}^{-1}\hat{P}_{11}$ is symmetric.

The proof of condition (c) follows from (Hughes, 2017a, Proof of Lem. D.4), noting in addition that $(\tilde{Q}^{-1}\tilde{P})(\xi) = (\hat{Q}_{11}^{-1}\hat{P}_{11})(\xi) - K\xi$, which is symmetric since $\hat{Q}_{11}^{-1}\hat{P}_{11}$ and K are symmetric. □

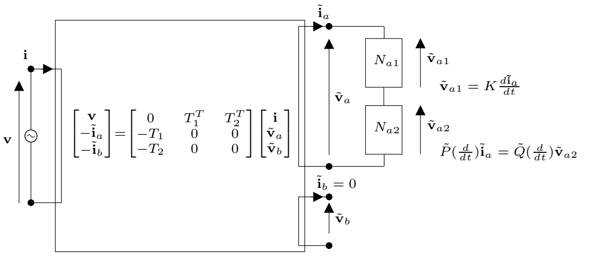

Fig. 2. RLCT network realization of the behavior in (7.1).

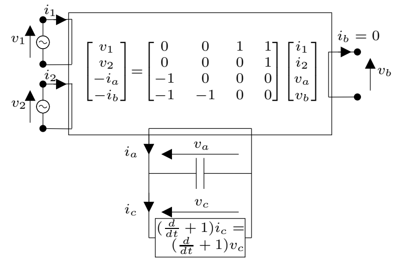

Fig. 4. RLCT network realization example 2.

8 Examples

First, consider the problem of realizing the set of solutions to the differential equation:

$$(\tfrac{d}{dt}+1)i = (\tfrac{d}{dt}+1)v. \tag{8.1}$$

We begin by finding a $\mathcal{B}_s$ as in (2.2) and matrices $X, S \in \mathbb{R}^{d\times d}$ as in Lemma 19, and we then find a passive and reciprocal realization for this behavior as in Theorem 18. Finally, we obtain an RLCT network realization from this passive and reciprocal realization using results in Anderson and Vongpanitlerd (2006). This realization procedure works in the general case, using algorithms for computing: 1. a state-space realization for a behavior (see Remark 2); 2. the available energy of a passive system (see (Hughes, 2017a, Rem. 15)); 3. the null space and column space of a real matrix; 4. a Cholesky decomposition of a positive-definite matrix; 5. an eigenvalue decomposition of a symmetric matrix; and 6. solutions to Lyapunov and Sylvester equations (as in Anderson and Vongpanitlerd, 2006, Ths. 3.7.3 and 3.7.4).

We first obtain a state-space realization for the behavior in (8.1), and we transform this into controller staircase form (see Polderman and Willems, 1998, Cor. 5.2.25). In this case, we find that $\mathcal{B}$ has a state-space realization $\mathcal{B} = \mathcal{B}_s^{(i,v)}$, where $\mathcal{B}_s$ is the set of solutions to

$$\tfrac{d\mathbf{x}}{dt} = \tilde{A}\mathbf{x} + \tilde{B}i,\ v = \tilde{C}\mathbf{x} + Di, \text{ where}$$
$$\tilde{A} = -1, \tilde{B} = 0, \tilde{C} = 1, \text{ and } D = 1.$$

Then, following Lemma B.1, we let $\Upsilon(K)$ and $A_\Upsilon(K)$ be as in (B.1) with $\tilde{A}, \tilde{B}$ and $\tilde{C}$ substituted for A, B and C. Following the terminology in the proof of Lemma B.1, we obtain $K_1 = 4$ and $K_- = 0$. Here, K_1 satisfies $K_1 > 0$ and $\Upsilon(K_1) \geq 0$, and is obtained by first finding $X = K_1^{-1}$ by computing the available energy for the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x} + Bi, v = C\mathbf{x} + Di$ (see Lemma B.1). Also, K_- is obtained by computing the available energy for the system $\frac{d\hat{\mathbf{x}}}{dt} = A^T\hat{\mathbf{x}} - C^Tu, y = -B^T\hat{\mathbf{x}} + D^Tu$ (see Lemma B.1), and satisfies $K_- \geq 0$, $\Upsilon(K_-) \geq 0$, and $\text{spec}(A_\Upsilon(K_-)) \in \bar{\mathbb{C}}_-$. Next, note from Lemma B.1 that there exists $\alpha > 0$ such that, for any given $0 < \epsilon \leq \alpha$, then $K_\epsilon := \epsilon K_1 + (1-\epsilon)K_-$ satisfies $K_\epsilon > 0$, $\Upsilon(K_\epsilon) \geq 0$, and $\text{spec}(A_\Upsilon(K_\epsilon)) \in \bar{\mathbb{C}}_-$. In this case, it can be verified that $K = 1$ satisfies $K > 0$, $\Upsilon(K) \geq 0$, and $\text{spec}(A_\Upsilon(K)) \in \bar{\mathbb{C}}_-$. Now, following the proof of Theorem 9, we augment the matrices $\tilde{A}, \tilde{B}$ and $\tilde{C}$ to obtain an unobservable state-space realization $\mathcal{B} = \hat{\mathcal{B}}_s^{(i,v)}$, where $\hat{\mathcal{B}}_s$ corresponds to the set of solutions to

