# Variational generalization of the Green–Naghdi and Whitham equations for fluid sloshing in three-dimensional rotating and translating coordinates

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**Abstract.** This paper derives an averaged Lagrangian functional for dynamic coupling between rigid-body motion and its interior shallow-water sloshing in three-dimensional rotating and translating coordinates; with a time-dependent rotation vector. A new set of variational shallow-water equations (SWEs) and generalized Green–Naghdi equations for the interior fluid sloshing with 3–D rotation vector and translations, and also the equations of motion for the linear momentum and angular momentum of the rigid-body containing shallow water, are derived from the averaged Lagrangian functional, which describes a *columnar motion*, by using Hamilton's principle and the Euler–Poincaré variational framework. The generalized Green–Naghdi equations have a form of potential vorticity (PV) conservation, which can be obtained from the particle-relabeling symmetry, and is a combination of the PV derived by Miles and Salmon (1985) and the PV derived by Dellar & Salmon (2005) for geophysical fluid dynamics problems, where the rotation vector varies spatially. By applying the assumption of zero-potential-vorticity flow to the averaged Lagrangian functional, a new set of Boussinesq-like evolution equations are derived, which are a generalization of the Whitham equations for fluid sloshing in three-dimensional rotating and translating coordinates. Moreover, the new variational principles are appended to Luke's variational principle to present a unified variational framework for the hydrodynamic problem of interactions between gravity-driven potential-flow water waves and a freely floating rigid-body, dynamically coupled to its interior weakly dispersive nonlinear shallow-water sloshing in three dimensions.

## 1 Introduction

Since the seminal works of Herivel (1955), Eckart (1960a), Luke (1967), Zakharov (1968), Arnold (1969), Bretherton (1970), Broer (1974), Lukovsky (1976), Miles (1977), Benjamin & Olver (1982), Olver (1982) and Salmon (1983), variational principles have been extensively used in mathematical formulation of the equations governing the motion of an inviscid fluid (e.g. Oliver 2006; Stewart & Dellar 2010; Dellar 2011; Oliver 2014), and in constructing variational, geometric and structure-preserving numerical schemes (e.g. Marsden & West 2001; Pavlov *et al.* 2011; Desbrun *et al.* 2014; Gagarina *et al.* 2014; Stewart & Dellar 2016).

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The Euler equations in fluid mechanics and their reduced variants such as the traditional and non-traditional single and multilayer shallow-water equations in geophysical fluid dynamics, and gravity-driven potential flow water waves, in the Lagrangian particle-path and Eulerian frameworks, can be expressed in variational, canonical and non-canonical Hamiltonian and Poisson-bracket formulations (e.g. Lewis *et al.* 1986; Bridges 1994; Morrison 1998; Dellar & Salmon 2005; Bokhove & Oliver 2006). The conservation laws are related to symmetries of the Lagrangian by Noether's theorem (e.g. Noether 1918; Hill 1951; Shepherd 1990), e.g., momentum and energy conservation arise from the translation symmetries in space and time, and potential vorticity conservation arises from the particle-relabeling symmetry. Variational principles for rigid-body dynamics with fluid-filled cavities are given by Moiseyev & Rumyantsev (1968), Lukovsky (2015, and references cited therein), and more recently by Alemi Ardakani (2019, 2020) and Alemi Ardakani *et al.* (2019). In the study of rigid-body dynamics, the Lie group SO$(3)$ is the configuration space and the symmetry group of the Lagrangian functional, which allows to introduce the *Euler–Poincaré reduction* framework for obtaining the reduced dynamics on the quotient space TSO$(3)$/SO$(3)$ (Holm, Schmah & Stoica 2009). The Euler–Poincaré reduction theorem for the motion of a free rigid-body and for a heavy top with a broken symmetry is given by Holm, Marsden & Ratiu (1998a). The Euler–Poincaré equations, the Lagrangian analogue of the Lie–Poisson Hamiltonian equations, for the motion of an ideal incompressible fluid, for ideal fluids with nonlinear dispersion, and for geophysical fluid dynamics problems are given by Holm, Marsden & Ratiu (1998a, 1998b, 1999).

Alemi Ardakani (2019) derived an Euler–Poincaré variational framework for the problem of interactions between potential-flow water waves and a freely floating rigid-body dynamically coupled to its interior inviscid and incompressible fluid sloshing described by the Euler equations in three-dimensional rotating and translating coordinates. In this paper, we are interested to derive a new *reduced shallow water* variant of the variational principle given by Alemi Ardakani (2019) for the coupled **(**rigid-body motion + interior fluid sloshing**)** dynamics with 3–D rotation vector and translations. The reduced shallow-water variational principle is used in formulation of the nonlinear partial differential equations governing the motion of a rigid-body in three dimensions, *dynamically coupled* to its interior *weakly dispersive nonlinear shallow-water sloshing* in two-horizontal space dimensions. A new set of generalized Green–Naghdi equations, a new set of variational shallow-water equations (SWEs), and a new set of generalized Whitham equations with 3–D rotation vector and translations are derived. The new generalized Whitham equations for the interior fluid sloshing in three-dimensional rotating and translating coordinates are derived by applying the assumption of zero-potential-vorticity flow to the reduced Lagrangian functional in Eulerian coordinates. Moreover, the new reduced variational principle is added to Luke's variational principle (Luke 1967) to develop a mathematical theory or a unified variational framework for the hydrodynamic problem of three-dimensional interactions between potential-flow water waves and a freely floating rigid-body, dynamically coupled to its interior weakly dispersive nonlinear fluid sloshing. For this purpose the variational Reynold's transport theorem is applied to take into account the time-dependent boundaries of the coupled **(**wave-body + interior dispersive shallow-water slosh**)** interactions.

Alemi Ardakani & Bridges (2011) presented a new derivation of shallow-water equations in two-horizontal space dimensions with complete Coriolis, centrifugal and translational force, for a 3–D inviscid but vortical fluid in a vessel undergoing *prescribed* rigid-body motion in three dimensions. These SWEs are reported below to compare them with the new *variational SWEs* of the current paper in §3, and to introduce the notation used in the following sections in studying the problem of *variational dynamic coupling* between rigid-body motion and its interior shallow-water sloshing in 3–D rotating-translating coordinates. The

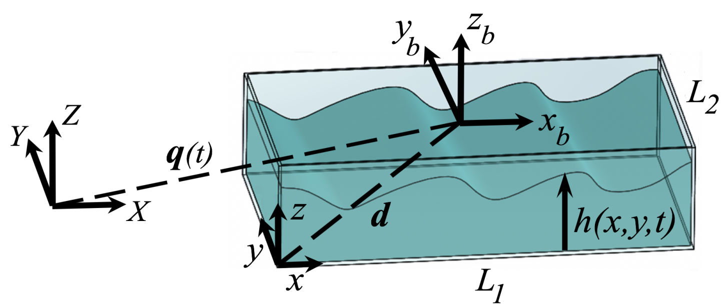


Figure 1: Schematic showing a configuration of the fixed coordinate system $\boldsymbol{X} = (X, Y, Z)$ relative to the body coordinate systems denoted by $\boldsymbol{x}_b = (x_b, y_b, z_b)$ and $\boldsymbol{x} = (x, y, z)$, attached to the moving rigid-body. The distance between the origin of the fixed (laboratory) frame $\boldsymbol{X}$ and the point of rotation is denoted by the vector $\boldsymbol{q}\,(t)$. The distance from the point of rotation, i.e. the origin of the body frame $\boldsymbol{x}_b$, to the origin of the body frame $\boldsymbol{x}$ is denoted by the constant vector $\boldsymbol{d}$.

fluid occupies the region

$$0 \leq x \leq L_1\,, \quad 0 \leq y \leq L_2\,, \quad 0 \leq z \leq h\,(x, y, t)\,, \tag{1.1}$$

where the lengths $L_1$ and $L_2$ are given positive constants, and $z = h\,(x, y, t)$ is the position of the free surface, which is a single-valued function. The configuration of the fluid in a rectangular rigid-body, which is free to rotate and translate in $\mathbb{R}^3$, is schematically shown in Figure 1. Three frames of reference are used. The laboratory or fixed frame has coordinates denoted by $\boldsymbol{X} = (X, Y, Z)$. The first body frame, which is placed at the centre of rotation of the moving body has coordinates denoted by $\boldsymbol{x}_b = (x_b, y_b, z_b)$. The second body frame, which is attached to the moving body and used for the analysis of the fluid motion inside the tank, has coordinates denoted by $\boldsymbol{x} = (x, y, z)$. The distance between the origin of the body frame $\boldsymbol{x}_b$ (the point of rotation) and the origin of the body frame $\boldsymbol{x}$, is denoted by the position vector $\boldsymbol{d} = (d_1, d_2, d_3)$ which is a constant vector. Hence, the position of a fluid particle relative to the body frame $\boldsymbol{x}_b$ is $\boldsymbol{x}_b = \boldsymbol{x} + \boldsymbol{d}$. The fluid-body system has a uniform translation $\boldsymbol{q}\,(t) = (q_1(t), q_2(t), q_3(t))$ relative to the laboratory frame $\boldsymbol{X}$, which is the vector from the origin of the laboratory frame $\boldsymbol{X}$ to the origin of the body frame $\boldsymbol{x}_b$. The position of a fluid particle in the body frame $\boldsymbol{x}$ is related to a point in the laboratory frame $\boldsymbol{X}$ by

$$\boldsymbol{X} = \boldsymbol{Q}\,(\boldsymbol{x} + \boldsymbol{d}) + \boldsymbol{q}\,, \tag{1.2}$$

where $\boldsymbol{Q}\,(t) \in \mathsf{SO}\,(3)$ is a proper rotation in $\mathbb{R}^3$, i.e. $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$ and $\det\,(\boldsymbol{Q}) = 1$. By reducing the Euler equations relative to the rotating and translating body frame $\boldsymbol{x}$ and using the vorticity equation, the surface SWEs take the form (Alemi Ardakani & Bridges 2011)

$$\left.\begin{aligned} U_t + UU_x + VU_y + a_{11}\,(x, y, t)\,h_x + a_{12}\,(x, y, t)\,h_y &= b_1\,(x, y, t)\,, \\ V_t + UV_x + VV_y + a_{21}\,(x, y, t)\,h_x + a_{22}\,(x, y, t)\,h_y &= b_2\,(x, y, t)\,, \\ h_t + (hU)_x + (hV)_y &= 0\,, \end{aligned}\right\} \tag{1.3}$$

where the free surface horizontal velocity field is

$$U\,(x, y, t) = u\,(x, y, z, t)\,\big|^h := u\,(x, y, h\,(x, y, t)\,, t) \quad \text{and} \quad V\,(x, y, t) = v\,(x, y, z, t)\,\big|^h\,, \tag{1.4a, b}$$

and the coefficients $a_{11}$, $a_{12}$, $b_1$, $a_{21}$, $a_{22}$ and $b_2$ are

$$\left.\begin{aligned}
a_{11}(x,y,t) &= 2\Omega_1 V + \boldsymbol{Q}\boldsymbol{e}_3\cdot\ddot{\boldsymbol{q}} + g\,\boldsymbol{Q}\boldsymbol{e}_3\cdot\boldsymbol{e}_3 - \left(\Omega_1^2+\Omega_2^2\right)(h+d_3)\\
&\quad - \left(\dot{\Omega}_2 - \Omega_1\Omega_3\right)(x+d_1) + \left(\dot{\Omega}_1 + \Omega_2\Omega_3\right)(y+d_2)\,,\\
a_{12}(x,y,t) &= 2\Omega_2 V\,,\\
b_1(x,y,t) &= -2\Omega_2 h_t + 2\Omega_3 V - \boldsymbol{Q}\boldsymbol{e}_1\cdot\ddot{\boldsymbol{q}} - g\,\boldsymbol{Q}\boldsymbol{e}_1\cdot\boldsymbol{e}_3 + \left(\Omega_2^2+\Omega_3^2\right)(x+d_1)\\
&\quad + \left(\dot{\Omega}_3 - \Omega_1\Omega_2\right)(y+d_2) - \left(\dot{\Omega}_2 + \Omega_1\Omega_3\right)(h+d_3)\,,\\
a_{21}(x,y,t) &= -2\Omega_1 U\,,\\
a_{22}(x,y,t) &= -2\Omega_2 U + \boldsymbol{Q}\boldsymbol{e}_3\cdot\ddot{\boldsymbol{q}} + g\,\boldsymbol{Q}\boldsymbol{e}_3\cdot\boldsymbol{e}_3 - \left(\Omega_1^2+\Omega_2^2\right)(h+d_3)\\
&\quad - \left(\dot{\Omega}_2 - \Omega_1\Omega_3\right)(x+d_1) + \left(\dot{\Omega}_1 + \Omega_2\Omega_3\right)(y+d_2)\,,\\
b_2(x,y,t) &= 2\Omega_1 h_t - 2\Omega_3 U - \boldsymbol{Q}\boldsymbol{e}_2\cdot\ddot{\boldsymbol{q}} - g\,\boldsymbol{Q}\boldsymbol{e}_2\cdot\boldsymbol{e}_3 + \left(\Omega_1^2+\Omega_3^2\right)(y+d_2)\\
&\quad - \left(\dot{\Omega}_3 + \Omega_1\Omega_2\right)(x+d_1) + \left(\dot{\Omega}_1 - \Omega_2\Omega_3\right)(h+d_3)\,.
\end{aligned}\right\} \tag{1.5}$$

The *body angular velocity* is a time-dependent vector $\boldsymbol{\Omega}(t) = (\Omega_1(t), \Omega_2(t), \Omega_3(t))$ relative to the body coordinate system $\boldsymbol{x}_b$ with entries determined from the rotation tensor $\boldsymbol{Q}(t)$ by

$$\boldsymbol{Q}^T\dot{\boldsymbol{Q}} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} := \widehat{\boldsymbol{\Omega}}\,, \tag{1.6}$$

where the skew-symmetric matrix $\widehat{\boldsymbol{\Omega}} \in \mathfrak{so}(3)$ satisfies the *hat map* (Marsden & Ratiu 1999; Holm, Schmah & Stoica 2009)

$$\widehat{\boldsymbol{\Omega}}\boldsymbol{r} = \boldsymbol{\Omega}\times\boldsymbol{r}\,, \quad \text{for any} \quad \boldsymbol{r}\in\mathbb{R}^3\,, \quad \boldsymbol{\Omega} := (\Omega_1,\Omega_2,\Omega_3)\,. \tag{1.7}$$

The body angular velocity is to be contrasted with the *spatial angular velocity*, the angular velocity viewed from the laboratory frame $\boldsymbol{X}$, which is $\widehat{\boldsymbol{\Omega}}^{\text{spatial}} := \dot{\boldsymbol{Q}}\boldsymbol{Q}^T$. As vectors the spatial and body angular velocities are related by $\boldsymbol{\Omega}^{\text{spatial}} = \boldsymbol{Q}\boldsymbol{\Omega}$. The use of the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ in (1.5) is to compactify notation such that $\boldsymbol{Q}\boldsymbol{e}_3\cdot\boldsymbol{e}_3 = Q_{33}$ where $Q_{ij}$ is the $(i,j)$th entry of the matrix representation of $\boldsymbol{Q}$, and $\boldsymbol{Q}\boldsymbol{e}_3\cdot\ddot{\boldsymbol{q}} = Q_{13}\ddot{q}_1 + Q_{23}\ddot{q}_2 + Q_{33}\ddot{q}_3$ with similar expressions for the other such terms. The surface SWEs (1.3) conserve a potential vorticity (PV) of the form

$$\widehat{\mathscr{P}} = \frac{V_x - U_y + 2\Omega_3 - 2\Omega_2 h_y - 2\Omega_1 h_x}{h}\,. \tag{1.8}$$

It can be proved that in two-horizontal space dimensions $\mathcal{D}\widehat{\mathscr{P}} = \widehat{\mathscr{P}}_t + U\widehat{\mathscr{P}}_x + V\widehat{\mathscr{P}}_y = 0$ (see Alemi Ardakani & Bridges 2011).

Dellar & Salmon (2005) derived a set of *obliquely rotating* SWEs and the Green–Naghdi equations with a complete Coriolis force, i.e. including the *non-traditional* components of the Coriolis force, and topography from a variational principle by using Hamilton's principle of least action applied to a two-dimensional vertically averaged Lagrangian functional. By restricting the fluid to move in columns, Dellar & Salmon (2005) reduced the three-dimensional Lagrangian functional (Eckart 1960a; Salmon 1982a)

$$\mathscr{L} = \iiint \left(\tfrac{1}{2}\left\|\dot{\boldsymbol{x}} + \boldsymbol{\Omega}\times\boldsymbol{x}\right\|^2 - \tfrac{1}{2}\left\|\boldsymbol{\Omega}\times\boldsymbol{x}\right\|^2 - gz + p(\boldsymbol{a},t)\left(\frac{\partial(x,y,z)}{\partial(a,b,c)} - 1\right)\right)\mathrm{d}\boldsymbol{a}\,, \tag{1.9}$$

which is expressed in the Lagrangian particle-path setting for an inviscid and incompressible fluid of unit density in a frame rotating about an arbitrary axis with angular velocity $\boldsymbol{\Omega}$. The horizontal components of the angular velocity vector $\boldsymbol{\Omega}$ can be arbitrary functions of $x$ and $y$, and its vertical component can be an arbitrary function of $x$, $y$ and $z$. This allows a variety of beta-plane approximations of the the rotation vector in geophysical fluid dynamics problems. However, the rotation vector $\boldsymbol{\Omega}$ must be non-divergent, i.e. $\boldsymbol{\nabla}\cdot\boldsymbol{\Omega}=0$, to ensure conservation of potential vorticity (Grimshaw 1975; Dellar & salmon 2005; Stewart & Dellar 2010). In our study in this paper for the problem of fluid sloshing in a container undergoing rigid-body motion in three dimensions the rotation vector $\boldsymbol{\Omega}$ is only a function of time, i.e. $\boldsymbol{\Omega}=\boldsymbol{\Omega}\left(t\right)$. In the Lagrangian particle-path description, a fluid particle is described by its position

$$\boldsymbol{x}\left(\boldsymbol{a},t\right)=\left(x\left(a,b,c,t\right),y\left(a,b,c,t\right),z\left(a,b,c,t\right)\right)\,, \tag{1.10}$$

which is marked by Lagrangian labels $\boldsymbol{a}=\left(a,b,c\right)$ at time $t$. The Lagrangian labels $\left(a,b,c\right)$ can be chosen such that the Jacobian of the label-to-particle map satisfies

$$J=\frac{\partial\left(x,y,z\right)}{\partial\left(a,b,c\right)}=x_a\left(y_bz_c-y_cz_b\right)+x_b\left(y_cz_a-y_az_c\right)+x_c\left(y_az_b-y_bz_a\right)=1\,. \tag{1.11}$$

This means that at an initial time $t_0$ the Lagrangian labels $\left(a,b,c\right)$ are physically possible coordinates, i.e. $\left(a,b,c\right)=\left(x_0,y_0,z_0\right)$. The second term $-\frac{1}{2}\left\|\boldsymbol{\Omega}\times\boldsymbol{x}\right\|^2$ in (1.9) is used to subtract out the contribution from the kinetic energy which gives rise to the centrifugal force, taking into account that in geophysical fluid dynamics and physical oceanography the centrifugal force is conventionally incorporated into the gravitational acceleration $g$ in the potential energy (see e.g. Dellar & Salmon 2005 and appendix C of Müller 1995). The last tem in (1.9) introduces pressure $p\left(\boldsymbol{a},t\right)$ as a Lagrange multiplier to enforce incompressibility of the fluid. Salmon (1983) and Miles & Salmon (1985) respectively derived the *traditional* shallow-water equations and the Green–Naghdi equations (Green & Naghdi 1976) from the 3–D Lagrangian (1.9), albeit with a *purely vertical rotation vector*, by restricting the fluid to move in columns and using Hamilton's principle. See Eckart (1960b) for the traditional approximation of the Coriolis force. Stewart & Dellar (2010) formulated an extended variant of the Lagrangian functional (1.9) for the flow of multiple superposed layers of inviscid and incompressible fluids with different constant densities over variable bottom topography in a rotating frame, and derived multilayer shallow-water equations with complete Coriolis force, i.e. on a non-traditional beta-plane.