$$\tfrac{d\hat{\mathbf{x}}}{dt} = \hat{A}\hat{\mathbf{x}} + \hat{B}i,\ v = \hat{C}\hat{\mathbf{x}} + Di, \text{ where}$$
$$\hat{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \hat{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and, as before, $D = 1$. Then, in Lemma B.2, we let $\tilde{A}_{11}, \tilde{A}_{21}, \tilde{A}_{22}, \tilde{B}_1, \tilde{B}_2$ and $\tilde{C}_1$ be obtained by partitioning $\hat{A}, \hat{B}$ and $\hat{C}$ (here, $\tilde{A}_{11}$ is the top left entry of $\hat{A}$, and so forth), and we find that (i) $K_{11} = 1$ satisfies $\tilde{\Upsilon}(K_{11}) = \frac{3}{2} \geq 0$ and $\tilde{A}_\Upsilon(K_{11}) = -\frac{1}{2}$ (Here, K_{11} is the matrix K obtained earlier in the proof, and satisfies $\text{spec}(\tilde{A}_{\tilde{\Upsilon}}(K_{11})) \in \bar{\mathbb{C}}_-$); (ii) $K_{12} = -\frac{1}{3}$ solves the Sylvester equation in Lemma B.2; and (iii) $\nabla = \frac{1}{3}$ solves the Lyapunov equation $\Psi(\nabla) = 0$ in Lemma B.2. We thus obtain

$$\hat{K} = \begin{bmatrix} I & 0 \\ K_{12}^T K_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} K_{11} & 0 \\ 0 & \nabla \end{bmatrix}\begin{bmatrix} I & K_{11}^{-1}K_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{4}{9} \end{bmatrix},$$

which satisfies $\hat{K} > 0$ and $-\hat{K}\hat{A}^T - \hat{A}\hat{K} - (\hat{K}\hat{C}^T -$

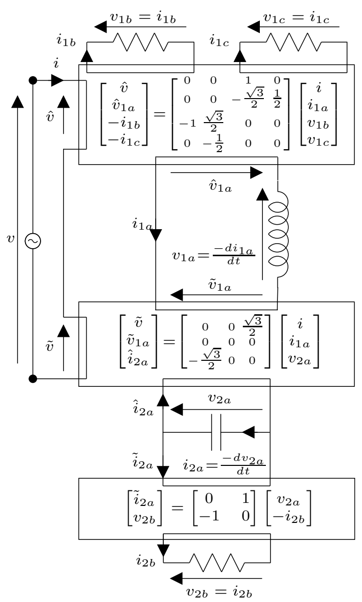

Fig. 3. RLCT network realization example 1.

$\hat{B})(D+D^T)^{-1}(\hat{C}\hat{K}-\hat{B}^T)$. We then let $\hat{X}=\hat{K}^{-1}$. Also, following Remark 12, we obtain the matrix

$$\hat{S}=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right],$$

which satisfies $\hat{S}\hat{A}=\hat{A}^T\hat{S}$ and $\hat{S}\hat{B}=\hat{C}^T$. We have thus obtained a state-space realization $\mathcal{B}_s$ as in (2.2) and matrices $X,S\in\mathbb{R}^{d\times d}$ as in Lemma 19.

Next, using a Cholesky decomposition we obtain

$$\hat{X}=R^TR,\ \text{with } R=\begin{bmatrix}\frac{2}{\sqrt{3}}&\frac{\sqrt{3}}{2}\\0&\frac{3}{2}\end{bmatrix}.$$

Then, following the proof of Theorem 18, we compute an eigenvalue decomposition of $(R^{-1})^T\hat{S}R^{-1}$ to obtain

$$\Sigma_i=\left[\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right],\ W=\begin{bmatrix}1&0\\0&\frac{1}{\sqrt{3}}\end{bmatrix},\ \text{and } V=\begin{bmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}.$$

Following the proof of Theorem 18, let $G:=R^{-1}VW^{-1/2}$, $A=G^{-1}\hat{A}G$, $B=G^{-1}\hat{B}$, and $C=\hat{C}G$, which gives

$$A=\left[\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right],\ B=\begin{bmatrix}\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}\end{bmatrix},\ \text{and } C=\begin{bmatrix}-\frac{\sqrt{3}}{2}&\frac{\sqrt{3}}{2}\end{bmatrix}.$$

These satisfy the conditions of Theorem 18.

Next, we use the results in (Anderson and Vongpanitlerd, 2006, Secs. 9.2 and 9.4) to obtain an RLCT network which realizes the behavior in (8.1). We let

$$M:=\left[\begin{smallmatrix}D&C\\-B&-A\end{smallmatrix}\right],\ \text{and } \Sigma=\left[\begin{smallmatrix}1&0\\0&-\Sigma_i\end{smallmatrix}\right],$$

and we conclude that $M+M^T\geq 0$ and ΣM is symmetric. It follows that M takes the form

$$M=\begin{bmatrix}M_{11}&-M_{21}^T\\M_{21}&M_{22}\end{bmatrix},$$

where $M_{11}\in\mathbb{R}^{2\times 2}$ and $M_{22}\in\mathbb{R}$ are symmetric, and $M_{11},M_{22}\geq 0$. In this case, by computing a Cholesky decomposition for M_{11}, we obtain

$$M_{11}=\begin{bmatrix}1&0\\-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}1&-\frac{\sqrt{3}}{2}\\0&\frac{1}{2}\end{bmatrix},M_{21}=\begin{bmatrix}-\frac{\sqrt{3}}{2}&0\end{bmatrix},M_{22}=1.$$

Finally, from (Anderson and Vongpanitlerd, 2006, Secs. 9.2 and 9.4), we find that $\mathcal{B}$ is the driving-point behavior of the RLCT network in Fig. 3.

We next consider the behavior $\mathcal{B}$ in (4.1), with

$$P(\xi)=\left[\begin{smallmatrix}\xi+1&0\\1&1\end{smallmatrix}\right]\ \text{and } Q(\xi)=\begin{bmatrix}\xi^2+2\xi+1&-(\xi^2+2\xi+1)\\0&0\end{bmatrix},$$

for which Q is singular. We use this example to illustrate both the proof of Theorem 17 and the inductive procedure described in Lemma 19. Again, the realization procedure works in the general case. In addition to the previously listed algorithms, it relies on the computation of an upper echelon form for a polynomial matrix.

First, following the proof of Theorem 17, we obtain matrices $T=[T_1\ T_2]^T$ and $\hat{Y}\in\mathbb{R}^{2\times 2}[\xi]$, where

$$T_1=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\ T_2=\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ \text{and } \hat{Y}(\xi)=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right],$$

and we find that

$$PT^{-1}=\hat{Y}\left[\begin{smallmatrix}P_1&0\\0&1\end{smallmatrix}\right]\ \text{and } QT^T=\hat{Y}\left[\begin{smallmatrix}Q_1&0\\0&0\end{smallmatrix}\right],\ \text{where}$$
$$P_1(\xi)=\xi+1\ \text{and } Q_1(\xi)=\xi^2+2\xi+1.$$

Here, for any given $\lambda>0$, then $T\in\mathbb{R}^{2\times 2}$ is a nonsingular matrix such that T_2 is a basis for the right nullspace of $Q(\lambda)$. Also (in general), $\hat{Y}$ and Q_1 are obtained by computing an upper echelon form for QT^T. It then follows from the proof of Theorem 17 that $\mathcal{B}$ is realized by a network of the form shown in Fig. 2, where $N_{a,1}$ is a short circuit, and $N_{a,2}$ is a network whose driving-point behavior is the set of solutions to $P_1(\frac{d}{dt})\mathbf{i}_1=Q_1(\frac{d}{dt})\mathbf{v}_1$.

Next, note that $\lim_{\xi\to\infty}((Q_1^{-1}P_1)(\xi))=0$, which is singular. Thus, (P_1,Q_1) satisfies conditions (R1)–(R2) on p. 8, but not condition (R3). Then, following Lemma B.6, we find that $\lim_{\xi\to\infty}(\frac{1}{\xi}(P_1^{-1}Q_1)(\xi))=1$, and accordingly we let $K=1$, $Q_2=P_1$, and $P_2(\xi)=Q_1(\xi)-\xi P_1(\xi)=\xi+1$. It can then be verified that $\mathcal{B}$ is realized by a network of the form of Fig. 4. Here, a network realization for the set of solutions to the differential equation $(\frac{d}{dt}+1)i_c=(\frac{d}{dt}+1)v_c$ was obtained in the first example.

9 Conclusions

This paper developed a theory of reciprocal systems which does not assume controllability. Necessary and sufficient algebraic conditions were established for a system to be reciprocal, both in terms of the high order differential equations describing the system, and in terms of a state-space realization. Analogous results were obtained for systems that are both passive and reciprocal. Notably, we answered the first open problem in (Çamlibel et al., 2003) by proving that a behavior is realizable as the driving-point behavior of an RLCT network if and only if it is passive and reciprocal.

A The elimination theorem

Let $\hat{\mathcal{B}}=\{(\mathbf{w}_1,\mathbf{w}_2)\in\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{n_1})\times\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{n_2})\mid\hat{R}(\frac{d}{dt})\mathrm{col}(\mathbf{w}_1\ \mathbf{w}_2)\}$. From (Polderman and Willems, 1998, Th. 6.2.6), there exists a unimodular U with

$$U\hat{R}=\left[\begin{smallmatrix}R_1&0\\R_2&M_2\end{smallmatrix}\right],\tag{A.1}$$

where the rightmost matrix is partitioned compatibly with $\mathrm{col}(\mathbf{w}_1\ \mathbf{w}_2)$, and M_2 has full row rank.