Alemi Ardakani (2019) derived an extended version of the Lagrangian functional (1.9) retaining the term which gives rise to the centrifugal force, however, for the motion of a rigid-body, which is free to undergo three-dimensional rotational and translational motions, dynamically coupled to its interior fluid sloshing described by the Euler equations with 3–D rotation vector and translations, which is schematically shown in Figure 1. The Lagrangian action functional takes the form (Alemi Ardakani 2019)

$$\left.\begin{aligned}&\mathscr{L}\left(\boldsymbol{\Omega},\boldsymbol{Q},\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{x},\dot{\boldsymbol{x}}\right)=\int_{t_1}^{t_2}\bigg(\iiint\Big(\tfrac{1}{2}\left\|\dot{\boldsymbol{x}}\right\|^2+\dot{\boldsymbol{x}}\boldsymbol{\cdot}\left(\boldsymbol{\Omega}\times\left(\boldsymbol{x}+\boldsymbol{d}\right)+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\\&+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\boldsymbol{\cdot}\left(\boldsymbol{\Omega}\times\left(\boldsymbol{x}+\boldsymbol{d}\right)\right)+\tfrac{1}{2}\left\|\dot{\boldsymbol{q}}\right\|^2-g\left(\boldsymbol{Q}\left(\boldsymbol{x}+\boldsymbol{d}\right)+\boldsymbol{q}\right)\boldsymbol{\cdot}\hat{\boldsymbol{z}}+p\left(\boldsymbol{a},t\right)\left(J-1\right)\Big)\rho\,\mathrm{d}\boldsymbol{a}\\&+\tfrac{1}{2}\boldsymbol{\Omega}\boldsymbol{\cdot}\boldsymbol{I}_f\boldsymbol{\Omega}+\tfrac{1}{2}m_v\left\|\dot{\boldsymbol{q}}\right\|^2+\left(\boldsymbol{\Omega}\times m_v\overline{\boldsymbol{x}}_v\right)\boldsymbol{\cdot}\boldsymbol{Q}^T\dot{\boldsymbol{q}}+\tfrac{1}{2}\boldsymbol{\Omega}\boldsymbol{\cdot}\boldsymbol{I}_v\boldsymbol{\Omega}-m_vg\left(\boldsymbol{Q}\overline{\boldsymbol{x}}_v+\boldsymbol{q}\right)\boldsymbol{\cdot}\hat{\boldsymbol{z}}\bigg)\,\mathrm{d}t\,,\end{aligned}\right\} \tag{1.12}$$

where the integral is over the volume of the reference or label space $\boldsymbol{a}$, $\rho$ is the density of the interior inviscid and incompressible fluid, $\hat{\boldsymbol{z}}$ is the unit vector in the $Z$ direction, $\boldsymbol{I}_v$ is the mass moment of inertia of the dry rigid-body relative to the point of rotation, $m_v$ is the mass of the dry rigid-body, $\overline{\boldsymbol{x}}_v=\left(\overline{x}_v,\overline{y}_v,\overline{z}_v\right)$ is the centre of mass of the dry body relative to the body frame $\boldsymbol{x}_b$, and

$$\boldsymbol{I}_f=\iiint\left(\left\|\boldsymbol{x}+\boldsymbol{d}\right\|^2\boldsymbol{I}-\left(\boldsymbol{x}+\boldsymbol{d}\right)\otimes\left(\boldsymbol{x}+\boldsymbol{d}\right)\right)\rho\,\mathrm{d}\boldsymbol{a}\,, \tag{1.13}$$

is the mass moment of inertia of the fluid relative to the point of rotation, i.e. the origin of the body frame $\boldsymbol{x}_b$, $\otimes$ denotes the tensor product, and $\boldsymbol{I}$ is the $3 \times 3$ identity matrix. It can be concluded that the term $\frac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_f \boldsymbol{\Omega}$ in (1.12), with $\boldsymbol{\Omega}(t) = (\Omega_1(t), \Omega_2(t), \Omega_3(t))$, reads

$$\tfrac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_f \boldsymbol{\Omega} = \iiint \tfrac{1}{2} \left\| \boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d}) \right\|^2 \rho \, \mathrm{d}\boldsymbol{a} \,. \tag{1.14}$$

This term gives rise to the centrifugal force in the fluid sloshing problem. The aim in the current paper is to derive a shallow water approximation of the Lagrangian action (1.12), which leads to a new set of variational SWEs and a new Green–Naghdi model for fluid sloshing with complete Coriolis, centrifugal, and translational force, and also gives the equations of motion for the rigid-body containing shallow water. To clarify the notation used, if $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) = (u(\boldsymbol{x}, t), v(\boldsymbol{x}, t), w(\boldsymbol{x}, t))$ denotes the Eulerian velocity of a fluid particle relative to the body frame with $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t)$ the corresponding flow map, the fluid particle initially at position $\boldsymbol{a}$ is at position $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t)$ at time $t$, then the Lagrangian velocity of the fluid particle is $\dot{\boldsymbol{x}}(\boldsymbol{a}, t) = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{a}, t), t)$, and the Lagrangian acceleration of the fluid particle is $\ddot{\boldsymbol{x}}(\boldsymbol{a}, t) = D\boldsymbol{u}/Dt = \boldsymbol{u}_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u}$ with $\boldsymbol{\nabla} \equiv (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. For an incompressible fluid the Jacobian $J$ of the label-to-particle mapping $(a, b, c) \to (x, y, z)$ is the motion invariant, i.e. $\partial J/\partial t = 0$, which is the continuity equation $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ in the Lagrangian particle-path formulation.

The paper starts with the derivation of a reduced shallow-water Lagrangian by restricting the fluid to move in columns in the 3–D Lagrangian action (1.12) in §2. In §3 a new set of generalized Green–Naghdi equations and variational SWEs in two-horizontal space dimensions are derived using Hamilton's variational principle. The variational SWEs are compared with the surface SWEs (1.3) derived by Alemi Ardakani & Bridges (2011). The material conservation of potential vorticity for the new generalized Green–Naghdi equations and the variational SWEs with 3–D rotation vector and translations are studied in §4. In §5 by applying the assumption zero-potential-vorticity flow, first introduced by Miles & Salmon (1985), a new generalized Whitham model is derived for the problem of fluid sloshing in a vessel undergoing prescribed rigid-body motion in three dimensions. The Euler–Poincaré equations for the linear momentum and angular momentum of the rigid-body containing shallow water are presented in §6. In §7 a unified variational framework is presented for the problem of three-dimensional interactions between potential-flow water waves and a freely floating rigid-body dynamically coupled to its interior weakly dispersive nonlinear fluid sloshing. The paper ends with concluding remarks in §8.

# 2 Restriction to columnar motion: the shallow-water and Green–Naghdi Lagrangian functional for coupled fluid and rigid-body dynamics in three dimensions

The aim in this section is to derive a reduced shallow water variant of the Lagrangian functional (1.12) for dynamic coupling between rigid-body motion and its interior inviscid and incompressible *shallow-water sloshing* in three-dimensional rotating and translating coordinates. We follow Salmon (1983, 1988), Miles & Salmon (1985) and Dellar & Salmon (2005) and restrict the fluid to columnar motion by assuming that

$$x = x(a, b, t) \quad \text{and} \quad y = y(a, b, t) \,, \tag{2.1a, b}$$

with no dependence on the third Lagrangian label $c$. The Jacobian of the label-to-particle map (1.11) then simplifies to

$$\frac{\partial\left(x,y,z\right)}{\partial\left(a,b,c,\right)}=\frac{\partial\left(x,y\right)}{\partial\left(a,b\right)}\frac{\partial z}{\partial c}=1\,. \tag{2.2}$$

Choosing $c=0$ at the bottom $z=0$, and $c=h_0\left(a,b\right)$ at the free surface $z=h\left(x,y,t\right)$, we may integrate $(2.2)$ with respect to $c$ to determine $z$:

$$z=\frac{\partial\left(a,b\right)}{\partial\left(x,y\right)}c=\frac{h\left(x,y,t\right)}{h_0}c=\frac{1}{\mathcal{J}}c\,, \tag{2.3}$$

and noting that

$$h\left(x,y,t\right)=\frac{h_0\left(a,b\right)}{\mathcal{J}}\quad\text{with}\quad\mathcal{J}=\frac{\partial\left(x,y\right)}{\partial\left(a,b\right)}\,, \tag{2.4}$$

where $h_0\left(a,b\right)$ is the initial condition for the wave height inside the container at $t=t_0$, and $\mathcal{J}$ is the horizontal Jacobian. The horizontal and vertical components of the particle velocity are given by

$$\boldsymbol{u}_2\left(x,y,t\right)\equiv\left(u\left(x,y,t\right),v\left(x,y,t\right)\right)=\left(\dot{x},\dot{y}\right)=\dot{\boldsymbol{x}}_2\quad\text{and}\quad\dot{z}=\frac{\dot{h}}{h_0}c\,, \tag{2.5$a,b$}$$

where

$$\dot{h}=\frac{\partial h}{\partial t}+\boldsymbol{u}_2\cdot\boldsymbol{\nabla}_2 h\equiv\mathcal{D}h\quad\text{and}\quad\boldsymbol{\nabla}_2 h\equiv\left(\frac{\partial h}{\partial x},\frac{\partial h}{\partial y}\right)\,, \tag{2.6$a,b$}$$

and $\mathcal{D}\equiv\mathcal{D}/\mathcal{D}t=\partial/\partial t+\boldsymbol{u}_2\cdot\boldsymbol{\nabla}_2$ is the Lagrangian or material time derivative. Differentiating (2.4) with respect to $t$ gives

$$\begin{cases}\dot{h}=-h_0\dfrac{\dot{\mathcal{J}}}{\mathcal{J}^2}=-h_0\,\mathcal{J}^{-2}\left(\dfrac{\partial\left(\dot{x},y\right)}{\partial\left(a,b\right)}+\dfrac{\partial\left(x,\dot{y}\right)}{\partial\left(a,b\right)}\right)=-h\,\mathcal{J}^{-1}\left(\dfrac{\partial\left(\dot{x},y\right)}{\partial\left(a,b\right)}+\dfrac{\partial\left(x,\dot{y}\right)}{\partial\left(a,b\right)}\right)\\[2ex]=-h\left(\dfrac{\partial\left(\dot{x},y\right)}{\partial\left(x,y\right)}+\dfrac{\partial\left(x,\dot{y}\right)}{\partial\left(x,y\right)}\right)=-h\left(\dfrac{\partial\dot{x}}{\partial x}+\dfrac{\partial\dot{y}}{\partial y}\right)=-h\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)=-h\boldsymbol{\nabla}_2\cdot\boldsymbol{u}_2\,.\end{cases} \tag{2.7}$$

From (2.6) and (2.7), we obtain the continuity equation in the Eulerian form

$$h_t+\boldsymbol{\nabla}_2\cdot\left(h\boldsymbol{u}_2\right)=h_t+\left(hu\right)_x+\left(hv\right)_y=0\,. \tag{2.8}$$

Now substituting (2.3) and (2.5) into the Lagrangian action (1.12) allow the $c$ integration to be completed, which gives the reduced shallow-water Lagrangian for dynamic coupling between rigid-body motion and its interior shallow-water sloshing. Note that the incompressibility constraint $p\left(\boldsymbol{a},t\right)\left(J-1\right)$ in (1.12) is automatically satisfied and can be discarded. Integration of the first term in (1.12) gives

$$\left.\begin{aligned}\iiint_0^{h_0}\tfrac{1}{2}\left\|\dot{\boldsymbol{x}}\right\|^2\rho\,\mathrm{d}\boldsymbol{a}\;&=\;\iiint_0^{h_0}\tfrac{1}{2}\left(\left\|\dot{\boldsymbol{x}}_2\right\|^2+\frac{\dot{h}^2}{h_0^2}c^2\right)\rho\,\mathrm{d}c\,\mathrm{d}\boldsymbol{a}_2\\&=\;\iint\tfrac{1}{2}\left(\left\|\dot{\boldsymbol{x}}_2\right\|^2+\frac{1}{3}\dot{h}^2\right)\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,,\end{aligned}\right\} \tag{2.9}$$

where $\mathrm{d}\boldsymbol{a}_2=\left(\mathrm{d}a,\mathrm{d}b\right)$. Integration of the second term in the Lagrangian (1.12) gives

$$\iiint_0^{h_0}\dot{\boldsymbol{x}}\cdot\left(\boldsymbol{\Omega}\times\left(\boldsymbol{x}+\boldsymbol{d}\right)+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\rho\,\mathrm{d}\boldsymbol{a}=\iint\left\langle\dot{\mathcal{X}}\,,\,\boldsymbol{\Omega}\times\left(\mathcal{X}+\boldsymbol{d}\right)+\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\right\rangle\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,, \tag{2.10}$$

where $\mathfrak{X}$ and $\dot{\mathfrak{X}}$ are defined by

$$\mathfrak{X} = \left(x, y, \frac{1}{2}h\right) \quad \text{and} \quad \dot{\mathfrak{X}} = \left(\dot{x}, \dot{y}, \frac{1}{2}\dot{h}\right) . \qquad (2.11a, b)$$

See (A.1) in appendix A for the proof of (2.10). Restriction of the third and fourth terms in the Lagrangian (1.12) to columnar motion gives, respectively

$$\left.\begin{aligned} \iiint_0^{h_0} \boldsymbol{Q}^T\dot{\boldsymbol{q}} \cdot (\boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d}))\, \rho\, \mathrm{d}c\, \mathrm{d}\boldsymbol{a}_2 &= \iint \left\langle \boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\, ,\, \boldsymbol{\Omega} \times (\mathfrak{X} + \boldsymbol{d}) \right\rangle \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\, , \\ \iiint_0^{h_0} \tfrac{1}{2} \left\|\dot{\boldsymbol{q}}\right\|^2 \rho\, \mathrm{d}c\, \mathrm{d}\boldsymbol{a}_2 &= \iint \tfrac{1}{2} \left\|\dot{\boldsymbol{q}}\right\|^2 \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\, . \end{aligned}\right\} \qquad (2.12)$$

Restriction of the fifth term in (1.12) to columnar motion gives

$$\iiint_0^{h_0} -g\left(\boldsymbol{Q}\left(\boldsymbol{x} + \boldsymbol{d}\right) + \boldsymbol{q}\right) \cdot \hat{\boldsymbol{z}}\, \rho\, \mathrm{d}c\, \mathrm{d}\boldsymbol{a}_2 = \iint -g\left(\boldsymbol{Q}\left(\mathfrak{X} + \boldsymbol{d}\right) + \boldsymbol{q}\right) \cdot \hat{\boldsymbol{z}}\, \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\, . \qquad (2.13)$$

Finally, restriction of the term containing $\boldsymbol{I}_f$ in (1.12) to columnar motion gives

$$\begin{cases} \iiint_0^{h_0} \tfrac{1}{2} \left\|\boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d})\right\|^2 \rho\, \mathrm{d}c\, \mathrm{d}\boldsymbol{a}_2 = \iint \Big[ \tfrac{1}{2}\left(\Omega_1^2 + \Omega_2^2\right)\left(\dfrac{1}{3}h^2 + d_3^2 + d_3 h\right) \\ +\tfrac{1}{2}\left(\Omega_1^2 + \Omega_3^2\right)(y + d_2)^2 + \tfrac{1}{2}\left(\Omega_2^2 + \Omega_3^2\right)(x + d_1)^2 - \Omega_1\Omega_2\,(x + d_1)\,(y + d_2) \\ -\left(\tfrac{1}{2}h + d_3\right)\left(\Omega_1\Omega_3\,(x + d_1) + \Omega_2\Omega_3\,(y + d_2)\right)\Big]\, \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2 = \tfrac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_f^{SW}\boldsymbol{\Omega}\, , \end{cases} \qquad (2.14)$$

where $\boldsymbol{I}_f^{SW}$, which is a symmetric matrix, is the reduced shallow water version of the mass moment of inertia of the interior fluid relative to the point of rotation. The entries of $\boldsymbol{I}_f^{SW}$, in the Lagrangian particle-path and Eulerian settings, are given in appendix A.

Now, having derived the reduced terms (2.9), (2.10), (2.12), (2.13) and (2.14), the reduced shallow-water (SW) or Green–Naghdi (GN) variant of the 3–D Lagrangian action (1.12) for dynamic coupling between rigid-body motion in three dimensions and its interior shallow-water sloshing in two-horizontal space dimensions takes the form

$$\begin{cases} \mathscr{L}_{SW/\mathsf{GN}}\left(\boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}, \mathfrak{X}, \dot{\mathfrak{X}}\right) = \displaystyle\int_{t_1}^{t_2} \bigg( \iint \Big( \tfrac{1}{2}\left\|\dot{\boldsymbol{x}}_2\right\|^2 + \dfrac{1}{6}\dot{h}^2 + \dot{\mathfrak{X}} \cdot \left(\boldsymbol{\Omega} \times (\mathfrak{X} + \boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right) \\ +\boldsymbol{Q}^T\dot{\boldsymbol{q}} \cdot \left(\boldsymbol{\Omega} \times (\mathfrak{X} + \boldsymbol{d})\right) + \tfrac{1}{2}\left\|\dot{\boldsymbol{q}}\right\|^2 - g\left(\boldsymbol{Q}\left(\mathfrak{X} + \boldsymbol{d}\right) + \boldsymbol{q}\right) \cdot \hat{\boldsymbol{z}} \Big)\, \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2 + \tfrac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_f^{SW}\boldsymbol{\Omega} \\ + \tfrac{1}{2}m_v \left\|\dot{\boldsymbol{q}}\right\|^2 + \left(\boldsymbol{\Omega} \times m_v\overline{\boldsymbol{x}}_v\right) \cdot \boldsymbol{Q}^T\dot{\boldsymbol{q}} + \tfrac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_v\boldsymbol{\Omega} - m_v g\left(\boldsymbol{Q}\overline{\boldsymbol{x}}_v + \boldsymbol{q}\right) \cdot \hat{\boldsymbol{z}} \bigg)\, \mathrm{d}t\, . \end{cases} \qquad (2.15)$$

Taking the first variations of the shallow-water Lagrangian action (2.15) with respect to $\boldsymbol{\Omega}$, $\boldsymbol{Q}$, $\boldsymbol{q}$, and $\dot{\boldsymbol{q}}$ yields the Euler–Poincaré equations for the angular momentum and linear momentum of the rigid-body *containing shallow water*. Moreover, taking the first variation of the action integral (2.15) with respect to $\mathfrak{X}$ and $\dot{\mathfrak{X}}$ gives the three-dimensional rotating-translating SWEs for the motion of the interior shallow water relative to the body frame $\boldsymbol{x}$.

# 3 Derivation of the variational shallow-water and generalized Green–Naghdi equations for fluid sloshing in three-dimensional rotating and translating coordinates

The Lagrangian SWEs for the position $\boldsymbol{x}_2\,(t)$ of fluid particles in the body frame $\boldsymbol{x}$ can be provided by Hamilton's variational principle

$$\delta\mathscr{L}_{SW/\mathsf{GN}}\left(\boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}, \mathfrak{X}, \dot{\mathfrak{X}}\right) = 0\, , \qquad (3.1)$$

where the action integral $\mathscr{L}_{SW}$ is given in (2.15), by taking the variations $\delta\mathcal{X}$ and $\delta\dot{\mathcal{X}}$, with fixed endpoints $\delta\mathcal{X}(t_1) = \delta\mathcal{X}(t_2) = \mathbf{0}$, assuming that $\boldsymbol{\Omega}$, $\boldsymbol{Q}$, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are constants. The Lagrangian action (2.15) depends on $\boldsymbol{x}_2 = (x, y)$ not only explicitly, but also implicitly via $h(x, y, t)$. From (2.4) it can be concluded that $\delta h$ takes the form (Miles & Salmon 1985)

$$\delta h = -h\,\boldsymbol{\nabla}_2 \cdot \delta\boldsymbol{x}_2\,, \tag{3.2}$$

which leads to the following variational identity (Miles & Salmon 1985)

$$\iint \mathcal{F}\,\delta h\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2 = \iint \frac{1}{h}\,\boldsymbol{\nabla}_2\left(h^2\mathcal{F}\right) \cdot \delta\boldsymbol{x}_2\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,, \tag{3.3}$$

where $\mathcal{F}$ is any differentiable function of $\boldsymbol{x}_2$ and $t$. See appendix B for the proof of (3.2) and (3.3). Due to lengthy derivations, here we calculate the first variation of each term in (2.15) with respect to $\mathcal{X}$ and $\dot{\mathcal{X}}$ separately, and present some derivations in appendix C.