Then, from (Polderman and Willems, 1998, Th. 2.5.4), $\hat{\mathcal{B}}$ is the set of locally integrable solutions to $R_1(\frac{d}{dt})\mathbf{w}_1 = 0$ and $M_2(\frac{d}{dt})\mathbf{w}_2 = -R_2(\frac{d}{dt})\mathbf{w}_1$. Now, let $\mathcal{B} := \{\mathbf{w}_1 \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{n_1}) \mid R_1(\frac{d}{dt})\mathbf{w}_1 = 0\}$. Since M_2 has full row rank, then it is easily shown that for any $\mathbf{w}_1 \in \mathcal{D}_+(\mathbb{R}, \mathbb{R}^{n_1})$ there exists $\mathbf{w}_2 \in \mathcal{D}_+(\mathbb{R}, \mathbb{R}^{n_2})$ such that $M_2(\frac{d}{dt})\mathbf{w}_2 = -R_2(\frac{d}{dt})\mathbf{w}_1$, whence $(\hat{\mathcal{B}} \cap (\mathcal{D}_+(\mathbb{R}, \mathbb{R}^{n_1}) \times \mathcal{D}_+(\mathbb{R}, \mathbb{R}^{n_2})))^{(\mathbf{w}_1)} = \mathcal{B} \cap \mathcal{D}_+(\mathbb{R}, \mathbb{R}^{n_1})$. But it may not be the case that $\hat{\mathcal{B}}^{(\mathbf{w}_1)} = \mathcal{B}$ (see, e.g., Polderman, 1997, Example 2.1). If $\hat{\mathcal{B}}^{(\mathbf{w}_1)} = \mathcal{B}$, then $\mathbf{w}_2$ is called *properly eliminable* (Polderman, 1997). From (Polderman, 1997, Example 3.1), if $\mathcal{B}_s$ is as in (2.2), then $\mathbf{x}$ is properly eliminable. Also, the internal currents and voltages in any given RLCT network are always properly eliminable (Hughes, 2017b, Sec. 6).

Finally, if $\mathcal{B}$ is as in (2.1) and $T \in \mathbb{R}^{q \times q}$ is a nonsingular real matrix, then it is easily shown that $\mathcal{B}^{(T\mathbf{w})} = \{\mathbf{z} \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid (RT^{-1})(\frac{d}{dt})\mathbf{z} = 0\}$.

B The passive and reciprocal behavior theorem, supplementary lemmas

This appendix contains four supplementary lemmas used to prove the results in Sections 6 and 7. In the first two lemmas, for any given symmetric $K \in \mathbb{R}^{d \times d}$, we let

$$\Upsilon(K) := -KA^T - AK - (KC^T - B)(D + D^T)^{-1}(CK - B^T),$$
$$\text{and } A_\Upsilon(K) := A^T - C^T(D + D^T)^{-1}(B^T - CK). \quad \text{(B.1)}$$

Lemma B.1 *Let $\mathcal{B}_s$ be as in (2.2), and let $\hat{\mathcal{B}} := \mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$ be passive, (C, A) be observable, $D + D^T > 0$, and $\Upsilon(K), A_\Upsilon(K)$ be as in (B.1). Then there exists $K \in \mathbb{R}^{d \times d}$ such that $K > 0$, $\Upsilon(K) \geq 0$, and spec$(A_\Upsilon(K)) \in \overline{\mathbb{C}}_-$.*

PROOF. Since (C, A) is observable then there exists $X \in \mathbb{R}^{d \times d}$ such that $X > 0$ and $-A^T X - XA - (C^T - XB)(D + D^T)^{-1}(C - B^T X) = 0$ (see Hughes, 2017c, Th. 13). Now, let $K_1 := X^{-1} \in \mathbb{R}^{d \times d}$, so $K_1 > 0$ and $\Upsilon(K_1) = 0$. Thus, from (Hughes, 2017a, Ths. 10 and 11), there exists $K_- \geq 0$ such that $\Upsilon(K_-) = 0$, spec$(A_\Upsilon(K_-)) \in \overline{\mathbb{C}}_-$, and $K_- \leq K_1$ (here, the available energy S_a for the system $\frac{d\mathbf{x}}{dt} = A^T\mathbf{x} - C^T\mathbf{u}, \mathbf{y} = -B^T\mathbf{x} + D^T\mathbf{u}$ satisfies $S_a(\mathbf{x}_0) = \mathbf{x}_0^T K_- \mathbf{x}_0$ for all $\mathbf{x}_0 \in \mathbb{R}^d$). Now, let ϵ be a fixed but arbitrary real number in the interval $0 < \epsilon < 1$, and let $K_\epsilon := (1-\epsilon)K_- + \epsilon K_1$. Since $\epsilon K_1 > 0$ and $(1-\epsilon)K_- \geq 0$, then $K_\epsilon > 0$. Also,

$$\Upsilon(K_\epsilon) = (1-\epsilon)\Upsilon(K_-) + \epsilon\Upsilon(K_1) + \epsilon(1-\epsilon)(K_- - K_1)C^T(D + D^T)^{-1}C(K_- - K_1) \geq 0,$$

and so $\Upsilon(K_\epsilon) \geq 0$. To complete the proof of the present theorem, we will show that there exists $0 < \alpha < 1$ such that spec$(A_\Upsilon(K_\epsilon)) \in \overline{\mathbb{C}}_-$ for all $0 < \epsilon \leq \alpha$. To see this, note that $Z := K_1 - K_-$ satisfies $Z \geq 0$ and

$$-ZA_\Upsilon(K_-) - A_\Upsilon(K_-)^T Z - ZC^T(D + D^T)^{-1}CZ = \Upsilon(K_1) - \Upsilon(K_-) = 0.$$