For the variations $\delta\dot{\boldsymbol{x}}_2$ of the first term in (2.15) we have

$$\delta\int_{t_1}^{t_2}\iint \tfrac{1}{2}\big\langle \dot{\boldsymbol{x}}_2\,,\,\dot{\boldsymbol{x}}_2\big\rangle\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t = \int_{t_1}^{t_2}\iint \big\langle \delta\boldsymbol{x}_2\,,\,-\ddot{\boldsymbol{x}}_2\big\rangle\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t\,, \tag{3.4}$$

where, when integrating by parts, we used the condition that the variations vanish at the endpoints in time. For the variation $\delta\dot{h}$ of the second term in (2.15), we have

$$\left\{\begin{aligned} &\delta\int_{t_1}^{t_2}\iint \frac{1}{6}\dot{h}^2\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t = \int_{t_1}^{t_2}\iint \frac{1}{3}\dot{h}\,\delta\dot{h}\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t = \int_{t_1}^{t_2}\iint -\frac{1}{3}\ddot{h}\,\delta h\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t \\ &= \int_{t_1}^{t_2}\iint -\frac{1}{3}\mathcal{D}^2 h\,\delta h\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t = \int_{t_1}^{t_2}\iint \left\langle \delta\boldsymbol{x}_2\,,\,-\frac{1}{3}\frac{1}{h}\boldsymbol{\nabla}_2\left(h^2\,\mathcal{D}^2 h\right)\right\rangle\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t\,, \end{aligned}\right. \tag{3.5}$$

where the variational identity (3.3) is used, and

$$\left.\begin{aligned} \ddot{h} &= \mathcal{D}^2 h = \mathcal{D}\left(h_t + \boldsymbol{u}_2\cdot\boldsymbol{\nabla}_2 h\right) = -\mathcal{D}\left(h\boldsymbol{\nabla}_2\cdot\boldsymbol{u}_2\right) \\ &= -\mathcal{D}h\boldsymbol{\nabla}_2\cdot\boldsymbol{u}_2 - h\boldsymbol{\nabla}_2\cdot\mathcal{D}\boldsymbol{u}_2 = h\left(\left(\boldsymbol{\nabla}_2\cdot\boldsymbol{u}_2\right)^2 - \boldsymbol{\nabla}_2\cdot\mathcal{D}\boldsymbol{u}_2\right)\,. \end{aligned}\right\} \tag{3.6}$$

The term $-(1/3)(1/h)\,\boldsymbol{\nabla}_2\left(h^2\,\mathcal{D}^2 h\right)$ gives rise to weakly dispersive nonlinear terms, i.e. the Green–Naghdi model, in the resulting variational SWEs.

For the variations $\delta\mathcal{X}$ and $\delta\dot{\mathcal{X}}$ of the first component of the third term in (2.15), assuming that $\boldsymbol{\Omega}$ is constant, we have

$$\delta\iiint\left\langle\dot{\mathcal{X}}\,,\,\boldsymbol{\Omega}\times(\mathcal{X}+\boldsymbol{d})\right\rangle\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t = \iiint\left\langle\delta\mathcal{X}\,,\,\underbrace{-\dot{\boldsymbol{\Omega}}\times(\mathcal{X}+\boldsymbol{d})}_{1}\,\underbrace{-2\boldsymbol{\Omega}\times\dot{\mathcal{X}}}_{2}\right\rangle\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t\,, \tag{3.7}$$

where, when integrating by parts, we used the condition that the variations vanish at the endpoints in time. The term denoted by 1 in (3.7) takes the form (see (C.1) in appendix C)

$$\left\{\begin{aligned} &\int_{t_1}^{t_2}\iint\left\langle\delta\mathcal{X}\,,\,-\dot{\boldsymbol{\Omega}}\times(\mathcal{X}+\boldsymbol{d})\right\rangle\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t = \\ &= \iiint\left\langle\delta\boldsymbol{x}_2\,,\,\begin{bmatrix}\dot{\Omega}_3(y+d_2) + \Big(\dot{\Omega}_2(x+d_1) - \dot{\Omega}_1(y+d_2)\Big)\,h_x - \dot{\Omega}_2 d_3 \\ -\dot{\Omega}_3(x+d_1) + \Big(\dot{\Omega}_2(x+d_1) - \dot{\Omega}_1(y+d_2)\Big)\,h_y + \dot{\Omega}_1 d_3\end{bmatrix}\right\rangle\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t\,, \end{aligned}\right. \tag{3.8}$$

and the term denoted by 2 in (3.7) simplifies to

$$
\left\{
\begin{aligned}
&\int_{t_1}^{t_2}\iint \left\langle \delta\boldsymbol{\mathfrak{X}}\,,\, -2\boldsymbol{\Omega}\times\dot{\boldsymbol{\mathfrak{X}}}\right\rangle \rho\, h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t = \int_{t_1}^{t_2}\iint \Bigg\langle \delta\boldsymbol{x}_2\,, \\
&\qquad \begin{bmatrix} -\Omega_2\dot{h} + 2\Omega_3\dot{y} + 2h_x\left(-\Omega_1\dot{y}+\Omega_2\dot{x}\right) + h\left(-\Omega_1\dot{y}_x + \Omega_2\dot{x}_x\right) \\ -2\Omega_3\dot{x} + \Omega_1\dot{h} + 2h_y\left(-\Omega_1\dot{y}+\Omega_2\dot{x}\right) + h\left(-\Omega_1\dot{y}_y + \Omega_2\dot{x}_y\right) \end{bmatrix} \Bigg\rangle\, \rho\, h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t\,,
\end{aligned}
\right.
\tag{3.9}
$$

and thus (3.7) becomes

$$
\left\{
\begin{aligned}
&\delta \int_{t_1}^{t_2}\iint \left\langle \dot{\boldsymbol{\mathfrak{X}}}\,,\, \boldsymbol{\Omega}\times\left(\boldsymbol{\mathfrak{X}}+\boldsymbol{d}\right)\right\rangle \rho\, h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t = \\
&= \int_{t_1}^{t_2}\iint \Bigg\langle \delta\boldsymbol{x}_2\,, \begin{bmatrix} \dot{\Omega}_3\left(y+d_2\right) + \Big(\dot{\Omega}_2\left(x+d_1\right) - \dot{\Omega}_1\left(y+d_2\right)\Big)\, h_x - \dot{\Omega}_2 d_3 \\ -\dot{\Omega}_3\left(x+d_1\right) + \Big(\dot{\Omega}_2\left(x+d_1\right) - \dot{\Omega}_1\left(y+d_2\right)\Big)\, h_y + \dot{\Omega}_1 d_3 \end{bmatrix} \\
&\qquad + \begin{bmatrix} 2\Omega_3\dot{y} - \Omega_2\dot{h} + 2\left(\Omega_2\dot{x} - \Omega_1\dot{y}\right)h_x + h\left(\Omega_2\dot{x}_x - \Omega_1\dot{y}_x\right) \\ -2\Omega_3\dot{x} + \Omega_1\dot{h} + 2\left(\Omega_2\dot{x} - \Omega_1\dot{y}\right)h_y + h\left(\Omega_2\dot{x}_y - \Omega_1\dot{y}_y\right) \end{bmatrix} \Bigg\rangle\, \rho\, h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t\,.
\end{aligned}
\right.
\tag{3.10}
$$

For the variations $\delta\dot{\boldsymbol{\mathfrak{X}}}$ of the second component of the third term in (2.15), assuming that $\boldsymbol{Q}$ and $\dot{\boldsymbol{q}}$ are constants, we have

$$
\left\{
\begin{aligned}
&\int_{t_1}^{t_2}\iint \left\langle \delta\dot{\boldsymbol{\mathfrak{X}}}\,,\, \boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\right\rangle \rho\, h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t = \int_{t_1}^{t_2}\iint -\left\langle \delta\boldsymbol{\mathfrak{X}}\,,\, \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\right)\right\rangle \rho\, h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t \\
&= \int_{t_1}^{t_2}\iint -\left\langle \delta\boldsymbol{\mathfrak{X}}\,,\, \underbrace{-\overbrace{\boldsymbol{Q}^{-1}\dot{\boldsymbol{Q}}}^{=\,\widehat{\boldsymbol{\Omega}}}\boldsymbol{Q}^{-1}}_{=\,(\mathrm{d}/\mathrm{d}t)\left(\boldsymbol{Q}^{-1}\right)}\,\dot{\boldsymbol{q}} + \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}\right\rangle \rho\, h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t \\
&= \int_{t_1}^{t_2}\iint \left\langle \delta\boldsymbol{\mathfrak{X}}\,,\, \overbrace{\underbrace{\boldsymbol{\Omega}\times\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}}_{\hookrightarrow\,\boxed{1}} - \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}}^{\boxed{2}\,\hookleftarrow}\right\rangle \rho\, h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t \qquad \boxed{1}:\ \text{using the hat map} \\
&= \int_{t_1}^{t_2}\iint \left\langle \delta\boldsymbol{x}_2\,, \begin{bmatrix} \Omega_2\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3 - \Omega_3\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2 - \ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1 \\ \Omega_3\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1 - \Omega_1\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3 - \ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2 \end{bmatrix}\right\rangle \rho\, h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t \\
&\quad + \int_{t_1}^{t_2}\iint \Bigg\langle \underbrace{\delta\boldsymbol{x}_2\,,\, \tfrac{1}{2}\frac{1}{h}\boldsymbol{\nabla}_2\left(h^2\left(\Omega_1\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2 - \Omega_2\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1 - \ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)\right)}_{\hookrightarrow\ \text{using the variational identity (3.3)}} \Bigg\rangle \rho\, h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t \\
&= \int_{t_1}^{t_2}\iint \Bigg\langle \delta\boldsymbol{x}_2\,, \begin{bmatrix} \Omega_2\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3 - \Omega_3\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2 - \ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1 \\ \Omega_3\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1 - \Omega_1\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3 - \ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2 \end{bmatrix} \\
&\qquad\qquad + \begin{bmatrix} \left(\Omega_1\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2 - \Omega_2\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1 - \ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right) h_x \\ \left(\Omega_1\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2 - \Omega_2\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1 - \ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right) h_y \end{bmatrix} \Bigg\rangle \rho\, h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t\,,
\end{aligned}
\right.
\tag{3.11}
$$

where

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}^{-1}\right) = -\boldsymbol{Q}^{-1}\dot{\boldsymbol{Q}}\boldsymbol{Q}^{-1}\,,
\tag{3.12}
$$

and, when integrating by parts, we used the condition that $\delta\boldsymbol{\mathfrak{X}}$ vanishes at the endpoints in time. See Marsden & Ratiu (1999) and Holm *et al.* (2009) for the proof of (3.12). Note that the terms denoted by $\boxed{\mathbf{2}}$ in (3.11) take the form

$$\boldsymbol{\Omega} \times \boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} - \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} = \begin{bmatrix} \Omega_2\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3 - \Omega_3\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_2 - \ddot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 \\ \Omega_3\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 - \Omega_1\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3 - \ddot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_2 \\ \Omega_1\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_2 - \Omega_2\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 - \ddot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3 \end{bmatrix}. \tag{3.13}$$

Similarly, for the variations $\delta\mathfrak{X}$ of the fourth in the Lagrangian action (2.15), assuming that $\boldsymbol{\Omega}$, $\boldsymbol{Q}$ and $\dot{\boldsymbol{q}}$ are constants, we have

$$\left.\begin{aligned} &\delta \int_{t_1}^{t_2} \iint \left\langle \boldsymbol{Q}^T\dot{\boldsymbol{q}}\,,\, \boldsymbol{\Omega} \times (\mathfrak{X} + \boldsymbol{d}) \right\rangle \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\, \mathrm{d}t = \int_{t_1}^{t_2} \iint \Bigg\langle \delta\boldsymbol{x}_2\,, \\ &\begin{bmatrix} \Omega_3\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_2 - \Omega_2\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3 + (\Omega_2\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 - \Omega_1\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_2)\, h_x \\ \Omega_1\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3 - \Omega_3\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 + (\Omega_2\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 - \Omega_1\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_2)\, h_y \end{bmatrix} \Bigg\rangle \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\, \mathrm{d}t\,. \end{aligned}\right\} \tag{3.14}$$

For the variations $\delta\mathfrak{X}$ of the potential energy of the interior fluid in (2.15), assuming that $\boldsymbol{Q}$ and $\boldsymbol{q}$ are constants, we have (see (C.2) in appendix C)

$$\left.\begin{aligned} &\delta \int_{t_1}^{t_2} \iint -g \left\langle \hat{\boldsymbol{z}}\,,\, \boldsymbol{Q}\,(\mathfrak{X} + \boldsymbol{d}) + \boldsymbol{q} \right\rangle \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\, \mathrm{d}t \\ &= \int_{t_1}^{t_2} \iint \left\langle \delta\boldsymbol{x}_2\,,\, -g \begin{bmatrix} \boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_1 \\ \boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_2 \end{bmatrix} - g \begin{bmatrix} \boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_3\, h_x \\ \boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_3\, h_y \end{bmatrix} \right\rangle \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\, \mathrm{d}t\,. \end{aligned}\right\} \tag{3.15}$$

Taking the variations $\delta\boldsymbol{x}_2$ and $\delta h$ of the mass moment of inertia of the interior shallow water in the action integral (2.15), assuming that $\boldsymbol{\Omega}$ is constant, gives

$$\left.\begin{aligned} &\delta \int_{t_1}^{t_2} \tfrac{1}{2} \left\langle \boldsymbol{\Omega}\,,\, \boldsymbol{I}_f^{SW} \boldsymbol{\Omega} \right\rangle \mathrm{d}t = \int_{t_1}^{t_2} \iint \Bigg\langle \delta\boldsymbol{x}_2\,, \\ &\begin{bmatrix} ((h + d_3)\,(\Omega_1^2 + \Omega_2^2) - (x + d_1)\,\Omega_1\Omega_3 - (y + d_2)\,\Omega_2\Omega_3)\, h_x \\ + (x + d_1)\,(\Omega_2^2 + \Omega_3^2) - (y + d_2)\,\Omega_1\Omega_2 - (h + d_3)\,\Omega_1\Omega_3 \\ \\ ((h + d_3)\,(\Omega_1^2 + \Omega_2^2) - (x + d_1)\,\Omega_1\Omega_3 - (y + d_2)\,\Omega_2\Omega_3)\, h_y \\ + (y + d_2)\,(\Omega_1^2 + \Omega_3^2) - (x + d_1)\,\Omega_1\Omega_2 - (h + d_3)\,\Omega_2\Omega_3 \end{bmatrix} \Bigg\rangle \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\, \mathrm{d}t\,, \end{aligned}\right\} \tag{3.16}$$

where the proof of (3.16) is given in appendix C.

Now, since $\delta\boldsymbol{x}_2$ is arbitrary, from (3.4), (3.5), (3.10), (3.11), (3.14), (3.15), (3.16) and Hamilton's variational principle (3.1) it can be concluded that the new *variational SWEs* in two-horizontal space dimensions, i.e. the horizontal $x$– and $y$–momentum equations, in the Lagrangian particle-path formulation, take the form

$$\begin{cases} \ddot{x} + \dfrac{1}{3}\dfrac{1}{h}\dfrac{\partial}{\partial x}\left(h^2\ddot{h}\right) + \Omega_2\dot{h} + \Big[2\Omega_1\dot{y} - 2\Omega_2\dot{x} + \left(\dot{\Omega}_1 + \Omega_2\Omega_3\right)(y + d_2) \\ + \ddot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3 + g\,\boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_3 - (h + d_3)\left(\Omega_1^2 + \Omega_2^2\right) + \left(\Omega_1\Omega_3 - \dot{\Omega}_2\right)(x + d_1)\Big] \dfrac{\partial h}{\partial x} \\ = 2\Omega_3\dot{y} + (\Omega_2\dot{x}_x - \Omega_1\dot{y}_x)\,h + \left(\dot{\Omega}_3 - \Omega_1\Omega_2\right)(y + d_2) - \dot{\Omega}_2 d_3 - \ddot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 \\ + (x + d_1)\left(\Omega_2^2 + \Omega_3^2\right) - (h + d_3)\,\Omega_1\Omega_3 - g\,\boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_1\,, \end{cases} \tag{3.17}$$

and

$$
\begin{cases}
\ddot{y} + \dfrac{1}{3}\dfrac{1}{h}\dfrac{\partial}{\partial y}\left(h^2\ddot{h}\right) - \Omega_1\dot{h} + \Big[ -2\Omega_2\dot{x} + 2\Omega_1\dot{y} + \left(\dot{\Omega}_1 + \Omega_2\Omega_3\right)(y+d_2) \\
+\ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3 + g\,\boldsymbol{e}_3\cdot\boldsymbol{Q}\boldsymbol{e}_3 - (h+d_3)\left(\Omega_1^2+\Omega_2^2\right) + \left(\Omega_1\Omega_3 - \dot{\Omega}_2\right)(x+d_1)\Big]\dfrac{\partial h}{\partial y} \\
= -2\Omega_3\dot{x} + \left(\Omega_2\dot{x}_y - \Omega_1\dot{y}_y\right)h - \left(\dot{\Omega}_3 + \Omega_1\Omega_2\right)(x+d_1) + \dot{\Omega}_1 d_3 - \ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2 \\
+ (y+d_2)\left(\Omega_1^2+\Omega_3^2\right) - (h+d_3)\,\Omega_2\Omega_3 - g\,\boldsymbol{e}_3\cdot\boldsymbol{Q}\boldsymbol{e}_2\,.
\end{cases}
\tag{3.18}
$$

Transforming, the momentum equations (3.17) and (3.18) from the Lagrangian particle-path setting to Eulerian coordinates, replacing the Lagrangian variables $\ddot{x}$, $\ddot{y}$, $\dot{x}$, $\dot{y}$, $\dot{h}$ and $\ddot{h}$ by their respective Eulerian quantities $\mathcal{D}u/\mathcal{D}t$, $\mathcal{D}v/\mathcal{D}t$, $u$, $v$, $\mathcal{D}h$ and $\mathcal{D}^2h$ respectively, the SWEs (3.17) and (3.18) respectively take the form

$$
\begin{cases}
\dfrac{\mathcal{D}u}{\mathcal{D}t} + \overbrace{\dfrac{1}{3}\dfrac{1}{h}\dfrac{\partial}{\partial x}\left(h^2\mathcal{D}^2h\right)}^{\text{GN}} + \Omega_2\mathcal{D}h + \Big[2\Omega_1 v - 2\Omega_2 u + \left(\dot{\Omega}_1 + \Omega_2\Omega_3\right)(y+d_2) \\
+\ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3 + g\,\boldsymbol{e}_3\cdot\boldsymbol{Q}\boldsymbol{e}_3 - (h+d_3)\left(\Omega_1^2+\Omega_2^2\right) + \left(\Omega_1\Omega_3 - \dot{\Omega}_2\right)(x+d_1)\Big]\dfrac{\partial h}{\partial x} \\
= 2\Omega_3 v + \left(\Omega_2 u_x - \Omega_1 v_x\right)h + \left(\dot{\Omega}_3 - \Omega_1\Omega_2\right)(y+d_2) - \dot{\Omega}_2 d_3 - \ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1 \\
+ (x+d_1)\left(\Omega_2^2+\Omega_3^2\right) - (h+d_3)\,\Omega_1\Omega_3 - g\,\boldsymbol{e}_3\cdot\boldsymbol{Q}\boldsymbol{e}_1\,,
\end{cases}
\tag{3.19}
$$

and

$$
\begin{cases}
\dfrac{\mathcal{D}v}{\mathcal{D}t} + \overbrace{\dfrac{1}{3}\dfrac{1}{h}\dfrac{\partial}{\partial y}\left(h^2\mathcal{D}^2h\right)}^{\text{GN}} - \Omega_1\mathcal{D}h + \Big[ -2\Omega_2 u + 2\Omega_1 v + \left(\dot{\Omega}_1 + \Omega_2\Omega_3\right)(y+d_2) \\
+\ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3 + g\,\boldsymbol{e}_3\cdot\boldsymbol{Q}\boldsymbol{e}_3 - (h+d_3)\left(\Omega_1^2+\Omega_2^2\right) + \left(\Omega_1\Omega_3 - \dot{\Omega}_2\right)(x+d_1)\Big]\dfrac{\partial h}{\partial y} \\
= -2\Omega_3 u + \left(\Omega_2 u_y - \Omega_1 v_y\right)h - \left(\dot{\Omega}_3 + \Omega_1\Omega_2\right)(x+d_1) + \dot{\Omega}_1 d_3 - \ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2 \\
+ (y+d_2)\left(\Omega_1^2+\Omega_3^2\right) - (h+d_3)\,\Omega_2\Omega_3 - g\,\boldsymbol{e}_3\cdot\boldsymbol{Q}\boldsymbol{e}_2\,,
\end{cases}
\tag{3.20}
$$

where the terms denoted by GN (an auxiliary acceleration) in the horizontal momentum equations (3.19) and (3.20) lead to higher-order dispersive terms (see e.g. Miles & Salmon 1985). In summary, the candidate SWEs for $(h,u,v)$ for fluid sloshing in a vessel undergoing rigid-body motion in three dimensions are the continuity equation (2.8) and the momentum equations (3.19) and (3.20) in Eulerian coordinates. The new variational SWEs (2.8), (3.19) and (3.20) with the dispersive terms denoted by GN in (3.19) and (3.20) are the *generalized Green–Naghdi equations for shallow-water sloshing in three-dimensional rotating and translating coordinates*. See Miles & Salmon (1985) for derivation of the Green–Naghdi equations (Green & Naghdi 1976) from Hamilton's principle, and see Dellar & Salmon (2005) for derivation of the Green–Naghdi equations for the non-traditional rotating SWEs from the shallow-water variant of the Lagrangian functional (1.9).

Next, discarding the Green–Naghdi terms, i.e. the GN terms, and substituting for $\mathcal{D}h$ from (2.6) in the momentum equations (3.19) and (3.20), the 3–D rotating–translating variational SWEs may be written as

$$\left.\begin{aligned} u_t + uu_x + vu_y + \alpha_{11}(x,y,t)\,h_x + \alpha_{12}(x,y,t)\,h_y &= \beta_1(x,y,t)\ , \\ v_t + uv_x + vv_y + \alpha_{21}(x,y,t)\,h_x + \alpha_{22}(x,y,t)\,h_y &= \beta_2(x,y,t)\ , \\ h_t + (hu)_x + (hv)_y &= 0\,, \end{aligned}\right\} \tag{3.21}$$

where the coefficients $\alpha_{11}$, $\alpha_{12}$, $\beta_1$, $\alpha_{21}$, $\alpha_{22}$ and $\beta_2$ are

$$\left.\begin{aligned} \alpha_{11}(x,y,t) &= 2\Omega_1 v - \Omega_2 u + \left(\dot{\Omega}_1 + \Omega_2\Omega_3\right)(y+d_2) + g\,\boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_3 \\ &\quad - \left(\Omega_1^2 + \Omega_2^2\right)(h+d_3) + \left(\Omega_1\Omega_3 - \dot{\Omega}_2\right)(x+d_1) + \ddot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3\,, \\ \alpha_{12}(x,y,t) &= \Omega_2 v\,, \\ \beta_1(x,y,t) &= -\Omega_2 h_t + 2\Omega_3 v + (\Omega_2 u_x - \Omega_1 v_x)\,h + \left(\dot{\Omega}_3 - \Omega_1\Omega_2\right)(y+d_2) \\ &\quad -\dot{\Omega}_2 d_3 - \ddot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 - g\,\boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_1 + \left(\Omega_2^2 + \Omega_3^2\right)(x+d_1) \\ &\quad -\Omega_1\Omega_3(h+d_3)\ , \\ \alpha_{21}(x,y,t) &= -\Omega_1 u\,, \\ \alpha_{22}(x,y,t) &= -2\Omega_2 u + \Omega_1 v + \left(\dot{\Omega}_1 + \Omega_2\Omega_3\right)(y+d_2) + g\,\boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_3 \\ &\quad - \left(\Omega_1^2 + \Omega_2^2\right)(h+d_3) + \left(\Omega_1\Omega_3 - \dot{\Omega}_2\right)(x+d_1) + \ddot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3\,, \\ \beta_2(x,y,t) &= \Omega_1 h_t - 2\Omega_3 u + (\Omega_2 u_y - \Omega_1 v_y)\,h - \left(\dot{\Omega}_3 + \Omega_1\Omega_2\right)(x+d_1) \\ &\quad +\dot{\Omega}_1 d_3 - \ddot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_2 - g\,\boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_2 + \left(\Omega_1^2 + \Omega_3^2\right)(y+d_2) \\ &\quad -\Omega_2\Omega_3(h+d_3)\ . \end{aligned}\right\} \tag{3.22}$$

Now, if we set $U(x,y,t) = u(x,y,t)$ and $V(x,y,t) = v(x,y,t)$ in the surface SWEs (1.3), which is consistent with the theory of shallow-water equations, it can be concluded that the coefficients of the new SWEs (3.22) are related to the coefficients of the surface SWEs (1.5) by

$$\left.\begin{aligned} \alpha_{11}(x,y,t) &= a_{11}(x,y,t) - \Omega_2 u\,, \quad \alpha_{12}(x,y,t) = \tfrac{1}{2}a_{12}(x,y,t)\ , \\ \beta_1(x,y,t) &= b_1(x,y,t) + \Omega_2 h_t + \left(\Omega_2 u_x - \Omega_1 v_x + \dot{\Omega}_2\right)h\,, \end{aligned}\right\} \tag{3.23}$$

and

$$\left.\begin{aligned} \alpha_{22}(x,y,t) &= a_{22}(x,y,t) + \Omega_1 v\,, \quad \alpha_{21}(x,y,t) = \tfrac{1}{2}a_{21}(x,y,t)\ , \\ \beta_2(x,y,t) &= b_2(x,y,t) - \Omega_1 h_t + \left(\Omega_2 u_y - \Omega_1 v_y - \dot{\Omega}_1\right)h\,. \end{aligned}\right\} \tag{3.24}$$

The surface SWEs (1.3) are derived using a reduction method applied to the three-dimensional rotating Euler equations relative to the body frame $\boldsymbol{x}$ (Alemi Ardakani & Bridges 2011). In non-variational approaches, the conservation laws associated with the Eulerian equations may remain hidden and their derivations are often tedious and unrevealing. In the variational or Hamiltonian approach, the conservation laws are known to exist if the Lagrangian functional reveals the corresponding symmetry property. Preserving conservation laws is a primary advantage of approximation methods based on Hamilton's variational principle compared with some approximation methods applied directly to the equations of motion. The new variational SWEs (3.21) and the Green-Naghdi equations (3.19), (3.20) and (2.8) retain conservation laws because the approximate shallow-water Lagrangian (2.15) do not violate the symmetry properties of the exact Lagrangian functional (1.12) for the three-dimensional problem. See section §4 for the particle relabeling symmetry property of the Lagrangian (2.15), and the material conservation of potential-vorticity for the proposed variational SWEs and the Green–Naghdi equations.

The shallow-water/Green–Naghdi Lagrangian action (2.15) is described in the Lagrangian particle-path setting. Transformation of the action integral (2.15) to the Eulerian setting gives

$$\begin{cases} \mathscr{L}_{SW/\mathsf{GN}}\left(\boldsymbol{\Omega},\boldsymbol{Q},\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{u}_2,h\right)=\displaystyle\int_{t_1}^{t_2}\left(\iint\left(\tfrac{1}{2}\left\|\boldsymbol{u}_2\right\|^2+\frac{1}{6}h^2\left(\boldsymbol{\nabla}_2\cdot\boldsymbol{u}_2\right)^2\right.\right.\\ +\boldsymbol{\mathfrak{U}}\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{\mathfrak{X}}+\boldsymbol{d})+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{\mathfrak{X}}+\boldsymbol{d})\right)+\tfrac{1}{2}\left\|\dot{\boldsymbol{q}}\right\|^2\\ \left.-g\left(\boldsymbol{Q}\left(\boldsymbol{\mathfrak{X}}+\boldsymbol{d}\right)+\boldsymbol{q}\right)\cdot\hat{\boldsymbol{z}}\right)\rho\,h\,\mathrm{d}\boldsymbol{x}_2+\tfrac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_f^{SW}\boldsymbol{\Omega}+\tfrac{1}{2}m_v\left\|\dot{\boldsymbol{q}}\right\|^2\\ \left.+\left(\boldsymbol{\Omega}\times m_v\overline{\boldsymbol{x}}_v\right)\cdot\boldsymbol{Q}^T\dot{\boldsymbol{q}}+\tfrac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_v\boldsymbol{\Omega}-m_vg\left(\boldsymbol{Q}\overline{\boldsymbol{x}}_v+\boldsymbol{q}\right)\cdot\hat{\boldsymbol{z}}\right)\mathrm{d}t\,, \end{cases} \tag{3.25}$$

where

$$\boldsymbol{\mathfrak{U}}=\left(\boldsymbol{u}_2,-\tfrac{1}{2}h\boldsymbol{\nabla}_2\cdot\boldsymbol{u}_2\right)=\left(u,v,-\tfrac{1}{2}h\left(u_x+v_y\right)\right)\,, \tag{3.26}$$

and entries of the symmetric matrix $\boldsymbol{I}_f^{SW}$ in the Eulerian setting are defined in appendix A. The Lagrangian variations $\delta\boldsymbol{x}_2$ induce variations of the Eulerian quantities $\boldsymbol{u}_2$ and $h$ via the *continuity equation* (Oliver 2006)

$$\delta h+\boldsymbol{\nabla}_2\cdot\left(\boldsymbol{w}_2h\right)=0\,, \tag{3.27}$$

and the so-called *Lin constraint* (Lin 1963; Bretherton 1970; Oliver 2006)

$$\delta\boldsymbol{u}_2=\dot{\boldsymbol{w}}_2+\boldsymbol{\nabla}_2\boldsymbol{w}_2\,\boldsymbol{u}_2-\boldsymbol{\nabla}_2\boldsymbol{u}_2\,\boldsymbol{w}_2\,, \tag{3.28}$$

where $\boldsymbol{w}_2$ is a vector-valued free variation which is defined by $\delta\boldsymbol{x}_2=\boldsymbol{w}_2\circ\boldsymbol{x}_2$. In this section we derived the Eulerian form of the generalized Green–Naghdi momentum equations (3.19) and (3.20) by taking the variations $\delta\boldsymbol{\mathfrak{X}}$ and $\delta\dot{\boldsymbol{\mathfrak{X}}}$ of the Lagrangian action (2.15) in the Lagrangian particle-path description, and transforming the equations of motion to Eulerian coordinates. To derive the momentum equations we could alternatively take the variations $\delta\boldsymbol{u}_2$ and $\delta h$ of the Lagrangian action (3.25) in the Eulerian setting by using the Eulerian variations (3.27) and (3.28).

# 4 Potential vorticity for the generalized Green–Naghdi equations and variational SWEs with 3–D rotation vector

The aim in this section is to derive a conservation law for the material conservation of potential vorticity for the variational SWEs (3.21) and the generalized Green–Naghdi equations (3.19), (3.20) and (2.8) for fluid sloshing in a container undergoing prescribed rigid-body motion in three dimensions. The existence of this conservation law is guaranteed by the variational formulation of §3, and Noether's theorem that relates symmetries in a variational principle to conservation laws (e.g. Noether 1918; Hill 1951; Goldstein 1980; Stewart & Dellar 2010).

Potential vorticity conservation arises from the particle-relabeling symmetry property of the shallow-water Lagrangian (2.15). The usual particle-relabeling symmetry arguments are given by Ripa (1981), Salmon (1982a, 1982b, 1983, 1988, 1998), Shepherd (1990), Müller (1995), and Padhye & Morrison (1996). Dellar & Salmon (2005) extended these arguments to derive a general expression for the conservation of potential vorticity in terms of the canonical momenta obtained from a Lagrangian functional. The variational SWEs (3.21) and the generalized Green–Naghdi equations (3.19), (3.20) and (2.8) possess a potential vorticity $\mathscr{P}$ that obeys the conservation law $\mathcal{D}\mathscr{P} = \mathscr{P}_t + \boldsymbol{u}_2 \cdot \boldsymbol{\nabla}_2 \mathscr{P} = 0$. The conserved potential vorticity in terms of the Eulerian spatial derivatives of the canonical momenta $\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2)$ takes the form (Dellar & Salmon 2005)

$$\mathscr{P} = \frac{\partial\,(x,y)}{\partial\,(a,b)} \left( \frac{\partial\,(\mathcal{P}_2, y)}{\partial\,(x,y)} + \frac{\partial\,(\mathcal{P}_1, x)}{\partial\,(x,y)} \right) = \frac{1}{h} \left( \frac{\partial \mathcal{P}_2}{\partial x} - \frac{\partial \mathcal{P}_1}{\partial y} \right) , \tag{4.1}$$

which is applicable to all Lagrangian functionals in which the particle labels $\boldsymbol{a}_2 = (a,b)$ only appear through the wave height defined in (2.4). The proof of (4.1) is given in appendix A of Dellar & Salmon (2005).

The canonical momenta $\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2)$ in (4.1) for the shallow-water Lagrangian (2.15) can be obtained from

$$\boldsymbol{\mathcal{P}} = \frac{\delta \widehat{\mathscr{L}}}{\delta \dot{\boldsymbol{\mathcal{X}}}} , \tag{4.2}$$

where $\widehat{\mathscr{L}}$ is the kinetic energy minus the potential energy of the fluid in the shallow-water Lagrangian (2.15), that is

$$\left.\begin{aligned} \widehat{\mathscr{L}} \;&=\; \tfrac{1}{2} \left\| \dot{\boldsymbol{x}}_2 \right\|^2 + \frac{1}{6} \dot{h}^2 + \dot{\boldsymbol{\mathcal{X}}} \cdot \left( \boldsymbol{\Omega} \times (\boldsymbol{\mathcal{X}} + \boldsymbol{d}) + \boldsymbol{Q}^T \dot{\boldsymbol{q}} \right) + \boldsymbol{Q}^T \dot{\boldsymbol{q}} \cdot \left( \boldsymbol{\Omega} \times (\boldsymbol{\mathcal{X}} + \boldsymbol{d}) \right) \\ &\quad - g \left( \boldsymbol{Q}\, (\boldsymbol{\mathcal{X}} + \boldsymbol{d}) + \boldsymbol{q} \right) \cdot \hat{\boldsymbol{z}} + \tfrac{1}{2} \boldsymbol{\Omega} \cdot \boldsymbol{I}_f^{SW} \boldsymbol{\Omega} \, . \end{aligned}\right\} \tag{4.3}$$

The variational derivatives defining $\boldsymbol{\mathcal{P}}$ are taken using a mass-weighted inner product for integrals with respect to $\mathrm{d}\boldsymbol{a}_2 = \mathrm{d}a\, \mathrm{d}b$, that is

$$\left\{\begin{aligned} &\delta \mathscr{L}_{SW/\mathsf{GN}} = \iint \left( \left\langle \frac{\delta \widehat{\mathscr{L}}}{\delta \boldsymbol{\mathcal{X}}} \, , \, \delta \boldsymbol{\mathcal{X}} \right\rangle + \left\langle \frac{\delta \widehat{\mathscr{L}}}{\delta \dot{\boldsymbol{\mathcal{X}}}} \, , \, \delta \dot{\boldsymbol{\mathcal{X}}} \right\rangle \right) \rho\, h_0 \, \mathrm{d}\boldsymbol{a}_2 \\ &= \iint \left( \left\langle \frac{\delta \widehat{\mathscr{L}}}{\delta \boldsymbol{x}_2} + \frac{1}{h} \boldsymbol{\nabla}_2 \left( h^2 \, \frac{\delta \widehat{\mathscr{L}}}{\delta h} + h \dot{h} \, \frac{\delta \widehat{\mathscr{L}}}{\delta \dot{h}} \right) , \, \delta \boldsymbol{x}_2 \right\rangle \right. \\ &\qquad \left. + \left\langle \frac{\delta \widehat{\mathscr{L}}}{\delta \dot{\boldsymbol{x}}_2} + \frac{1}{h} \, \boldsymbol{\nabla}_2 \left( h^2 \, \frac{\delta \widehat{\mathscr{L}}}{\delta \dot{h}} \right) , \, \delta \dot{\boldsymbol{x}}_2 \right\rangle \right) \rho\, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, . \end{aligned}\right. \tag{4.4}$$

See appendix D for the proof of (4.4). Hence, the canonical momenta $\boldsymbol{\mathcal{P}}$ for the shallow-water sloshing Lagrangian (2.15) takes the form

$$\boldsymbol{\mathcal{P}} = \frac{\delta \widehat{\mathscr{L}}}{\delta \dot{\boldsymbol{x}}_2} + \frac{1}{h} \, \boldsymbol{\nabla}_2 \left( h^2 \, \frac{\delta \widehat{\mathscr{L}}}{\delta \dot{h}} \right) . \tag{4.5}$$

Now, from (4.1) and (4.5) we infer that the conserved potential vorticity for the generalized Green–Naghdi equations in rotating coordinates (3.19), (3.20) and (2.8) reads

$$\left.\begin{aligned} \mathscr{P}_{\mathsf{GN}} \;&=\; \frac{1}{h} \left( \frac{\partial \mathcal{P}_2}{\partial x} - \frac{\partial \mathcal{P}_1}{\partial y} \right) \\ &=\; \mathscr{P}_{SW} \underbrace{- \frac{1}{3h} \boldsymbol{\nabla}_2 h \cdot \boldsymbol{\nabla}_2^{\perp} \boldsymbol{\nabla}_2 \cdot \boldsymbol{u}_2}_{= \frac{1}{3h} \widehat{\boldsymbol{k}} \cdot \boldsymbol{\nabla} \times (\mathcal{D}h \boldsymbol{\nabla}_2 h)} = \mathscr{P}_{SW} + \frac{1}{3h} \frac{\partial\,(\mathcal{D}h, h)}{\partial\,(x,y)} \, , \end{aligned}\right\} \tag{4.6}$$

where $\mathscr{P}_{SW}$ is the conserved potential vorticity for the rotating variational SWEs (3.21), that is

$$\mathscr{P}_{SW} = \frac{v_x - u_y + 2\Omega_3 - \boldsymbol{\Omega}_2 \cdot \boldsymbol{\nabla}_2 h}{h}\,, \tag{4.7}$$

and $\boldsymbol{\nabla}_2^{\perp} = (-\partial/\partial y, \partial/\partial x)$, $\boldsymbol{\Omega}_2 = (\Omega_1, \Omega_2)$, and $\widehat{\boldsymbol{k}}$ is the unit vector in the $z$ direction. The potential vorticity $\mathscr{P}_{SW}$ in (4.7) is a simplified form of the potential vorticity derived by Dellar & Salmon (2005) for the shallow-water equations with a complete Coriolis force and topography (see equation (1) in Dellar & Salmon 2005). The second term in the generalized Green–Naghdi potential vorticity (4.6) is a pseudovorticity that is derived from the term $(1/6)\,\dot{h}^2$ in the shallow-water Lagrangian (2.15). The pseudovorticity term in (4.6) was first derived by Miles & Salmon (1985) for the Green–Naghdi equations (Green & Naghdi 1976). See equation (5.5) in Miles & Salmon (1985). To summarise, the potential vorticity (4.6) for the new generalized Green–Naghdi equations with 3–D rotation vector is a combination of the potential vorticity expressions given by Miles & Salmon (1985) and Dellar & Salmon (2005).

Finally, if we write the potential vorticity (1.8) for the surface SWEs (1.3) (Alemi Ardakani & Bridges 2011) in the form

$$\widehat{\mathscr{P}} = \frac{V_x - U_y + 2\Omega_3 - 2\boldsymbol{\Omega}_2 \cdot \boldsymbol{\nabla}_2 h}{h}\,, \tag{4.8}$$

and set $U = u$ and $V = v$, we conclude that the potential vorticity expressions for the variational SWEs (3.21) and the surface SWEs (1.3) are related by

$$\mathscr{P}_{SW} = \widehat{\mathscr{P}} + \frac{\boldsymbol{\Omega}_2 \cdot \boldsymbol{\nabla}_2 h}{h}\,. \tag{4.9}$$

# 5 Zero-potential-vorticity flow and a generalization of the Whitham equations for fluid sloshing in three-dimensional rotating and translating coordinates

The Green–Naghdi system is a long-wave model for gravity-driven surface water waves, which are long but may not have small amplitude. The assumption that the idealised fluid moves in columns is equivalent to that made by Green & Naghdi (1976) which implies the restriction (Miles & Salmon 1985)

$$\beta = (h_0/L)^2 \ll 1\,, \tag{5.1}$$

where $h_0$ is the mean water depth and $L$ is a horizontal lengthscale. The assumption (5.1) implies that dispersion in the Green–Naghdi model is weak (Miles & Salmon 1985). The Green–Naghdi equations can also be derived by depth-averaging the Euler equations using a scaling argument and asymptotics, and retaining only first-order terms in $\beta$ in the resulting set of equations (e.g. Gavrilyuk *et al.* 2015). Miles & Salmon (1985) used the assumption of *zero-potential-vorticity* flow in the Eulerian form of Hamilton's variational principle for the Green–Naghdi system to derive a canonical generalization of Boussinesq's equations derived by Whitham (1967), which fully accommodates nonlinearity. In Whitham's equations (see equation (12) in Whitham (1967) or equations (1.8*a*) and (1.8*b*) in Miles & Salmon (1985)) nonlinearity is of the same order of dispersion, i.e. the amplitude parameter $a/h_0 = \mathcal{O}\,(\beta)$.