Next, let $T \in \mathbb{R}^{d \times d}$ be nonsingular with $TA_\Upsilon(K_-)^T T^{-1} = \text{diag}(A_1 \;\; A_2)$ where spec$(A_1) \in \mathbb{C}_-$ and spec$(A_2) \in j\mathbb{R}$ (here, the rows of T_1 span the stable left eigenspace of $A_\Upsilon(K_-)^T$), and partition T compatibly as $T = \text{col}(T_1 \;\; T_2)$. Then the row space of T_2 is spanned by the left Jordan chains corresponding to the imaginary axis eigenvalues of $A_\Upsilon(K_-)$. Consider one such Jordan chain:

$$\mathbf{z}_1^T A_\Upsilon(K_-)^T = j\omega\mathbf{z}_1^T, \text{ and}$$
$$\mathbf{z}_k^T A_\Upsilon(K_-)^T = j\omega\mathbf{z}_k^T + \mathbf{z}_{k-1}^T \; (k = 2, 3, \ldots, N).$$

Then, by taking the complex conjugate transpose, we obtain $A_\Upsilon(K_-)\bar{\mathbf{z}}_1 = -j\omega\bar{\mathbf{z}}_1$ and $A_\Upsilon(K_-)\bar{\mathbf{z}}_k = -j\omega\bar{\mathbf{z}}_k + \bar{\mathbf{z}}_{k-1}$ $(k = 2, 3, \ldots, N)$. Thus, for $k = 1$,

$$\begin{aligned}\mathbf{z}_k^T ZC^T(D + D^T)^{-1}CZ\bar{\mathbf{z}}_k &= \mathbf{z}_k^T(-ZA_\Upsilon(K_-) - A_\Upsilon(K_-)^T Z)\bar{\mathbf{z}}_k \\ &= \mathbf{z}_k^T Z\bar{\mathbf{z}}_k(j\omega - j\omega) = 0, \quad \text{(B.2)}\end{aligned}$$

whence $CZ\bar{\mathbf{z}}_1 = 0$. This implies that $(-ZA_\Upsilon(K_-) - A_\Upsilon(K_-)^T Z)\bar{\mathbf{z}}_1 = ZC^T(D + D^T)^{-1}CZ\bar{\mathbf{z}}_1 = 0$, so $A_\Upsilon(K_-)^T Z\bar{\mathbf{z}}_1 = -ZA_\Upsilon(K_-)\bar{\mathbf{z}}_1 = j\omega Z\bar{\mathbf{z}}_1$. It follows that $C(Z\bar{\mathbf{z}}_1) = 0$ and $A(Z\bar{\mathbf{z}}_1) = A_\Upsilon(K_-)^T(Z\bar{\mathbf{z}}_1) = j\omega Z\bar{\mathbf{z}}_1$, and so $Z\bar{\mathbf{z}}_1 = 0$ since (C, A) is observable. Next, note that (B.2) holds for $k = 2$, and similar to before we find that $Z\bar{\mathbf{z}}_2 = 0$. Proceeding by induction, we obtain $Z\bar{\mathbf{z}}_k = 0$, whence $\mathbf{z}_k^T Z = 0$ $(k = 1, 2, \ldots, N)$. Since the vectors $\mathbf{z}_1 \ldots \mathbf{z}_N$ span the row space of T_2, then $T_2 Z = 0$. Thus, by partitioning $\hat{T} := T^{-1}$ compatibly with T as $\hat{T} = [\hat{T}_1 \;\; \hat{T}_2]$, noting that $A_\Upsilon(K_\epsilon)^T = A_\Upsilon(K_-)^T + \epsilon ZC^T(D + D^T)^{-1}C$, and letting $\hat{A}_{12} := \epsilon T_1 ZC^T(D + D^T)^{-1}C\hat{T}_2$, we find that

$$TA_\Upsilon(K_\epsilon)^T T^{-1} = \begin{bmatrix} A_1 + \epsilon T_1 ZC^T(D + D^T)^{-1}C\hat{T}_1 & \hat{A}_{12} \\ 0 & A_2 \end{bmatrix}.$$

Thus, spec$(A_\Upsilon(K_\epsilon)) = $ spec$(A_1 + \epsilon T_1 ZC^T(D + D^T)^{-1}C\hat{T}_1)$ $\cup$spec(A_2). Since spec$(A_1) \in \mathbb{C}_-$, then there exists a $0 < \alpha < 1$ such that spec$(A_1 + \epsilon T_1 ZC^T(D + D^T)^{-1}C\hat{T}_1) \in \overline{\mathbb{C}}_-$ for all $0 < \epsilon \leq \alpha$. For any such ϵ, then $K := K_\epsilon$ satisfies the conditions of the present theorem statement. □

Lemma B.2 *Let $\mathcal{B}_s$ be as in (2.2), and let $\hat{\mathcal{B}} := \mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$ be passive, (C, A) be detectable (i.e., col$(C \;\; \lambda I - A)$ has full column rank for all $\lambda \in \overline{\mathbb{C}}_+$), $D + D^T > 0$, and $\Upsilon(K)$ be as in (B.1). Then there exists $K \in \mathbb{R}^{d \times d}$ such that $K > 0$ and $\Upsilon(K) \geq 0$.*