The aim in this section is to apply the assumption of zero-potential-vorticity flow to the fluid component of the Green–Naghdi Lagrangian action (3.25) in *Eulerian coordinates* to derive new Boussinesq-like evolution equations, which are a generalization of the Whitham equations for fluid sloshing in three-dimensional rotating and translating coordinates. By appending the constraint of continuity to the fluid component of the Lagrangian (3.25), the variational principle

$$\delta \mathscr{L}_{\mathsf{GN}}\left(\boldsymbol{u}_2, h, \lambda\right) = 0\,, \tag{5.2}$$

with the Lagrangian action

$$\left\{\begin{aligned}
&\mathscr{L}_{\mathsf{GN}}\left(\boldsymbol{u}_2, h, \lambda\right) = \int_{t_1}^{t_2}\iint\Big(\tfrac{1}{2}\left\|\boldsymbol{u}_2\right\|^2 + \frac{1}{6}h^2\left(\boldsymbol{\nabla}_2\cdot\boldsymbol{u}_2\right)^2 \\
&\qquad + \boldsymbol{\mathcal{U}}\cdot\left(\boldsymbol{\Omega}\times\left(\boldsymbol{\mathcal{X}}+\boldsymbol{d}\right) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\cdot\left(\boldsymbol{\Omega}\times\left(\boldsymbol{\mathcal{X}}+\boldsymbol{d}\right)\right) + \tfrac{1}{2}\left\|\dot{\boldsymbol{q}}\right\|^2 \\
&\qquad - g\left(\boldsymbol{Q}\left(\boldsymbol{\mathcal{X}}+\boldsymbol{d}\right)+\boldsymbol{q}\right)\cdot\hat{\boldsymbol{z}}\Big)\,\rho\, h\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t + \int_{t_1}^{t_2}\tfrac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_f^{SW}\boldsymbol{\Omega}\,\mathrm{d}t \\
&\left\{\begin{aligned}
&\underbrace{+\int_{t_1}^{t_2}\iint\lambda\left(\mathcal{D}h + h\boldsymbol{\nabla}_2\cdot\boldsymbol{u}_2\right)\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\,,}_{} \\
&\downarrow \text{ (integrating the constraint term by parts)} \\
&= -\int_{t_1}^{t_2}\iint h\left(\lambda_t + \boldsymbol{u}_2\cdot\boldsymbol{\nabla}_2\lambda\right)\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\,,
\end{aligned}\right\} \rightarrow \left\{\begin{aligned}&\text{Taking into account that}\\ &u=0 \text{ at } x=0, L_1\\ &\text{and } v=0 \text{ at } y=0, L_2.\end{aligned}\right.
\end{aligned}\right. \tag{5.3}$$

for the variations $\delta\boldsymbol{u}_2$, $\delta h$ and $\delta\lambda$ yields differential equations whose solutions also satisfy the rotating Green–Naghdi equations (3.19), (3.20) and (2.8), but with zero potential vorticity $\mathscr{P}_{\mathsf{GN}} = 0$. In (5.3) $\lambda$ is the Lagrange multiplier of the continuity equation (2.8).

Following Miles & Salmon (1985) we may write the Green–Naghdi potential vorticity (4.6) in the form

$$\mathscr{P}_{\mathsf{GN}} = \frac{1}{h}\left(\mathbb{P} + \mathbb{P}^{\star}\right)\,, \tag{5.4}$$

where

$$\left.\begin{aligned}
\mathbb{P} &= \widehat{\boldsymbol{k}}\cdot\boldsymbol{\nabla}\times\left(\boldsymbol{u}_2 + \boldsymbol{R}_2\right) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + 2\Omega_3 - \boldsymbol{\Omega}_2\cdot\boldsymbol{\nabla}_2 h\,,\\
\mathbb{P}^{\star} &= \frac{1}{3}\widehat{\boldsymbol{k}}\cdot\boldsymbol{\nabla}\times\left(\mathcal{D}h\boldsymbol{\nabla}_2 h\right)\,,
\end{aligned}\right\} \tag{5.5}$$

and $\boldsymbol{R}_2$ is a vector potential, yet to be determined, such that

$$\widehat{\boldsymbol{k}}\cdot\boldsymbol{\nabla}\times\boldsymbol{R}_2 = 2\Omega_3 - \boldsymbol{\Omega}_2\cdot\boldsymbol{\nabla}_2 h\,. \tag{5.6}$$

The pseudovorticity $\mathbb{P}^{\star}$ may be recast in the alternative form (Miles & Salmon 1985)

$$\mathbb{P}^{\star} = \frac{1}{3}n^{-1}\widehat{\boldsymbol{k}}\cdot\boldsymbol{\nabla}\times\left(h^{-n}\boldsymbol{\nabla}_2\left(h^{n+1}\mathcal{D}h\right)\right)\,, \tag{5.7}$$

where $n$ is an arbitrary parameter. The material conservation of the potential vorticity $\mathcal{D}\mathscr{P}_{\mathsf{GN}} = 0$ implies that if $\mathbb{P} + \mathbb{P}^{\star} = 0$ at $t = 0$ it remains so, and hence there exists potentials $\Phi_n$ such that $\boldsymbol{u}_2 + \boldsymbol{R}_2$ has the one-parameter family of representations (see Miles & Salmon 1985)

$$\boldsymbol{u}_2 + \boldsymbol{R}_2 = \boldsymbol{\nabla}_2\Phi_n - \frac{1}{3}n^{-1}h^{-n}\boldsymbol{\nabla}_2\left(h^{n+1}\mathcal{D}h\right) = \boldsymbol{\nabla}_2\widehat{\Phi} - \frac{1}{3}\mathcal{D}h\boldsymbol{\nabla}_2 h\,, \tag{5.8}$$

where $\widehat{\Phi}$ is related to $\Phi_n$ via

$$\Phi_n = \widehat{\Phi} + \frac{1}{3}n^{-1}h\mathcal{D}h\,. \tag{5.9}$$

The variation of the Lagrangian action (5.3) with respect to $\lambda$ yields the conservation of mass equation (2.8). Taking the variations $\delta \boldsymbol{u}_2$ of the Lagrangian action (5.3) yields

$$\iiint \left\langle \delta \boldsymbol{u}_2 \, , \, h \left( \boldsymbol{u}_2 - \boldsymbol{\nabla}_2 \lambda - \frac{1}{3} h^{-1} \boldsymbol{\nabla}_2 \left( h^3 \boldsymbol{\nabla}_2 \cdot \boldsymbol{u}_2 \right) + \boldsymbol{R}_2 \right) \right\rangle \rho \, \mathrm{d}\boldsymbol{x}_2 \, \mathrm{d}t = \boldsymbol{0} \, , \tag{5.10}$$

where $\boldsymbol{R}_2$ takes the form

$$\boldsymbol{R}_2 = \begin{bmatrix} \left( \Omega_1 \left( y + d_2 \right) - \Omega_2 \left( x + d_1 \right) + \dot{\boldsymbol{q}} \cdot \boldsymbol{Q} \boldsymbol{e}_3 \right) \dfrac{\partial h}{\partial x} + \Omega_2 d_3 - \Omega_3 \left( y + d_2 \right) + \dot{\boldsymbol{q}} \cdot \boldsymbol{Q} \boldsymbol{e}_1 \\ \left( \Omega_1 \left( y + d_2 \right) - \Omega_2 \left( x + d_1 \right) + \dot{\boldsymbol{q}} \cdot \boldsymbol{Q} \boldsymbol{e}_3 \right) \dfrac{\partial h}{\partial y} - \Omega_1 d_3 + \Omega_3 \left( x + d_1 \right) + \dot{\boldsymbol{q}} \cdot \boldsymbol{Q} \boldsymbol{e}_2 \end{bmatrix} , \tag{5.11}$$

which satisfies (5.6). Now, if we set

$$\mathbf{V} = \boldsymbol{u}_2 - \boldsymbol{\nabla}_2 \lambda - \frac{1}{3} h^{-1} \boldsymbol{\nabla}_2 \left( h^3 \boldsymbol{\nabla}_2 \cdot \boldsymbol{u}_2 \right) + \boldsymbol{R}_2 \, , \tag{5.12}$$

and substitute the Lin constraint (3.28) into the variational principle (5.10), we obtain

$$\iiint \left\langle \dot{\boldsymbol{w}}_2 + \boldsymbol{\nabla}_2 \boldsymbol{w}_2 \, \boldsymbol{u}_2 - \boldsymbol{\nabla}_2 \boldsymbol{u}_2 \, \boldsymbol{w}_2 \, , \, h \mathbf{V} \right\rangle \rho \, \mathrm{d}\boldsymbol{x}_2 \, \mathrm{d}t = \boldsymbol{0} \, . \tag{5.13}$$

Integrating (5.13) by parts (in space and time) and imposing the endpoint conditions $\boldsymbol{w}_2 \left( t_1 \right) = \boldsymbol{w}_2 \left( t_2 \right) = \boldsymbol{0}$, and the boundary conditions $u \left( 0, y, t \right) = u \left( L_1, y, t \right) = 0$, $v \left( x, 0, t \right) = v \left( x, L_2, t \right) = 0$, $w_1 \left( 0, y, t \right) = w_1 \left( L_1, y, t \right) = 0$ and $w_2 \left( x, 0, t \right) = w_2 \left( x, L_2, t \right) = 0$, and applying the continuity equation (2.8), we obtain

$$\iiint \left\langle - h \boldsymbol{w}_2 \, , \, \mathbf{V}_t + \boldsymbol{u}_2 \cdot \boldsymbol{\nabla}_2 \mathbf{V} + \left( \boldsymbol{\nabla}_2 \boldsymbol{u}_2 \right)^T \mathbf{V} \right\rangle \rho \, \mathrm{d}\boldsymbol{x}_2 \, \mathrm{d}t = \boldsymbol{0} \, . \tag{5.14}$$

Now, since $\boldsymbol{w}_2$ is arbitrary, from the variational principle (5.14) it can be concluded that

$$\left( \frac{\mathcal{D}}{\mathcal{D}t} + \left( \boldsymbol{\nabla}_2 \boldsymbol{u}_2 \right)^T \right) \mathbf{V} = \boldsymbol{0} \, , \tag{5.15}$$

from which it can be inferred that

$$\boldsymbol{u}_2 = \boldsymbol{\nabla}_2 \lambda + \frac{1}{3} h^{-1} \boldsymbol{\nabla}_2 \left( h^3 \boldsymbol{\nabla}_2 \cdot \boldsymbol{u}_2 \right) - \boldsymbol{R}_2 \, , \tag{5.16}$$

for which $\mathscr{P}_{\mathsf{GN}} = 0$ for any choice of $\lambda$. The columnar approximation (2.1*a,b*) implies an $\mathcal{O} \left( \beta^2 \right)$ error on the right-hand side of (5.16). Comparing (5.8) and (5.16) after invoking (2.6*a*) and (2.7), we conclude that

$$\lambda = \Phi_1 + f \left( t \right) \, , \tag{5.17}$$

where $f$ is an arbitrary function of $t$, and hence

$$\boldsymbol{u}_2 = \boldsymbol{\nabla}_2 \Phi_1 + \frac{1}{3} h^{-1} \boldsymbol{\nabla}_2 \left( h^3 \boldsymbol{\nabla}_2 \cdot \boldsymbol{u}_2 \right) - \boldsymbol{R}_2 \, . \tag{5.18}$$

It is consistent with the columnar motion to approximate $\boldsymbol{u}_2$ by $\boldsymbol{\nabla}_2 \Phi_1$ in the second term on the right-hand side of (5.18) (Miles & Salmon 1985), which is of $\mathcal{O} \left( \beta \right)$, to obtain the following expression after dropping the subscript $1$ from $\Phi$

$$\boldsymbol{u}_2 = \boldsymbol{\nabla}_2 \Phi + \frac{1}{3} h^{-1} \boldsymbol{\nabla}_2 \left( h^3 \nabla_2^2 \Phi \right) - \boldsymbol{R}_2 \, , \tag{5.19}$$

where $\nabla_2^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Taking the variation $\delta h$ in Hamilton's principle (5.2), substituting for $\delta h$ from (3.27) in the resulting variational principle, integrating by parts, imposing the boundary conditions $w_1(0, y, t) = w_1(L_1, y, t) = 0$ and $w_2(x, 0, t) = w_2(x, L_2, t) = 0$, and taking into account that $\boldsymbol{w}_2$ is a vector-valued free variation yields

$$\begin{cases} \lambda_t + \boldsymbol{u}_2 \cdot \boldsymbol{\nabla}_2 \lambda - \frac{1}{2}\boldsymbol{u}_2 \cdot \boldsymbol{u}_2 - \frac{1}{2}\left(h \boldsymbol{\nabla}_2 \cdot \boldsymbol{u}_2\right)^2 - \widehat{\mathfrak{U}} \cdot \left(\boldsymbol{\Omega} \times \left(\widehat{\mathfrak{X}} + \boldsymbol{d}\right) + \boldsymbol{Q}^T \dot{\boldsymbol{q}}\right) \\ -\boldsymbol{Q}^T \dot{\boldsymbol{q}} \cdot \boldsymbol{\Omega} \times \left(\widehat{\mathfrak{X}} + \boldsymbol{d}\right) - \frac{1}{2}\dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}} + g \boldsymbol{\Sigma} \cdot \left(\widehat{\mathfrak{X}} + \boldsymbol{d} + \boldsymbol{Q}^T \boldsymbol{q}\right) - \frac{1}{2}\left\| \boldsymbol{\Omega} \times \left(\widehat{\mathfrak{X}} + \boldsymbol{d}\right) \right\|^2 = 0\,, \end{cases} \tag{5.20}$$

where $\boldsymbol{\Sigma}$ is defined in (6.6) and

$$\widehat{\mathfrak{X}} = (x, y, h) \quad \text{and} \quad \widehat{\mathfrak{U}} = (u, v, -h \boldsymbol{\nabla}_2 \cdot \boldsymbol{u}_2)\,. \tag{5.21$a, b$}$$

Introduce the standard shallow water scaling (e.g. Dingemans 1997; Alemi Ardakani & Bridges 2011; Gavrilyuk *et al.* 2015)

$$\begin{cases} \widetilde{x} = \dfrac{x}{L}\,, \quad \widetilde{y} = \dfrac{y}{L}\,, \quad \widetilde{u} = \dfrac{u}{c_0}\,, \quad \widetilde{v} = \dfrac{v}{c_0}\,, \quad \widetilde{t} = \dfrac{c_0}{L}t\,, \quad \widetilde{h} = \dfrac{h}{h_0}\,, \quad \widetilde{\boldsymbol{q}} = \dfrac{\boldsymbol{q}}{L}\,, \\ \widetilde{d}_1 = \dfrac{d_1}{L}\,, \quad \widetilde{d}_2 = \dfrac{d_2}{L}\,, \quad \widetilde{d}_3 = \dfrac{d_3}{h_0}\,, \quad \widetilde{\Phi} = \dfrac{\Phi}{c_0 L}\,, \quad \widetilde{\boldsymbol{\Omega}} = \dfrac{L}{c_0}\boldsymbol{\Omega}\,, \end{cases} \tag{5.22}$$

where $c_0 = \sqrt{g h_0}$ represents the horizontal velocity scale. Now, if we substitute the velocity field (5.19) in the Lagrangian functional (5.3), and use the scaling (5.22) to obtain the non-dimensional form the resulting Lagrangian action, and retain only terms of $\mathcal{O}(\beta)$, then Hamilton's variational principle (5.2) for *the generalized Whitham (GW) equations* for fluid sloshing in three-dimensional rotating and translating coordinates takes the form

$$\delta \mathscr{L}_{\mathsf{GW}}(h, \Phi) = 0\,, \tag{5.23}$$

with the Lagrangian action

$$\begin{cases} \mathscr{L}_{\mathsf{GW}}(h, \Phi) = \displaystyle\int_{t_1}^{t_2} \iint \Bigg( \Phi_t + \tfrac{1}{2}\boldsymbol{\nabla}_2\Phi \cdot \boldsymbol{\nabla}_2\Phi - \frac{1}{6}\left(h \nabla_2^2 \Phi\right)^2 - \tfrac{1}{2}\boldsymbol{R}_2 \cdot \boldsymbol{R}_2 \\ \qquad + \left(\boldsymbol{R}_2 - \boldsymbol{\nabla}_2\Phi\right) \cdot \begin{bmatrix} \Omega_2\left(\frac{1}{2}h + d_3\right) - \Omega_3(y + d_2) + \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 \\ \Omega_3(x + d_1) - \Omega_1\left(\frac{1}{2}h + d_3\right) + \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_2 \end{bmatrix} \\ \qquad + \tfrac{1}{2}h\left(\nabla_2^2\Phi - \boldsymbol{\nabla}_2 \cdot \boldsymbol{R}_2\right)\left(\Omega_1(y + d_2) - \Omega_2(x + d_1) + \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3\right) - \tfrac{1}{2}\dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}} \\ -\boldsymbol{Q}^T\dot{\boldsymbol{q}} \cdot \boldsymbol{\Omega} \times (\mathfrak{X} + \boldsymbol{d}) + g\left(\boldsymbol{Q}(\mathfrak{X} + \boldsymbol{d}) + \boldsymbol{q}\right) \cdot \hat{\boldsymbol{z}} \Bigg) \rho\, h\, \mathrm{d}\boldsymbol{x}_2\, \mathrm{d}t - \displaystyle\int_{t_1}^{t_2} \tfrac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_f^{SW}\boldsymbol{\Omega}\, \mathrm{d}t\,. \end{cases} \tag{5.24}$$

Now, taking the variations $\delta h$ and $\delta\Phi$ in the variational principle (5.23) yields a new set of Boussinesq-like evolution equations, for $h(x, y, t)$ and $\Phi(x, y, t)$, for fluid sloshing inside a container undergoing prescribed rigi-body motion in three dimensions. Taking the variation

$\delta h$ in Hamilton's principle (5.23) yields

$$
\begin{cases}
\Phi_t + \frac{1}{2}\boldsymbol{\nabla}_2\Phi \cdot \boldsymbol{\nabla}_2\Phi - \frac{1}{2}\left(h\nabla_2^2\Phi\right)^2 + h\left(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)\nabla_2^2\Phi \\
-\left(\Omega_2\left(h+d_3\right) - \Omega_3\left(y+d_2\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\right)\Phi_x - \left(\Omega_3\left(x+d_1\right) - \Omega_1\left(h+d_3\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\right)\Phi_y \\
-2h\left(\Omega_1 h_y - \Omega_2 h_x\right)\left(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right) \\
-\left(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)^2\left(\frac{1}{2}\boldsymbol{\nabla}_2 h\cdot\boldsymbol{\nabla}_2 h + h\nabla_2^2 h\right) \\
-\frac{1}{2}\Omega_1^2\left(\left(y+d_2\right)^2 + h^2\right) - \frac{1}{2}\Omega_2^2\left(\left(x+d_1\right)^2 + h^2\right) \\
+\Omega_1\Omega_2\left(x+d_1\right)\left(y+d_2\right) - \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\left(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right)\right) \\
-\frac{1}{2}\left(\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)^2 + g\left(\boldsymbol{Q}\left(\widehat{\mathfrak{X}} + \boldsymbol{d}\right) + \boldsymbol{q}\right)\cdot\hat{\boldsymbol{z}} = 0\,,
\end{cases}
\tag{5.25}
$$

and taking the variation $\delta\Phi$ yields

$$
\begin{cases}
h_t + \boldsymbol{\nabla}_2\cdot\left(h\,\boldsymbol{\nabla}_2\Phi\right) + \nabla_2^2\left(\dfrac{1}{3}h^3\nabla_2^2\Phi\right) \\
-\left(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)\left(\boldsymbol{\nabla}_2 h\cdot\boldsymbol{\nabla}_2 h + h\nabla_2^2 h\right) \\
+\left(\Omega_2\left(h-d_3\right) + \Omega_3\left(y+d_2\right) - \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\right)h_x \\
-\left(\Omega_3\left(x+d_1\right) + \Omega_1\left(h-d_3\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\right)h_y = 0\,.
\end{cases}
\tag{5.26}
$$

Derivation of the evolution equations (5.25) and (5.26) is given in appendix E. The variational principle (5.23) also recovers the rigid-wall boundary conditions $u\left(x,y,t\right) = 0$ at $x = 0, L_1$ and $v\left(x,y,t\right) = 0$ at $y = 0, L_2$ for the interior fluid, which are

$$
\begin{cases}
\boldsymbol{\nabla}_2\Phi + \dfrac{1}{3}h^{-1}\boldsymbol{\nabla}_2\left(h^3\nabla_2^2\Phi\right) = \\
\begin{bmatrix}
\Omega_2 d_3 - \Omega_3\left(y+d_2\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1 + \left(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)h_x \\
\Omega_3\left(x+d_1\right) - \Omega_1 d_3 + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2 + \left(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)h_y
\end{bmatrix}
\end{cases}
\quad \text{on} \quad \mathsf{S}\left(t\right)\,,
\tag{5.27}
$$

where $\mathsf{S}\left(t\right)$ is the wetted tank surface. See appendix E for the proof of (5.27). The evolution equations (5.25) and (5.26) with the boundary conditions (5.27) are *a generalization of the Whitham equations for inviscid and incompressible fluid sloshing in a container undergoing prescribed rigid-body motion in three dimensions*.