PROOF. By the observer staircase form (see Hughes, 2017c, note D2), there exists a $T \in \mathbb{R}^{d\times d}$ such that

$$TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},\ TB = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix},\ \text{and } CT^{-1} = [\tilde{C}_1\ 0],$$

with $(\tilde{C}_1, \tilde{A}_{11})$ observable. As (C, A) is detectable, then it is easily shown that $\text{spec}(\tilde{A}_{22}) \in \mathbb{C}_-$. Now, let

$$\begin{aligned}
\tilde{\mathcal{B}}_s &= \{(\mathbf{u},\mathbf{y},\tilde{\mathbf{x}}) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^n) \times \mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^n) \times \mathcal{L}_1^{\text{loc}}\left(\mathbb{R},\mathbb{R}^{\tilde{d}}\right) \mid \\
&\qquad \tfrac{d\tilde{\mathbf{x}}}{dt} = \tilde{A}_{11}\tilde{\mathbf{x}} + \tilde{B}_1\mathbf{u} \text{ and } \mathbf{y} = \tilde{C}_1\tilde{\mathbf{x}} + D\mathbf{u}\},\\
\tilde{\Upsilon}(\tilde{K}) &:= -\tilde{K}\tilde{A}_{11}^T - \tilde{A}_{11}\tilde{K} \\
&\qquad - (\tilde{K}\tilde{C}_1^T - \tilde{B}_1)(D + D^T)^{-1}(\tilde{C}_1\tilde{K} - \tilde{B}_1^T),
\end{aligned}$$

and $\tilde{A}_{\tilde{\Upsilon}}(\tilde{K}) = \tilde{A}_{11}^T - \tilde{C}_1^T(D + D^T)^{-1}(\tilde{B}_1^T - \tilde{C}_1\tilde{K})$.

It follows from (Hughes, 2017c, Note D3) that $\tilde{\mathcal{B}}_s^{(\mathbf{u},\mathbf{y})} = \mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$, which is passive, so from Lemma B.1 there exists $K_{11} \in \mathbb{R}^{\tilde{d}\times\tilde{d}}$ such that $K_{11} > 0$, $\tilde{\Upsilon}(K_{11}) \geq 0$, and $\text{spec}(\tilde{A}_{\tilde{\Upsilon}}(K_{11})) \in \overline{\mathbb{C}}_-$. Also, $\text{spec}(\tilde{A}_{22}) \in \mathbb{C}_-$, so by (Anderson and Vongpanitlerd, 2006, Th. 3.7.4) there exists a unique real K_{12} that satisfies the Sylvester equation

$$\begin{aligned}
&\tilde{A}_{22}K_{12}^T + K_{12}^T\tilde{A}_{\tilde{\Upsilon}}(K_{11}) \\
&\quad = -\tilde{A}_{21}K_{11} - \tilde{B}_2(D + D^T)^{-1}(\tilde{B}_1^T - \tilde{C}_1K_{11});
\end{aligned}$$

and there exists a (non-unique) $\nabla > 0$ that satisfies

$$\begin{aligned}
&\Psi(\nabla) := -\nabla\tilde{A}_{22}^T - \tilde{A}_{22}\nabla - K_{12}^TK_{11}^{-1}\tilde{\Upsilon}(K_{11})K_{11}^{-1}K_{12} \\
&-(\tilde{B}_2 - K_{12}^TK_{11}^{-1}\tilde{B}_1)(D+D^T)^{-1}(\tilde{B}_2 - K_{12}^TK_{11}^{-1}\tilde{B}_1)^T \geq 0.
\end{aligned}$$

It can then be verified that

$$\begin{aligned}
&K := T^{-1}\begin{bmatrix} I & 0 \\ K_{12}^TK_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} K_{11} & 0 \\ 0 & \nabla \end{bmatrix}\begin{bmatrix} I & K_{11}^{-1}K_{12} \\ 0 & I \end{bmatrix}(T^{-1})^T > 0,\\
&\text{and } \Upsilon(K) = T^{-1}\begin{bmatrix} \tilde{\Upsilon}(K_{11}) & 0 \\ 0 & \Psi(\nabla) \end{bmatrix}(T^{-1})^T \geq 0. \qquad \square
\end{aligned}$$

Remark B.3 It is easily shown that K in Lemma B.2 satisfies $\text{spec}(A_\Upsilon(K)) = \text{spec}(A_{\tilde{\Upsilon}}(K_{11})) \cup \text{spec}(A_{22}) \in \overline{\mathbb{C}}_-$.

The final two lemmas concern the decomposition in the proof of Lemma 19. We refer to that proof for the definition of statements (R1)–(R3) and (S1)–(S3).

Lemma B.4 *Let $P_{k-1}, Q_{k-1} \in \mathbb{R}^{n_{k-1}\times n_{k-1}}[\xi]$ satisfy (R1) for $i = k-1$, and let $n_k :=$ normalrank(P_{k-1}), $m_k := n_{k-1} - n_k$, and $r_k :=$ rank$(\lim_{\xi\to\infty}(Q_{k-1}^{-1}P_{k-1}(\xi)))$. The following hold.*

1. *There exists a nonsingular $T \in \mathbb{R}^{n_{k-1}\times n_{k-1}}$; unimodular $W \in \mathbb{R}^{n_{k-1}\times n_{k-1}}[\xi]$ and $\tilde{Q}_{22} \in \mathbb{R}^{m_k\times m_k}[\xi]$; $\tilde{Q}_{12} \in \mathbb{R}^{n_k\times m_k}[\xi]$; and P_k, Q_k satisfying (R1) and (R2) for i=k, with*

$$WP_{k-1}T = \begin{bmatrix} P_k & 0 \\ 0 & 0 \end{bmatrix}, WQ_{k-1}(T^{-1})^T = \begin{bmatrix} Q_k & \tilde{Q}_{12} \\ 0 & \tilde{Q}_{22} \end{bmatrix}. \qquad \text{(B.3)}$$