Equations (5.25) and (5.26) may be obtained by substituting (5.17) and (5.19) into (2.8) and (5.20) and retaining only terms of $\mathcal{O}\left(\beta\right)$.

To derive the equations of motion for the angular momentum and linear momentum of the coupled **(**generalized Whitham equations for the interior fluid + 3–D rigid-body motion**)** dynamical system, we can take the variations $\delta\boldsymbol{\Omega}$, $\delta\boldsymbol{Q}$, $\delta\boldsymbol{q}$ and $\delta\dot{\boldsymbol{q}}$ of the coupled system

$$
\delta\mathscr{L}\left(\Phi, h, \boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}\right) = \delta\mathscr{L}_{\mathsf{GW}} + \delta\mathscr{L}_b = 0\,,
\tag{5.28}
$$

where $\mathscr{L}_b$ is the Lagrangian action of the dry rigid-body given by

$$
\mathscr{L}_b = \int\left(\tfrac{1}{2}m_v\left\|\dot{\boldsymbol{q}}\right\|^2 + \left(\boldsymbol{\Omega}\times m_v\overline{\boldsymbol{x}}_v\right)\cdot\boldsymbol{Q}^T\dot{\boldsymbol{q}} + \tfrac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_v\boldsymbol{\Omega} - m_v g\left(\boldsymbol{Q}\overline{\boldsymbol{x}}_v + \boldsymbol{q}\right)\cdot\hat{\boldsymbol{z}}\right)\mathrm{d}t\,.
\tag{5.29}
$$

## 6 The Euler–Poincaré equations for the motion of the rigid-body containing weakly dispersive shallow water

The Euler–Poincaré reduction theorem for rigid-body dynamics is given by Holm *et al.* (1998a). Alemi Ardakani (2019) applied this framework to the Lagrangian action (1.12) to derive the coupled Euler–Poincaré equations for the three-dimensional **(**rigid-body motion + interior inviscid and incompressible fluid sloshing**)** dynamical system. Here, we apply the Euler–Poincaré framework to the *coupled shallow-water* Lagrangian action (2.15) to derive the equations of motion for the angular momentum and linear momentum of the coupled **(**rigid-body motion + interior shallow-water sloshing**)** dynamical system.

The equation of motion for the *body angular velocity* $\boldsymbol{\Omega}\left(t\right)$ is provided by Hamilton's variational principle (3.1) by taking the variations $\delta\boldsymbol{Q}$ of the shallow-water Lagrangian action (2.15) among paths $\boldsymbol{Q}\left(t\right)\in\mathsf{SO}\left(3\right)$, $t\in\left[t_1,t_2\right]$, with fixed endpoints, so that $\delta\boldsymbol{Q}\left(t_1\right)=\delta\boldsymbol{Q}\left(t_2\right)=\boldsymbol{0}$. The variations $\delta\boldsymbol{\Omega}$ are induced by the variations $\delta\boldsymbol{Q}$ via (Holm *et al.* 2009)

$$\delta\widehat{\boldsymbol{\Omega}}=\frac{\mathrm{d}\widehat{\boldsymbol{\Gamma}}}{\mathrm{d}t}+\left[\,\widehat{\boldsymbol{\Omega}}\,,\,\widehat{\boldsymbol{\Gamma}}\,\right]=\frac{\mathrm{d}\widehat{\boldsymbol{\Gamma}}}{\mathrm{d}t}+\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\Gamma}}-\widehat{\boldsymbol{\Gamma}}\widehat{\boldsymbol{\Omega}}\,, \tag{6.1}$$

where $\left[\,\cdot\,,\,\cdot\,\right]$ is the matrix commutator, and $\widehat{\boldsymbol{\Gamma}}\in\mathfrak{so}\left(3\right)$ is defined by

$$\widehat{\boldsymbol{\Gamma}}=\boldsymbol{Q}^{-1}\delta\boldsymbol{Q}\,. \tag{6.2}$$

Since $\left[\,\widehat{\boldsymbol{\Omega}}\,,\,\widehat{\boldsymbol{\Gamma}}\,\right]=\widehat{\boldsymbol{\Omega}\times\boldsymbol{\Gamma}}$, the equivalent vector representation of (6.1) is

$$\delta\boldsymbol{\Omega}=\dot{\boldsymbol{\Gamma}}+\boldsymbol{\Omega}\times\boldsymbol{\Gamma}\,. \tag{6.3}$$

Also it can be proved that (Marsden & Ratiu 1999; Holm *et al.* 2009)

$$\delta\boldsymbol{Q}^{-1}=-\boldsymbol{Q}^{-1}\delta\boldsymbol{Q}\boldsymbol{Q}^{-1}\,. \tag{6.4}$$

Now the Euler–Poincaré equation for $\boldsymbol{\Omega}\left(t\right)$ can be obtained by taking the first variation of the action integral $\mathscr{L}_{SW/\mathsf{GN}}\left(\boldsymbol{\Omega},\boldsymbol{Q},\boldsymbol{q},\dot{\boldsymbol{q}},\mathfrak{X},\dot{\mathfrak{X}}\right)$ in (2.15) with respect to $\boldsymbol{\Omega}$ and $\boldsymbol{Q}$ using the variations (6.3), (6.4) and the hat map (1.7), and assuming that that $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\mathfrak{X}$ and $\dot{\mathfrak{X}}$ are constants. Applying similar calculus of variations presented in §3.1 of Alemi Ardakani (2019), it can be proved that Hamilton's variational principle (3.1) for the variations $\delta\boldsymbol{\Omega}$ and $\delta\boldsymbol{Q}$ reads

$$\begin{cases}\displaystyle\int_{t_1}^{t_2}\iint\Big\langle\boldsymbol{\Gamma}\,,\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\mathfrak{X}}\times\left(\mathfrak{X}+\boldsymbol{d}\right)\right)+\boldsymbol{\Omega}\times\left(\dot{\mathfrak{X}}\times\left(\mathfrak{X}+\boldsymbol{d}\right)\right)+\dot{\mathfrak{X}}\times\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\\ \displaystyle\quad+\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\times\left(\mathfrak{X}+\boldsymbol{d}\right)\right)+\left(\mathfrak{X}+\boldsymbol{d}\right)\times\left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\times\boldsymbol{\Omega}\right)-g\left(\mathfrak{X}+\boldsymbol{d}\right)\times\boldsymbol{\Sigma}\Big\rangle\,\rho\,h_0\,\mathrm{d}\boldsymbol{a}_2\,\mathrm{d}t\\ \displaystyle+\int_{t_1}^{t_2}\Big\langle\boldsymbol{\Gamma}\,,\,-\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{I}_f^{SW}\boldsymbol{\Omega}\right)+\boldsymbol{I}_f^{SW}\boldsymbol{\Omega}\times\boldsymbol{\Omega}-\frac{\mathrm{d}}{\mathrm{d}t}\left(m_v\overline{\boldsymbol{x}}_v\times\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\right)\\ \displaystyle\quad+m_v\overline{\boldsymbol{x}}_v\times\left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\times\boldsymbol{\Omega}\right)-\boldsymbol{I}_v\dot{\boldsymbol{\Omega}}+\boldsymbol{I}_v\boldsymbol{\Omega}\times\boldsymbol{\Omega}-m_vg\overline{\boldsymbol{x}}_v\times\boldsymbol{\Sigma}\Big\rangle\,\mathrm{d}t=\boldsymbol{0}\,,\end{cases} \tag{6.5}$$

where

$$\boldsymbol{\Sigma}=\boldsymbol{Q}^{-1}\hat{\boldsymbol{z}}\,. \tag{6.6}$$

Therefore, since the variational principle (6.5) holds for any curve $\boldsymbol{\Gamma}\left(t\right)$ in $\mathfrak{so}\left(3\right)$ such that $\boldsymbol{\Gamma}\left(t_1\right)=\boldsymbol{\Gamma}\left(t_2\right)=\boldsymbol{0}$, we find that the body angular velocity of the rigid-body containing *shallow* *water* is governed by the equation:

$$\begin{cases} \displaystyle\iint \Bigg( \frac{\mathrm{d}}{\mathrm{d}t} \Big( \dot{\mathcal{X}} \times (\mathcal{X} + \boldsymbol{d}) \Big) + \boldsymbol{\Omega} \times \Big( \dot{\mathcal{X}} \times (\mathcal{X} + \boldsymbol{d}) \Big) + \dot{\mathcal{X}} \times \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \\ \displaystyle + \frac{\mathrm{d}}{\mathrm{d}t} \left( \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \times (\mathcal{X} + \boldsymbol{d}) \right) + (\mathcal{X} + \boldsymbol{d}) \times \left( \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \times \boldsymbol{\Omega} \right) - g \left( \mathcal{X} + \boldsymbol{d} \right) \times \boldsymbol{\Sigma} \Bigg) \, \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \\ \displaystyle - \frac{\mathrm{d}}{\mathrm{d}t} \left( \boldsymbol{I}_f^{SW} \boldsymbol{\Omega} \right) + \boldsymbol{I}_f^{SW} \boldsymbol{\Omega} \times \boldsymbol{\Omega} - \frac{\mathrm{d}}{\mathrm{d}t} \left( m_v \overline{\boldsymbol{x}}_v \times \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \right) + m_v \overline{\boldsymbol{x}}_v \times \left( \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \times \boldsymbol{\Omega} \right) \\ - \boldsymbol{I}_v \dot{\boldsymbol{\Omega}} + \boldsymbol{I}_v \boldsymbol{\Omega} \times \boldsymbol{\Omega} - m_v g \overline{\boldsymbol{x}}_v \times \boldsymbol{\Sigma} = \boldsymbol{0} \, , \end{cases} \tag{6.7}$$

which is the Euler-Poincaré equation for $\boldsymbol{\Omega}\,(t)$ in the Lagrangian particle-path setting. Equation (6.7) after differentiating with respect to time and simplifying using (3.12) and the hat map (1.7) reduces to

$$\begin{cases} \displaystyle\iint \Bigg( \ddot{\mathcal{X}} \times (\mathcal{X} + \boldsymbol{d}) + \boldsymbol{\Omega} \times \Big( \dot{\mathcal{X}} \times (\mathcal{X} + \boldsymbol{d}) \Big) + \boldsymbol{Q}^{-1} \ddot{\boldsymbol{q}} \times (\mathcal{X} + \boldsymbol{d}) \\ \displaystyle - g \left( \mathcal{X} + \boldsymbol{d} \right) \times \boldsymbol{\Sigma} \Bigg) \, \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 - \dot{\boldsymbol{I}}_f^{SW} \boldsymbol{\Omega} - \left( \boldsymbol{I}_f^{SW} + \boldsymbol{I}_v \right) \dot{\boldsymbol{\Omega}} \\ + \left( \boldsymbol{I}_f^{SW} + \boldsymbol{I}_v \right) \boldsymbol{\Omega} \times \boldsymbol{\Omega} - m_v \overline{\boldsymbol{x}}_v \times \boldsymbol{Q}^{-1} \ddot{\boldsymbol{q}} - m_v g \overline{\boldsymbol{x}}_v \times \boldsymbol{\Sigma} = \boldsymbol{0} \, . \end{cases} \tag{6.8}$$

Now set the mass moment of inertia of the coupled **(**rigid-body motion + interior shallow-water sloshing**)** system as

$$\boldsymbol{I}_t = \boldsymbol{I}_f^{SW} + \boldsymbol{I}_v \, , \tag{6.9}$$

and take into account the columnar motion of the interior fluid to obtain

$$m_f \overline{\boldsymbol{x}}_f = \iint \left( \mathcal{X} + \boldsymbol{d} \right) \, \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 = \iint \left( \mathcal{X} + \boldsymbol{d} \right) \, \rho \, h \, \mathrm{d}\boldsymbol{x}_2 \, , \tag{6.10}$$

where $\overline{\boldsymbol{x}}_f\,(t)$ is the centre of mass of the interior shallow water relative to the body frame $\boldsymbol{x}_b$, and

$$m_f = \iint \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 = \iint \rho \, h \, \mathrm{d}\boldsymbol{x}_2 \, , \tag{6.11}$$

is the mass of the interior shallow water which is time independent. Also by setting

$$m = m_f + m_v \, , \tag{6.12}$$

which is the total mass of the coupled **(**rigid-body + interior shallow-water**)** system, we have

$$m \overline{\boldsymbol{x}}\,(t) = m_f \overline{\boldsymbol{x}}_f + m_v \overline{\boldsymbol{x}}_v \, , \tag{6.13}$$

where $\overline{\boldsymbol{x}}$ is the centre of mass of the coupled system which is time dependent. Now the $\Omega$-equation (6.8) simplifies to

$$\left. \begin{aligned} & \iint \Big( \ddot{\mathcal{X}} \times (\mathcal{X} + \boldsymbol{d}) + \boldsymbol{\Omega} \times \Big( \dot{\mathcal{X}} \times (\mathcal{X} + \boldsymbol{d}) \Big) \Big) \, \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \\ & - m \overline{\boldsymbol{x}} \times \boldsymbol{Q}^{-1} \ddot{\boldsymbol{q}} - \dot{\boldsymbol{I}}_f^{SW} \boldsymbol{\Omega} - \boldsymbol{I}_t \dot{\boldsymbol{\Omega}} + \boldsymbol{I}_t \boldsymbol{\Omega} \times \boldsymbol{\Omega} - m g \overline{\boldsymbol{x}} \times \boldsymbol{\Sigma} = \boldsymbol{0} \, . \end{aligned} \right\} \tag{6.14}$$

Transforming this equation from the Lagrangian particle-path setting to Eulerian coordinates, replacing the Lagrangian variables $\dot{\mathcal{X}}$ and $\ddot{\mathcal{X}}$ by their respective Eulerian quantities $\mathcal{U}$ and $\mathcal{DU}/\mathcal{D}t$ respectively, the Euler-Poincaré equation (6.14) takes the form

$$\left. \begin{aligned} & \iint \left( \frac{\mathcal{DU}}{\mathcal{D}t} \times (\mathcal{X} + \boldsymbol{d}) + \boldsymbol{\Omega} \times \left( \mathcal{U} \times (\mathcal{X} + \boldsymbol{d}) \right) \right) \rho \, h \, \mathrm{d}\boldsymbol{x}_2 \\ & - m \overline{\boldsymbol{x}} \times \boldsymbol{Q}^{-1} \ddot{\boldsymbol{q}} - \dot{\boldsymbol{I}}_f^{SW} \boldsymbol{\Omega} - \boldsymbol{I}_t \dot{\boldsymbol{\Omega}} + \boldsymbol{I}_t \boldsymbol{\Omega} \times \boldsymbol{\Omega} - m g \overline{\boldsymbol{x}} \times \boldsymbol{\Sigma} = \boldsymbol{0} \, , \end{aligned} \right\} \tag{6.15}$$

where $\mathcal{U}$ is defined in (3.26) and

$$\frac{\mathcal{D}\mathcal{U}}{\mathcal{D}t} = \left(\frac{\mathcal{D}\boldsymbol{u}_2}{\mathcal{D}t}, \tfrac{1}{2}h\left(\left(\boldsymbol{\nabla}_2\cdot\boldsymbol{u}_2\right)^2 - \boldsymbol{\nabla}_2\cdot\mathcal{D}\boldsymbol{u}_2\right)\right) \quad \text{where} \quad \mathcal{D}/\mathcal{D}t \equiv \mathcal{D}\,, \tag{6.16}$$

and $\dot{\boldsymbol{I}}_f^{SW}$, which is in Eulerian coordinates, is defined in appendix A. Hence, the equation of motion for the angular momentum of the rigid-body containing shallow water takes the form (6.15) in Eulerian coordinates. Alternatively, we could directly take the variations $\delta\boldsymbol{\Omega}$ and $\delta\boldsymbol{Q}$ of the Lagrangian action (3.25) in the Eulerian setting to derive (6.15).

The Euler–Poincaré equation for $\boldsymbol{q}\left(t\right)$ is provided by Hamilton's variational principle (3.1) by taking the variations $\delta\boldsymbol{q}$ and $\delta\dot{\boldsymbol{q}}$ of the shallow-water Lagrangian action (2.15) with fixed endpoints $\delta\boldsymbol{q}\left(t_1\right) = \delta\boldsymbol{q}\left(t_2\right) = \boldsymbol{0}$, and assuming that $\boldsymbol{\Omega}$, $\boldsymbol{Q}$, $\mathcal{X}$ and $\dot{\mathcal{X}}$ are constants. Applying similar calculus of variations presented in $\S3.1$ of Alemi Ardakani (2019), it can be proved that Hamilton's principle leads to

$$\begin{cases} \displaystyle\iint \left(-\ddot{\mathcal{X}} - \boldsymbol{\Omega}\times\dot{\mathcal{X}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\Omega}\times\left(\mathcal{X}+\boldsymbol{d}\right)\right) - \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\left(\mathcal{X}+\boldsymbol{d}\right)\right) - \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - g\boldsymbol{\Sigma}\right)\rho\, h_0\,\mathrm{d}\boldsymbol{a}_2 \\ \displaystyle -m_v\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\Omega}\times m_v\overline{\boldsymbol{x}}_v\right) - \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times m_v\overline{\boldsymbol{x}}_v\right) - m_v g\boldsymbol{\Sigma} = \boldsymbol{0}\,, \end{cases} \tag{6.17}$$

which is the Euler-Poincaré equation, in the Lagrangian particle-path setting, for the translational motion $\boldsymbol{q}\left(t\right)$ of the rigid-body relative to the spatial frame $\boldsymbol{X}$. This equation, after differentiating with respect to time and applying (6.10), (6.11) and (6.13), simplifies to

$$\iint \left(-\ddot{\mathcal{X}} - 2\boldsymbol{\Omega}\times\dot{\mathcal{X}}\right)\rho\, h_0\,\mathrm{d}\boldsymbol{a}_2 - m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - \dot{\boldsymbol{\Omega}}\times m\overline{\boldsymbol{x}} - \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times m\overline{\boldsymbol{x}}\right) - mg\boldsymbol{\Sigma} = \boldsymbol{0}\,. \tag{6.18}$$

Transforming this equation from the Lagrangian particle-path setting to Eulerian coordinates, replacing the Lagrangian variables $\dot{\mathcal{X}}$ and $\ddot{\mathcal{X}}$ by their respective Eulerian quantities $\mathcal{U}$ and $D\mathcal{U}/Dt$ respectively, the $\boldsymbol{q}$-equation (6.18) reduces to

$$\iint \left(\frac{\mathcal{D}\mathcal{U}}{\mathcal{D}t} + 2\boldsymbol{\Omega}\times\mathcal{U}\right)\rho\, h\,\mathrm{d}\boldsymbol{x}_2 + m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{\Omega}}\times m\overline{\boldsymbol{x}} + \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times m\overline{\boldsymbol{x}}\right) + mg\boldsymbol{\Sigma} = \boldsymbol{0}\,. \tag{6.19}$$

Hence, the equation of motion for the linear momentum of the rigid-body containing shallow water takes the form (6.19) in Eulerian coordinates.

The evolutionary system for the rigid-body motion (6.15) and (6.19) is completed by the reconstruction formula

$$\dot{\boldsymbol{Q}} = \boldsymbol{Q}\,\widehat{\boldsymbol{\Omega}}\,, \tag{6.20}$$

and the constraint equation

$$\dot{\boldsymbol{\Sigma}}\left(t\right) = \boldsymbol{\Sigma}\left(t\right)\times\boldsymbol{\Omega}\left(t\right) \quad \text{with} \quad \boldsymbol{\Sigma}\left(0\right) = \boldsymbol{Q}^{-1}\left(0\right)\hat{\boldsymbol{z}}\,. \tag{6.21}$$

The solution of (6.20) yields the integral curve $\boldsymbol{Q}\left(t\right) \in \mathrm{SO}\left(3\right)$ for the orientation of the rigid-body containing shallow water.

The Euler–Poincaré equations (6.15) and (6.19) can be derived in terms of $\left(h, \Phi\right)$ for the interior fluid variables by substituting for $\mathcal{U}$ and $\mathcal{D}\mathcal{U}/\mathcal{D}t$ using the velocity field (5.19), and retaining only terms of $\mathcal{O}\left(\beta\right)$. The resulting equations are coupled to the generalized Whitham equations (5.25) and (5.26) with the boundary conditions (5.27) for the interior weakly dispersive nonlinear fluid sloshing. Alternatively, we can directly take the variations $\delta\boldsymbol{\Omega}$, $\delta\boldsymbol{Q}$, $\delta\boldsymbol{q}$ and $\delta\dot{\boldsymbol{q}}$ in the coupled variational principle (5.28) to find the equations of motion for the rigid-body motion.