2. *Let A_k, B_k, C_k, D_k satisfy (S1) for $i = k$; and let $A_{k-1} := A_k$, $B_{k-1} := [B_k\ 0]T^{-1}$, $C_{k-1} := (T^{-1})^T$ col$(C_k\ 0)$, and $D_{k-1} := (T^{-1})^T$ diag$(D_k\ 0)T^{-1}$. Then:*
   (a) *(S1) holds for $i = k-1$.*
   (b) *Let X_k satisfy (S2) for $i = k$; and let $X_{k-1} := X_k$. Then (S2) holds for $i = k-1$.*
   (c) *Let S_k satisfy (S3) for $i = k$; and let $S_{k-1} := S_k$. Then (S3) holds for $i = k-1$.*

PROOF. Condition 1 follows from (Hughes, 2017a, Lem. D.3, condition 1), noting that $T^TQ_{k-1}^{-1}P_{k-1}T = \text{diag}(Q_k^{-1}P_k\ 0)$, so $Q_k^{-1}P_k$ is symmetric since $Q_{k-1}^{-1}P_{k-1}$ is. To see condition 2a, we let $\mathcal{A}_k, Y_k, Z_k, U_k, V_k, E_k, F_k$ and G_k be as in (S1) for $i = k$. Following (Hughes, 2017a, Lem. D.3, proof of condition 2), we let

$$\begin{bmatrix} Y_{k-1} & Z_{k-1} \\ U_{k-1} & V_{k-1} \end{bmatrix} := \begin{bmatrix} W^{-1} & 0 \\ 0 & I \end{bmatrix}\left[\begin{array}{cc|c} Y_k & \tilde{Q}_{12} & Z_k \\ 0 & \tilde{Q}_{22} & 0 \\ \hline U_k & 0 & V_k \end{array}\right]\begin{bmatrix} T^T & 0 \\ 0 & I \end{bmatrix}.$$

It can be verified that each of the above matrices is unimodular. Also, with $\mathcal{A}_{k-1}(\xi) := \xi I - A_{k-1}$, $E_{k-1} := [E_k\ 0]T^{-1}$, $F_{k-1} := [F_k\ 0]T^T$, and $G_{k-1} := G_k$, it can be verified that (S1) holds for $i = k-1$.

The proof of condition 2b follows from (Hughes, 2017a, Lem. D.3, proof of condition 2(c)): with $R := \text{diag}(I\ T^{-1})$, then $\Omega_{k-1}(X_{k-1}) = R^T\text{diag}(\Omega_k(X_k)\ 0)R$. Finally, condition 2c is straightforward to check. $\square$

Remark B.5 With P_k, Q_k, P_{k-1} and Q_{k-1} as in the above lemma, then the driving-point behavior $P_{k-1}(\frac{d}{dt})\mathbf{i} = Q_{k-1}(\frac{d}{dt})\mathbf{v}$ is realized by a transformer terminated on a network with driving-point behavior $P_k(\frac{d}{dt})\hat{\mathbf{i}} = Q_k(\frac{d}{dt})\hat{\mathbf{v}}$.

Lemma B.6 *Let $P_{k-1}, Q_{k-1} \in \mathbb{R}^{n_{k-1}\times n_{k-1}}[\xi]$ satisfy (R1)–(R2) for i=k−1 (so P_k is nonsingular and $\lim_{\xi\to\infty}((Q_{k-1}^{-1}P_{k-1})(\xi)) = \text{diag}(I_{r_{k-1}}\ 0)$), and let $m_k := n_{k-1} - r_{k-1} > 0$. The following hold.*

1. *There exists $0 < K \in \mathbb{R}^{m_k\times m_k}$ such that diag$(0\ K) = \lim_{\xi\to\infty}(\frac{1}{\xi}P_{k-1}^{-1}Q_{k-1}(\xi))$.*
2. *Let $P_k(\xi) := Q_{k-1}(\xi) - P_{k-1}(\xi)\text{diag}(0\ K\xi)$, and $Q_k := P_{k-1}$. Then (R1) holds for $i = k$; $\deg(\det(Q_k)) < \deg(\det(Q_{k-1}))$; and there exist $\hat{D}_{12} \in \mathbb{R}^{r_{k-1}\times m_k}$, $\hat{D}_{21} \in \mathbb{R}^{m_k\times r_{k-1}}$, $\hat{D}_{22} \in \mathbb{R}^{m_k\times m_k}$ such that*

$$\lim_{\xi\to\infty}(Q_k^{-1}P_k(\xi)) =: D_k = \begin{bmatrix} I_{r_{k-1}} & \hat{D}_{12} \\ \hat{D}_{21} & \hat{D}_{22} \end{bmatrix}. \qquad \text{(B.4)}$$

3. *Let A_k, B_k, C_k, D_k satisfy (S1) for $i = k$; partition B_k, C_k compatibly with D_k as $B_k = [\hat{B}_1\ \hat{B}_2]$, $C_k =$ col$(\hat{C}_1\ \hat{C}_2)$; and let*

$$\begin{aligned}
A_{k-1} &:= \begin{bmatrix} A_k - \hat{B}_1\hat{C}_1 & \hat{B}_2K^{-1} - \hat{B}_1\hat{D}_{12}K^{-1} \\ \hat{D}_{21}\hat{C}_1 - \hat{C}_2 & \hat{D}_{21}\hat{D}_{12}K^{-1} - \hat{D}_{22}K^{-1} \end{bmatrix},\\
B_{k-1} &:= \begin{bmatrix} \hat{B}_1 & 0 \\ -\hat{D}_{21} & I \end{bmatrix},\ \textit{and } C_{k-1} := \begin{bmatrix} -\hat{C}_1 & -\hat{D}_{12}K^{-1} \\ 0 & K^{-1} \end{bmatrix}.
\end{aligned}$$

Then:

(a) (S1) holds for $i = k-1$.