# 7 A variational framework for three-dimensional interactions between potential-flow water waves and a freely floating rigid-body dynamically coupled to its interior weakly dispersive nonlinear shallow-water sloshing

The classical water-wave problem in three dimensions is described by the partial differential equations

$$\left.\begin{aligned}&\Delta\,\phi := \phi_{XX} + \phi_{YY} + \phi_{ZZ} = 0 \quad \text{for} \quad -H\left(X,Y\right) < Z < \eta\left(X,Y,t\right)\,,\\&\phi_t + \tfrac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + gZ = 0 \quad \text{on} \quad Z = \eta\left(X,Y,t\right)\,,\\&\phi_Z = \eta_t + \phi_X\eta_X + \phi_Y\eta_Y \quad \text{on} \quad Z = \eta\left(X,Y,t\right)\,,\\&\phi_Z + \phi_X H_X + \phi_Y H_Y = 0 \quad \text{on} \quad Z = -H\left(X,Y\right)\,,\end{aligned}\right\} \tag{7.1}$$

where $(X, Y, Z)$ is the spatial (laboratory) coordinate system, $\phi\left(X, Y, Z, t\right)$ is the velocity potential of an irrotational fluid lying between $Z = -H\left(X, Y\right)$ and $Z = \eta\left(X, Y, t\right)$ with the gravity acceleration $g$ acting in the negative $Z$ direction. In the horizontal directions $X$ and $Y$, the fluid domain is cut off by a cylindrical vertical surface $\mathbb{S}$ of infinite radius which extends from the bottom to the free surface. Luke's variational principle for three-dimensional gravity driven water waves reads (Luke 1967; Van Daalen *et al.* 1993)

$$\delta\mathscr{L}_w\left(\phi,\eta\right) \;=\; \delta\int_{t_1}^{t_2}\iiint_{V(t)} -\rho\left(\phi_t + \tfrac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + gZ\right)\,\mathrm{d}V\,\mathrm{d}t = 0\,, \tag{7.2}$$

where the Bernoulli pressure, playing the role of the Lagrangian density, is integrated over the transient fluid domain $V\left(t\right)$, with variations in $\phi\left(X, Y, Z, t\right)$ and $\eta\left(X, Y, t\right)$ subject to the restrictions $\delta\phi = 0$ at the end points of the time interval, $t_1$ and $t_2$. In (7.2) $\rho$ is the water density. The variational principle (7.2) recovers the complete set of equations of motion for the water wave problem described by (7.1).

In §§§3, 5 and 6, a variational framework is developed for the problem of dynamic coupling between rigid-body motion and its interior inviscid and incompressible fluid sloshing in three dimensions. The motion of the interior fluid of the rigid-body is governed by the shallow-water equations or the generalized Green–Naghdi equations in three-dimensional rotating and translating coordinates given in §3, or by the generalized Whitham equations presented in §5. The variational framework can be extended to the problem of hydrodynamic interactions between 3–D potential-flow water waves governed by (7.1) and a freely floating rigid-body dynamically coupled to its interior weakly dispersive nonlinear fluid sloshing. The extended variational principle can be obtained by the addition of Luke's variational principle (7.2) to Hamilton's variational principle (3.1) in the Lagrangian particle-path formulation or in the Eulerian setting for **(**the intetior SWEs/generalized Green–Naghdi equations + rigid-body motion**)** dynamical system, or to Hamilton's principle (5.28) for **(**the interior generalized Whitham equations + rigid-body motion**)** system. The unified variational principle for the coupled **(**exterior potential-flow water waves + floating rigid-body motion + interior SWEs/generalized Green–Naghdi equations**)** system takes the form

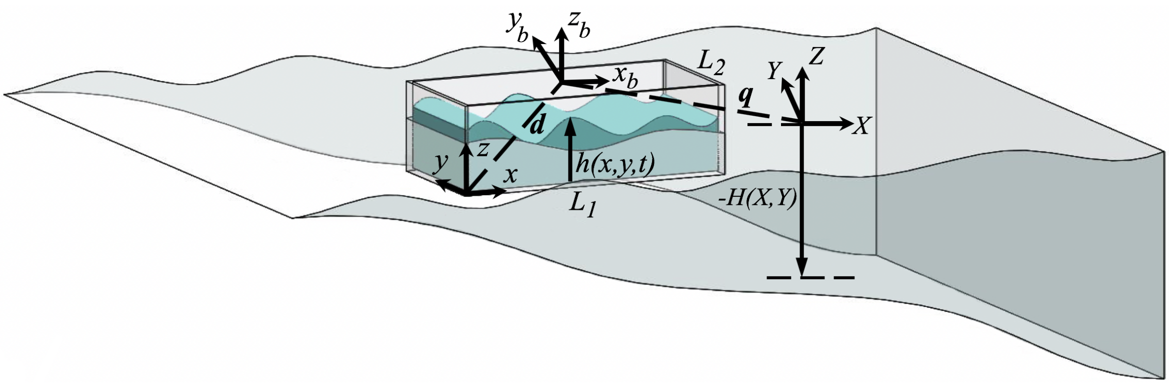


Figure 2: Schematic showing a freely floating rigid-body containing fluid in hydrodynamic interaction with exterior ocean surface waves.

$$\begin{cases} \delta \mathscr{L}\left(\phi, \eta, \boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}, \mathfrak{X}, \dot{\mathfrak{X}}\right) = \delta \displaystyle\int_{t_1}^{t_2} \iiint_{V(t)} -\rho\left(\phi_t + \tfrac{1}{2}\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi + gZ\right) \mathrm{d}V\,\mathrm{d}t \\ +\delta \displaystyle\int_{t_1}^{t_2} \Bigg( \iint \Big( \tfrac{1}{2}\left\|\dot{\boldsymbol{x}}_2\right\|^2 + \frac{1}{6}\dot{h}^2 + \dot{\mathfrak{X}} \cdot \left(\boldsymbol{\Omega} \times (\mathfrak{X} + \boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right) \\ +\boldsymbol{Q}^T\dot{\boldsymbol{q}} \cdot \left(\boldsymbol{\Omega} \times (\mathfrak{X} + \boldsymbol{d})\right) + \tfrac{1}{2}\left\|\dot{\boldsymbol{q}}\right\|^2 - g\left(\boldsymbol{Q}\left(\mathfrak{X} + \boldsymbol{d}\right) + \boldsymbol{q}\right) \cdot \hat{\boldsymbol{z}} \Big) \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2 + \tfrac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_f^{SW}\boldsymbol{\Omega} \\ +\tfrac{1}{2}m_v\left\|\dot{\boldsymbol{q}}\right\|^2 + \left(\boldsymbol{\Omega} \times m_v\overline{\boldsymbol{x}}_v\right) \cdot \boldsymbol{Q}^T\dot{\boldsymbol{q}} + \tfrac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_v\boldsymbol{\Omega} - m_v g\left(\boldsymbol{Q}\overline{\boldsymbol{x}}_v + \boldsymbol{q}\right) \cdot \hat{\boldsymbol{z}} \Bigg) \mathrm{d}t = 0\,, \end{cases} \tag{7.3}$$

where the Lagrangian of the interior fluid sloshing is represented in the Lagrangian particle-path setting. Alternatively, the unified variational principle can be formulated in Eulerian coordinates given by

$$\begin{cases} \delta \mathscr{L}\left(\phi, \eta, \boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u}_2, h, \lambda\right) = \delta \displaystyle\int_{t_1}^{t_2} \iiint_{V(t)} -\rho\left(\phi_t + \tfrac{1}{2}\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi + gZ\right) \mathrm{d}V\,\mathrm{d}t \\ +\delta \displaystyle\int_{t_1}^{t_2} \Bigg( \iint \Big( \tfrac{1}{2}\left\|\boldsymbol{u}_2\right\|^2 + \frac{1}{6}h^2\left(\boldsymbol{\nabla}_2 \cdot \boldsymbol{u}_2\right)^2 + \mathfrak{U} \cdot \left(\boldsymbol{\Omega} \times (\mathfrak{X} + \boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right) \\ +\boldsymbol{Q}^T\dot{\boldsymbol{q}} \cdot \left(\boldsymbol{\Omega} \times (\mathfrak{X} + \boldsymbol{d})\right) + \tfrac{1}{2}\left\|\dot{\boldsymbol{q}}\right\|^2 - g\left(\boldsymbol{Q}\left(\mathfrak{X} + \boldsymbol{d}\right) + \boldsymbol{q}\right) \cdot \hat{\boldsymbol{z}} \Big) \rho\, h\, \mathrm{d}\boldsymbol{x}_2 + \tfrac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_f^{SW}\boldsymbol{\Omega} \\ +\tfrac{1}{2}m_v\left\|\dot{\boldsymbol{q}}\right\|^2 + \left(\boldsymbol{\Omega} \times m_v\overline{\boldsymbol{x}}_v\right) \cdot \boldsymbol{Q}^T\dot{\boldsymbol{q}} + \tfrac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_v\boldsymbol{\Omega} - m_v g\left(\boldsymbol{Q}\overline{\boldsymbol{x}}_v + \boldsymbol{q}\right) \cdot \hat{\boldsymbol{z}} \Bigg) \mathrm{d}t \\ + \displaystyle\int_{t_1}^{t_2} \iint \lambda\left(\mathcal{D}h + h\boldsymbol{\nabla}_2 \cdot \boldsymbol{u}_2\right) \rho\, \mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t = 0\,. \end{cases} \tag{7.4}$$

The unified variational principle for the coupled (exterior potential-flow water waves + floating rigid-body motion + interior generalized Whitham equations**)** system can be formulated by

$$\delta \mathscr{L}_w\left(\phi, \eta\right) + \delta \mathscr{L}_{\mathsf{GW}}\left(\Phi, h, \boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}\right) + \delta \mathscr{L}_b\left(\boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}\right) = 0\,, \tag{7.5}$$

where $\mathscr{L}_w$, $\mathscr{L}_{\mathsf{GW}}$ and $\mathscr{L}_b$ are defined in (7.2), (5.24) and (5.29), respectively. $V(t)$ in (7.3), (7.4) or (7.5) cosists of a fluid bounded by the impermeable bottom $S_b$ defined by the equation $Z = -H(X, Y)$, the free surface $S_\eta$ defined by the equation $Z = \eta(X, Y, t)$, the vertical surface $\mathcal{S}$ and the wetted surface $S_w$ of the rigid body interacting with exterior water waves. The configuration of the fluid in a freely floating rigid-body interacting with exterior water waves is schematically shown in Figure 2.

In order to take the variations in (7.3), (7.4) or (7.5), the variational Reynold's transport theorem should be used, since the domain of integration $V(t)$ is time-dependent. The background mathematics on the variational analogue of Reynold's transport theorem is presented by Flanders (1973), Daniliuk (1976) and Gagarina, van der Vegt & Bokhove (2013). Taking the variations $\delta\phi$, $\delta\eta$, $\delta\boldsymbol{\Omega}$, $\delta\boldsymbol{Q}$, $\delta\boldsymbol{q}$, $\delta\dot{\boldsymbol{q}}$, $\delta\mathfrak{X}$ and $\delta\dot{\mathfrak{X}}$ in Hamilton's principle (7.3) gives

$$
\left\{
\begin{aligned}
&\delta\mathscr{L}\left(\phi,\eta,\boldsymbol{\Omega},\boldsymbol{Q},\boldsymbol{q},\dot{\boldsymbol{q}},\mathfrak{X},\dot{\mathfrak{X}}\right) = \int_{t_1}^{t_2}\iint_{S_\eta} -\left(\phi_t+\tfrac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi+gZ\right)\Big|^{Z=\eta}\rho\,\delta\eta\,\ell^{-1}\,\mathrm{d}S\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\iint_{S_w} P\left(X,Y,Z,t\right)\left(\delta\boldsymbol{X}_w\cdot\boldsymbol{n}\right)\mathrm{d}S\,\mathrm{d}t - \int_{t_1}^{t_2}\iiint_{V(t)}\left(\delta\phi_t+\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\delta\phi\right)\rho\,\mathrm{d}V\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\iint\left\langle\boldsymbol{\Gamma}\,,\,\frac{\mathcal{D}\mathfrak{U}}{\mathcal{D}t}\times(\mathfrak{X}+\boldsymbol{d})+\boldsymbol{\Omega}\times(\mathfrak{U}\times(\mathfrak{X}+\boldsymbol{d}))\right\rangle\rho\,h\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\left\langle\boldsymbol{\Gamma}\,,\,-m\overline{\boldsymbol{x}}\times\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-\dot{\boldsymbol{I}}_f^{SW}\boldsymbol{\Omega}-\boldsymbol{I}_t\dot{\boldsymbol{\Omega}}+\boldsymbol{I}_t\boldsymbol{\Omega}\times\boldsymbol{\Omega}-mg\overline{\boldsymbol{x}}\times\boldsymbol{\Sigma}\right\rangle\mathrm{d}t\\
&+\int_{t_1}^{t_2}\iint\left\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{q}\,,\,-\frac{\mathcal{D}\mathfrak{U}}{\mathcal{D}t}-2\boldsymbol{\Omega}\times\mathfrak{U}\right\rangle\rho\,h\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\left\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{q}\,,\,-m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-\dot{\boldsymbol{\Omega}}\times m\overline{\boldsymbol{x}}-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times m\overline{\boldsymbol{x}})-mg\boldsymbol{\Sigma}\right\rangle\mathrm{d}t\\
&+\int_{t_1}^{t_2}\iint\left\langle\delta\boldsymbol{x}_2\,,\,\frac{\mathcal{D}\boldsymbol{u}_2}{\mathcal{D}t}+\frac{1}{3}\frac{1}{h}\boldsymbol{\nabla}_2\left(h^2\mathcal{D}^2h\right)-\begin{bmatrix}\dot{\Omega}_3\left(y+d_2\right)-\dot{\Omega}_2\left(\frac{1}{2}h+d_3\right)\\ \dot{\Omega}_1\left(\frac{1}{2}h+d_3\right)-\dot{\Omega}_3\left(x+d_1\right)\end{bmatrix}\right.\\
&\quad-\tfrac{1}{2}\frac{1}{h}\boldsymbol{\nabla}_2\left(h^2\left(\dot{\Omega}_2\left(x+d_1\right)-\dot{\Omega}_1\left(y+d_2\right)\right)\right)-\begin{bmatrix}-\Omega_2\mathcal{D}h+2\Omega_3v\\ -2\Omega_3u+\Omega_1\mathcal{D}h\end{bmatrix}\\
&\quad-\frac{1}{h}\boldsymbol{\nabla}_2\left(h^2\left(-\Omega_1v+\Omega_2u\right)\right)-\begin{bmatrix}\Omega_2\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3-\Omega_3\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2-\ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\\ \Omega_3\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1-\Omega_1\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3-\ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\end{bmatrix}\\
&\quad-\tfrac{1}{2}\frac{1}{h}\boldsymbol{\nabla}_2\left(h^2\left(\Omega_1\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2-\Omega_2\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1-\ddot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)\right)-\begin{bmatrix}\Omega_3\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2-\Omega_2\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\\ \Omega_1\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3-\Omega_3\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\end{bmatrix}\\
&\quad-\left(\Omega_2\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1-\Omega_1\,\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\right)\boldsymbol{\nabla}_2h+g\begin{bmatrix}\boldsymbol{e}_3\cdot\boldsymbol{Q}\boldsymbol{e}_1\\ \boldsymbol{e}_3\cdot\boldsymbol{Q}\boldsymbol{e}_2\end{bmatrix}+g\boldsymbol{e}_3\cdot\boldsymbol{Q}\boldsymbol{e}_3\boldsymbol{\nabla}_2h\\
&\left.-\begin{bmatrix}\left(\left(h+d_3\right)\left(\Omega_1^2+\Omega_2^2\right)-\left(x+d_1\right)\Omega_1\Omega_3-\left(y+d_2\right)\Omega_2\Omega_3\right)h_x\\ +\left(x+d_1\right)\left(\Omega_2^2+\Omega_3^2\right)-\left(y+d_2\right)\Omega_1\Omega_2-\left(h+d_3\right)\Omega_1\Omega_3\\ \\ \left(\left(h+d_3\right)\left(\Omega_1^2+\Omega_2^2\right)-\left(x+d_1\right)\Omega_1\Omega_3-\left(y+d_2\right)\Omega_2\Omega_3\right)h_y\\ +\left(y+d_2\right)\left(\Omega_1^2+\Omega_3^2\right)-\left(x+d_1\right)\Omega_1\Omega_2-\left(h+d_3\right)\Omega_2\Omega_3\end{bmatrix}\right\rangle\rho\,h\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t=0\,,
\end{aligned}
\right.
\tag{7.6}
$$

where the results of §3 and §6 are applied in taking the variations of the second component of the variational principle (7.3). Here after taking the variations $\delta\boldsymbol{\Omega}$, $\delta\boldsymbol{Q}$, $\delta\boldsymbol{q}$, $\delta\dot{\boldsymbol{q}}$, $\delta\mathfrak{X}$ and $\delta\dot{\mathfrak{X}}$ of the second component of (7.3), which is for the coupled **(**interior fluid sloshing + rigid-body motion**)** system, in the Lagrangian particle-path setting, the results are transformed to Eulerian coordinates. Alternatively, we could take the variations in the variational principle (7.4) in Eulerian coordinates. In (7.6), $\boldsymbol{X}_w$ denotes the position of a point on the wetted body surface $S_w$ relative to the spatial (laboratory) frame $\boldsymbol{X}$, $\boldsymbol{n}$ is the unit normal vector along $\partial V \supset S_w$ in the laboratory frame, $\ell = \left(1+\eta_X^2+\eta_Y^2\right)^{1/2}$ giving $\mathrm{d}S = \ell\,\mathrm{d}X\mathrm{d}Y$, and $P$ is the pressure field of the exterior water waves defined by

$$
P\left(X,Y,Z,t\right) = -\rho\left(\phi_t+\tfrac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi+gZ\right)\quad\text{on}\quad S_w\,. \tag{7.7}
$$

These variations are subject to the restrictions that they vanish at the end points of the time interval. Moreover, the variations in $\eta$ and $\phi$ vanish on the vertical boundary at infinity, i.e.

on $\mathcal{S}$. The change in $\boldsymbol{X}_w$ due to the variations in $\boldsymbol{Q}$ and $\boldsymbol{q}$ is given by (Alemi Ardakani 2019)

$$\delta \boldsymbol{X}_w = \delta \boldsymbol{Q} \boldsymbol{x}_w + \delta \boldsymbol{q} \,, \tag{7.8}$$

where $\boldsymbol{x}_w$ is the position of a point on the wetted rigid body surface $S_w$ relative to the body frame $\boldsymbol{x}_b$. Using the variations (7.8), the second integral on the right-hand side of (7.6) simplifies to (see equation (5.7) in Alemi Ardakani 2019)

$$\int_{t_1}^{t_2} \iint_{S_w} P \left\langle \delta \boldsymbol{X}_w \,,\, \boldsymbol{n} \right\rangle \mathrm{d}S \, \mathrm{d}t = \int_{t_1}^{t_2} \iint_{S_w} \big( P \left\langle \boldsymbol{\Gamma} \,,\, \boldsymbol{x}_w \times \boldsymbol{n}_b \right\rangle + P \left\langle \boldsymbol{Q}^{-1} \delta \boldsymbol{q} \,,\, \boldsymbol{n}_b \right\rangle \big) \, \mathrm{d}S \, \mathrm{d}t \,, \tag{7.9}$$

where

$$\boldsymbol{n}_b = \boldsymbol{Q}^{-1} \boldsymbol{n} \,, \tag{7.10}$$

is the unit normal vector along $S_w$ in the body frame $\boldsymbol{x}_b$. Using the variational Reynold's transport theorem, it can be proved that (Alemi Ardakani 2019)

$$\left. \begin{aligned} -\int_{t_1}^{t_2} \iiint_{V(t)} \delta\phi_t \, \rho \, \mathrm{d}V \, \mathrm{d}t \;\; &= \;\; \int_{t_1}^{t_2} \iint_{S_\eta} \eta_t \, \delta\phi \bigg|^{Z=\eta} \rho \, \ell^{-1} \, \mathrm{d}S \, \mathrm{d}t \\ &\quad + \int_{t_1}^{t_2} \iint_{S_w} \Big( \dot{\boldsymbol{X}}_w \cdot \boldsymbol{n} \Big) \, \delta\phi \, \rho \, \mathrm{d}S \, \mathrm{d}t \,. \end{aligned} \right\} \tag{7.11}$$