(b) Let X_k *satisfy (S2) for* $i = k$*; and let* $X_{k-1} := \text{diag}(X_k \;\; K^{-1})$*. Then (S2) holds for* $i = k-1$.

(c) Let S_k *satisfy (S3) for* $i = k$*; and let* $S_{k-1} := \text{diag}(-S_k \;\; K^{-1})$*. Then (S3) holds for* $i = k-1$.

PROOF. First, note that $Q_{k-1}^{-1} P_{k-1} = P_{k-1}^T (Q_{k-1}^{-1})^T$ implies that $P_{k-1} Q_{k-1}^T = Q_{k-1} P_{k-1}^T$, and hence $P_{k-1}^{-1} Q_{k-1} = Q_{k-1}^T (P_{k-1}^T)^{-1}$. Conditions 1 and 2 then follow from (Hughes, 2017a, Lem. D.4, conditions 1 and 2), as $Q_k^{-1} P_k(\xi) = P_{k-1}^{-1} Q_{k-1}(\xi) - \text{diag}(0 \;\; K\xi)$, so $Q_k^{-1} P_k$ is symmetric since $P_{k-1}^{-1} Q_{k-1}$ and $\text{diag}(0 \;\; K\xi)$ are. For condition 3a, we let $\mathcal{A}_k, Y_k, Z_k, U_k, V_k, E_k, F_k$ and G_k be as in (S1) for $i = k$. Following (Hughes, 2017a, Lem. D.4, proof of condition 3), we partition the two matrices on the left-hand side of (S1) compatibly as

$$\left[\begin{array}{cc|c} \hat{Y}_{11} & \hat{Y}_{12} & \hat{Z}_1 \\ \hat{Y}_{21} & \hat{Y}_{22} & \hat{Z}_2 \\ \hline \hat{U}_1 & \hat{U}_2 & \hat{V} \end{array}\right] \text{ and } \left[\begin{array}{cc|c|c} -I & -\hat{D}_{12} & \multirow{2}{*}{I} & -\hat{C}_1 \\ -\hat{D}_{21} & -\hat{D}_{22} & & -\hat{C}_2 \\ \hline -\hat{B}_1 & -\hat{B}_2 & 0 & \mathcal{A}_k \end{array}\right], \tag{B.5}$$

and we let

$$\begin{bmatrix} Y_{k-1} & Z_{k-1} \\ U_{k-1} & V_{k-1} \end{bmatrix} = \left[\begin{array}{cc|cc} \hat{Y}_{11} & \hat{Y}_{12} & \hat{Z}_1 & 0 \\ \hat{Y}_{21} & \hat{Y}_{22} & \hat{Z}_2 & 0 \\ \hline -\hat{U}_1 & -\hat{U}_2 & -\hat{V} & 0 \\ 0 & I & 0 & -I \end{array}\right] \left[\begin{array}{cc|cc} I & \hat{D}_{12} & 0 & 0 \\ \hat{D}_{21} & \hat{D}_{22}+K\xi & 0 & I \\ \hline \hat{B}_1 & \hat{B}_2 & -I & 0 \\ \hat{D}_{21} & \hat{D}_{22}+K(1+\xi) & 0 & I \end{array}\right].$$

It can be verified that each of the above matrices is unimodular. Then, with $E_{k-1} := \text{col}(F_k \;\; 0)$, $F_{k-1}(\xi) := \text{col}(E_k(\xi) \;\; 0) + \text{col}(\xi \hat{U}_2(\xi) \;\; I)[0 \;\; K]$, and $G_{k-1} := \text{diag}(G_k \;\; I)$, we find that (S1) holds for $i = k-1$.

The proof of condition 3b is identical to (Hughes, 2017a, Lem. D.3, proof of condition 3(c)). Finally, condition 3c is straightforward to check (noting that $\lim_{\xi \to \infty}(Q_k^{-1} P_k(\xi))$ is symmetric, so $\hat{D}_{12} = \hat{D}_{21}^T$, and $\hat{D}_{11}$ and $\hat{D}_{22}$ are symmetric). □

Remark 20 With P_k, Q_k, P_{k-1} and Q_{k-1} as in the above lemma, then the driving-point behavior $P_{k-1}(\frac{d}{dt})\mathbf{i} = Q_{k-1}(\frac{d}{dt})\mathbf{v}$ can be realized by a parallel connection of networks with driving-point behaviors $\tilde{\mathbf{i}} = \text{diag}(0 \;\; K\frac{d}{dt})\tilde{\mathbf{v}}$ and $Q_k(\frac{d}{dt})\hat{\mathbf{i}} = P_k(\frac{d}{dt})\hat{\mathbf{v}}$.

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