Moreover, applying Green's first identity, we have

$$\left\{ \begin{aligned} &\int_{t_1}^{t_2} \iiint_{V(t)} \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla} \, \delta\phi \, \rho \, \mathrm{d}V \mathrm{d}t = -\int_{t_1}^{t_2} \iiint_{V(t)} \Delta \, \phi \, \delta\phi \, \rho \, \mathrm{d}V \mathrm{d}t \\ &+ \int_{t_1}^{t_2} \iint_{\partial V} \left( \boldsymbol{\nabla}\phi \cdot \boldsymbol{n} \right) \, \delta\phi \, \rho \, \mathrm{d}S \, \mathrm{d}t = -\int_{t_1}^{t_2} \iiint_{V(t)} \Delta \, \phi \, \delta\phi \, \rho \, \mathrm{d}V \mathrm{d}t \\ &+ \int_{t_1}^{t_2} \iint_{S_\eta} \big( -\eta_X \phi_X - \eta_Y \phi_Y + \phi_Z \big) \, \delta\phi \bigg|^{Z=\eta} \rho \, \ell^{-1} \, \mathrm{d}S \, \mathrm{d}t \\ &+ \int_{t_1}^{t_2} \iint_{S_b} \big( \phi_X H_X + \phi_Y H_Y + \phi_Z \big) \, \delta\phi \bigg|_{Z=-H} \rho \, \mathrm{d}S \, \mathrm{d}t + \int_{t_1}^{t_2} \iint_{S_w} \frac{\partial\phi}{\partial \boldsymbol{n}} \, \delta\phi \, \rho \, \mathrm{d}S \, \mathrm{d}t \,. \end{aligned} \right. \tag{7.12}$$

Now, substituting the expressions (7.9), (7.11) and (7.12) into the variational principle (7.6), we conclude that invariance of $\mathscr{L}$ with respect to a variation in the free-surface elevation $\eta$ yields the dynamic free-surface boundary condition in (7.1), invariance of $\mathscr{L}$ with respect to a variation in the velocity potential $\phi$ yields the field equation in (7.1) in the domain $V(t)$, invariance of $\mathscr{L}$ with respect to a variation in the velocity potential $\phi$ at $Z = -H(X, Y)$ gives the bottom boundary condition in (7.1), invariance of $\mathscr{L}$ with respect to a variation in the velocity potential $\phi$ at $Z = \eta(X, Y, t)$ gives the kinematic free-surface boundary condition in (7.1) and invariance of $\mathscr{L}$ with respect to a variation in the velocity potential $\phi$ on $S_w$ gives the contact condition on the wetted surface of the rigid-body,

$$\frac{\partial \Phi}{\partial \boldsymbol{n}} = \dot{\boldsymbol{X}}_w \cdot \boldsymbol{n} \quad \text{on} \quad S_w \,. \tag{7.13}$$

Invariance of $\mathscr{L}$ with respect to $\delta \boldsymbol{x}_2$ gives the generalized Green–Naghdi equations (3.19) and (3.20) for the interior weakly dispersive nonlinear fluid sloshing. Invariance of $\mathscr{L}$ with respect to $\boldsymbol{\Gamma}$ gives the hydrodynamic equation of motion for the rotational motion $\boldsymbol{\Omega}(t)$ of the coupled **(**floating rigid-body motion + interior weakly dispersive fluid sloshing**)** system interacting with exterior potential-flow water waves

$$\left.\begin{aligned}&\iint \left(\frac{\mathcal{D}\boldsymbol{\mathcal{U}}}{\mathcal{D}t}\times(\boldsymbol{\mathcal{X}}+\boldsymbol{d})+\boldsymbol{\Omega}\times(\boldsymbol{\mathcal{U}}\times(\boldsymbol{\mathcal{X}}+\boldsymbol{d}))\right)\rho\, h\,\mathrm{d}\boldsymbol{x}_2\\ &-m\overline{\boldsymbol{x}}\times\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-\dot{\boldsymbol{I}}_f^{SW}\boldsymbol{\Omega}-\boldsymbol{I}_t\dot{\boldsymbol{\Omega}}+\boldsymbol{I}_t\boldsymbol{\Omega}\times\boldsymbol{\Omega}-mg\overline{\boldsymbol{x}}\times\boldsymbol{\Sigma}\\ &+\iint_{S_w}P\left(X,Y,Z,t\right)\left(\boldsymbol{x}_w\times\boldsymbol{n}_b\right)\,\mathrm{d}S=\boldsymbol{0}\,,\end{aligned}\right\}\tag{7.14}$$

where $P\left(X,Y,Z,t\right)$ is defined in (7.7). Finally, the invariance of $\mathscr{L}$ with respect to $\boldsymbol{Q}^{-1}\delta\boldsymbol{q}$ gives the hydrodynamic equation of motion for the translational motion $\boldsymbol{q}\left(t\right)$ of the coupled **(**floating rigid-body motion + interior weakly dispersive fluid sloshing**)** system interacting with exterior potential-flow water waves

$$\left\{\begin{aligned}&\iint\left(\frac{\mathcal{D}\boldsymbol{\mathcal{U}}}{\mathcal{D}t}+2\boldsymbol{\Omega}\times\boldsymbol{\mathcal{U}}\right)\rho\, h\,\mathrm{d}\boldsymbol{x}_2+m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}+\dot{\boldsymbol{\Omega}}\times m\overline{\boldsymbol{x}}+\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times m\overline{\boldsymbol{x}}\right)\\ &+mg\boldsymbol{\Sigma}-\iint_{S_w}P\left(X,Y,Z,t\right)\boldsymbol{n}_b\,\mathrm{d}S=\boldsymbol{0}\,.\end{aligned}\right.\tag{7.15}$$

The terms including the pressure field $P\left(X,Y,Z,t\right)$ in the hydrodynamic equations of motion (7.14) and (7.15) are the moments and forces respectively acting on the rigid-body due to interactions with exterior potential-flow water waves. In summary, the equations of motion for the exterior potential-flow water waves in $V\left(t\right)$ are (7.1) with the contact boundary condition (7.13). The equations of motion for the interior fluid of the rigid-body are the generalized Green–Naghdi equations (3.19) and (3.20) and the conservation of mass equation (2.8) which are dynamically coupled to the hydrodynamic equations of motion for the floating rigid-body (7.14) and (7.15). The evolutionary system for the rigid-body motion is completed by (6.20) and (6.21).

If we take the variations in the variational principle (7.5), the interior fluid of the floating rigid-body in the coupled **(**exterior potential-flow water waves + floating rigid-body motion + interior weakly dispersive fluid sloshing**)** system would be described by the generalized Whitham equations (5.25), (5.26) and (5.27). Moreover, the first integral on the left-hand side in the equations of motion for the rigid-body motion (7.14) and (7.15) would be described in terms of the interior shallow water velocity potential $\Phi\left(x,y,t\right)$.

# 8 Concluding remarks

The paper is devoted to the derivation of *a reduced shallow-water* variational principle for dynamic coupling between a rigid-body, which is free to undergo three-dimensional rotational and traditional motions, and its interior *weakly dispersive* nonlinear shallow-water sloshing. The reduced variational principle gives rise to a new Green-Naghdi model for shallow-water sloshing in two-horizontal space dimensions with 3–D rotation vector and translations. Neglecting the higher-order dispersive terms in the generalized Green–Naghdi model gives rise to a new set of shallow-water equations in three-dimensional rotating and translating coordinates, which is the variational analogue of the surface shallow-water equations derived by Alemi Ardakani & Bridges 2011.

The material conservation of potential-vorticity for the generalized Green–Naghdi equations and the variational shallow-water equations is studied. A new generalization of the Whitham equations (Boussinesq-like evolution equations) for fluid sloshing in a vessel undergoing prescribed rigid-body motion in three dimensions is derived by applying the assumption of zero-potential-vorticity flow to the fluid component of the reduced variational principle in Eulerian coordinates.

The variational principles are extended to develop a mathematical theory for the problem of three-dimensional interactions between potential-flow water waves and a freely floating rigid-body dynamically coupled to its interior shallow-water sloshing, which can be described by the generalized Green–Naghdi equations, or by the variational shallow-water equations, or by the generalized Whitham equations. The exact nonlinear hydrodynamic equations of motion for the angular momentum and linear momentum of the floating rigid-body are derived.

The presented variational frameworks for the coupled **(**rigid-body motion + interior weakly dispersive shallow-water sloshing**)** system and for the coupled **(**exterior water waves + rigid-body motion + interior weakly dispersive shallow-water sloshing**)** interactions can be a starting point in constructing symplectic and structure-preserving numerical schemes for long-time computational modelling of these highly-coupled nonlinear systems. Gidel *et al.* (2017) developed a variational Galerkin finite-element method with a second-order Störmer–Verlet temporal scheme for the Benney–Luke equations (Benney & Luke 1964). These numerical methods can be extended for computational modelling of the variational frameworks presented in this paper.

## — Appendix —

# A Proof of (2.10) and the entries of $\boldsymbol{I}_f^{SW}$

Restriction of the second term in (1.12) to columnar motion gives

$$\begin{cases} \displaystyle\iiint_0^{h_0} \dot{\boldsymbol{x}} \cdot \left(\boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T \dot{\boldsymbol{q}}\right) \rho \, \mathrm{d}\boldsymbol{a} = \iiint_0^{h_0} \left[ \dot{x} \left( \Omega_2 \left( \frac{h}{h_0} c + d_3 \right) - \Omega_3 (y+d_2) \right) \right. \\ \displaystyle \left. + \dot{y} \left( \Omega_3 (x+d_1) - \Omega_1 \left( \frac{h}{h_0} c + d_3 \right) \right) + \frac{\dot{h}}{h_0} c \left( \Omega_1 (y+d_2) - \Omega_2 (x+d_1) \right) \right] \rho \, \mathrm{d}c \, \mathrm{d}\boldsymbol{a}_2 \\ \displaystyle + \iiint_0^{h_0} \left( \dot{x}\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 + \dot{y}\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_2 + \frac{\dot{h}}{h_0} c\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3 \right) \rho \, \mathrm{d}c \, \mathrm{d}\boldsymbol{a}_2 \\ \displaystyle = \iint \left( \dot{x} \left( \Omega_2 \left( \tfrac{1}{2} h + d_3 \right) - \Omega_3 (y+d_2) \right) + \dot{y} \left( \Omega_3 (x+d_1) - \Omega_1 \left( \tfrac{1}{2} h + d_3 \right) \right) \right. \\ \displaystyle \left. + \tfrac{1}{2} \dot{h} \left( \Omega_1 (y+d_2) - \Omega_2 (x+d_1) \right) + \dot{x}\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 + \dot{y}\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_2 + \tfrac{1}{2} \dot{h}\, \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3 \right) \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \\ \displaystyle = \iint \left\langle \dot{\mathfrak{X}} \, , \, \boldsymbol{\Omega} \times (\mathfrak{X} + \boldsymbol{d}) + \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \right\rangle \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, , \end{cases} \tag{A.1}$$

which recovers (2.10). The symmetric matrix $\boldsymbol{I}_f^{SW}$ is the reduced shallow water mass moment of inertia of the interior fluid relative to the point of rotation, i.e the origin of the body frame $\boldsymbol{x}_b$, with entries given by

$$\begin{cases} \displaystyle \boldsymbol{I}_{f\,11}^{SW} = \iint \left( (y+d_2)^2 + \frac{1}{3} h^2 + d_3^2 + d_3 h \right) \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \\ \displaystyle \qquad\quad = \iint \left( (y+d_2)^2 + \frac{1}{3} h^2 + d_3^2 + d_3 h \right) \rho \, h \, \mathrm{d}\boldsymbol{x}_2 \, , \\ \displaystyle \boldsymbol{I}_{f\,12}^{SW} = \iint - (x+d_1)(y+d_2) \, \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 = \iint - (x+d_1)(y+d_2) \, \rho \, h \, \mathrm{d}\boldsymbol{x}_2 \, , \\ \displaystyle \boldsymbol{I}_{f\,13}^{SW} = \iint - (x+d_1) \left( \tfrac{1}{2} h + d_3 \right) \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 = \iint - (x+d_1) \left( \tfrac{1}{2} h + d_3 \right) \rho \, h \, \mathrm{d}\boldsymbol{x}_2 \, , \end{cases}$$

$$
\begin{cases}
\boldsymbol{I}^{SW}_{f\,22} = \iint \left( (x+d_1)^2 + \dfrac{1}{3}h^2 + d_3^2 + d_3 h \right) \rho\, h_0 \,\mathrm{d}\boldsymbol{a}_2 \\
\qquad\quad = \iint \left( (x+d_1)^2 + \dfrac{1}{3}h^2 + d_3^2 + d_3 h \right) \rho\, h \,\mathrm{d}\boldsymbol{x}_2 \,, \\
\boldsymbol{I}^{SW}_{f\,23} = \iint -(y+d_2)\left(\tfrac{1}{2}h + d_3\right) \rho\, h_0 \,\mathrm{d}\boldsymbol{a}_2 = \iint -(y+d_2)\left(\tfrac{1}{2}h + d_3\right) \rho\, h \,\mathrm{d}\boldsymbol{x}_2 \,, \\
\boldsymbol{I}^{SW}_{f\,33} = \iint \left( (x+d_1)^2 + (y+d_2)^2 \right) \rho\, h_0 \,\mathrm{d}\boldsymbol{a}_2 = \iint \left( (x+d_1)^2 + (y+d_2)^2 \right) \rho\, h \,\mathrm{d}\boldsymbol{x}_2 \,.
\end{cases}
\tag{A.2}
$$

Differentiating $\boldsymbol{I}^{SW}_f$, in the Lagrangian particle-path setting, with respect to time and transforming the result to Eulerian coordinates gives

$$
\begin{cases}
\dot{\boldsymbol{I}}^{SW}_{f\,11} = \iint \left( 2v\,(y+d_2) - h\,(u_x + v_y) \left( \dfrac{2}{3}h + d_3 \right) \right) \rho\, h \,\mathrm{d}\boldsymbol{x}_2 \,, \\
\dot{\boldsymbol{I}}^{SW}_{f\,12} = \iint -\left( v\,(x+d_1) + u\,(y+d_2) \right) \rho\, h \,\mathrm{d}\boldsymbol{x}_2 \,, \\
\dot{\boldsymbol{I}}^{SW}_{f\,13} = \iint -\left( u\left(\tfrac{1}{2}h + d_3\right) - \tfrac{1}{2}h\,(u_x+v_y)\,(x+d_1) \right) \rho\, h \,\mathrm{d}\boldsymbol{x}_2 \,, \\
\dot{\boldsymbol{I}}^{SW}_{f\,22} = \iint \left( 2u\,(x+d_1) - h\,(u_x + v_y) \left( \dfrac{2}{3}h + d_3 \right) \right) \rho\, h \,\mathrm{d}\boldsymbol{x}_2 \,, \\
\dot{\boldsymbol{I}}^{SW}_{f\,23} = \iint -\left( v\left(\tfrac{1}{2}h + d_3\right) - \tfrac{1}{2}h\,(u_x+v_y)\,(y+d_2) \right) \rho\, h \,\mathrm{d}\boldsymbol{x}_2 \,, \\
\dot{\boldsymbol{I}}^{SW}_{f\,33} = \iint 2\left( u\,(x+d_1) + v\,(y+d_2) \right) \rho\, h \,\mathrm{d}\boldsymbol{x}_2 \,.
\end{cases}
\tag{A.3}
$$

# B Proof of $\delta h$ in (3.2) and the variational identity (3.3)

The following expressions are given by Miles & Salmon (1985), which are reviewed here:

$$
\begin{cases}
\delta h = -\dfrac{h_0}{\mathcal{J}}\mathcal{J}^{-1}\delta\mathcal{J} = -h\,\dfrac{\partial(a,b)}{\partial(x,y)} \left( \dfrac{\partial(\delta x, y)}{\partial(a,b)} + \dfrac{\partial(x,\delta y)}{\partial(a,b)} \right) \\
= -h \left( \dfrac{\partial(\delta x, y)}{\partial(x,y)} + \dfrac{\partial(x,\delta y)}{\partial(x,y)} \right) = -h \left( \dfrac{\partial \delta x}{\partial x} + \dfrac{\partial \delta y}{\partial y} \right) = -h\, \boldsymbol{\nabla}_2 \boldsymbol{\cdot} \delta\boldsymbol{x}_2 \,.
\end{cases}
\tag{B.1}
$$

Hence, the following variational identity can be concluded

$$
\begin{cases}
\displaystyle\iint \mathcal{F}\,\delta h\,\rho\, h_0 \,\mathrm{d}a\,\mathrm{d}b = \iint \mathcal{F}\,\delta h\,\rho\, h \,\mathrm{d}x\,\mathrm{d}y = \iint -\mathcal{F}\,h^2 \boldsymbol{\nabla}_2 \boldsymbol{\cdot} \delta\boldsymbol{x}_2\,\rho \,\mathrm{d}x\,\mathrm{d}y \\
\displaystyle = \iint \boldsymbol{\nabla}_2\left(\mathcal{F}\,h^2\right) \boldsymbol{\cdot} \delta\boldsymbol{x}_2\,\rho \,\mathrm{d}x\,\mathrm{d}y = \iint \frac{1}{h}\boldsymbol{\nabla}_2\left(\mathcal{F}\,h^2\right) \boldsymbol{\cdot} \delta\boldsymbol{x}_2\,\rho\, h_0 \,\mathrm{d}a\,\mathrm{d}b \,.
\end{cases}
\tag{B.2}
$$

Note that from the first to the second line in (B.2), we integrate by parts and impose the boundary conditions $\delta\boldsymbol{x}_2 = \boldsymbol{0}$ at the endpoints in space.

# C Proof of (3.8), (3.15), and (3.16)

Derivation of the variations in (3.8) is given below:

$$
\int_{t_1}^{t_2} \iint \left\langle \delta\boldsymbol{\mathcal{X}} \,,\, -\dot{\boldsymbol{\Omega}} \times (\boldsymbol{\mathcal{X}} + \boldsymbol{d}) \right\rangle \rho\, h_0 \,\mathrm{d}\boldsymbol{a}_2 \,\mathrm{d}t =
$$

$$
\left\{
\begin{aligned}
&= \int_{t_1}^{t_2} \iint \left( \left\langle \delta \boldsymbol{x}_2 \,,\, \begin{bmatrix} \dot{\Omega}_3 \left(y + d_2\right) - \dot{\Omega}_2 \left(\tfrac{1}{2}h + d_3\right) \\ \dot{\Omega}_1 \left(\tfrac{1}{2}h + d_3\right) - \dot{\Omega}_3 \left(x + d_1\right) \end{bmatrix} \right\rangle \right. \\
&\qquad\qquad \left. + \tfrac{1}{2} \Big( \dot{\Omega}_2 \left(x + d_1\right) - \dot{\Omega}_1 \left(y + d_2\right) \Big) \, \delta h \right) \, \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \iint \left\langle \delta \boldsymbol{x}_2 \,,\, \begin{bmatrix} \dot{\Omega}_3 \left(y + d_2\right) - \dot{\Omega}_2 \left(\tfrac{1}{2}h + d_3\right) \\ \dot{\Omega}_1 \left(\tfrac{1}{2}h + d_3\right) - \dot{\Omega}_3 \left(x + d_1\right) \end{bmatrix} \right\rangle \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \iint \left\langle \underbrace{\delta \boldsymbol{x}_2 \,,\, \tfrac{1}{2} \frac{1}{h} \boldsymbol{\nabla}_2 \Big( h^2 \Big( \dot{\Omega}_2 \left(x + d_1\right) - \dot{\Omega}_1 \left(y + d_2\right) \Big) \Big)}_{\hookrightarrow \text{ using the variational identity (3.3)}} \right\rangle \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \\
&= \iiint \left\langle \delta \boldsymbol{x}_2 \,,\, \begin{bmatrix} \dot{\Omega}_3 \left(y + d_2\right) + \Big( \dot{\Omega}_2 \left(x + d_1\right) - \dot{\Omega}_1 \left(y + d_2\right) \Big) h_x - \dot{\Omega}_2 d_3 \\ -\dot{\Omega}_3 \left(x + d_1\right) + \Big( \dot{\Omega}_2 \left(x + d_1\right) - \dot{\Omega}_1 \left(y + d_2\right) \Big) h_y + \dot{\Omega}_1 d_3 \end{bmatrix} \right\rangle \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \,.
\end{aligned}
\right.
\tag{C.1}
$$

Derivation of the variations in (3.15) is given below:

$$
\left.
\begin{aligned}
&\delta \int_{t_1}^{t_2} \iint -g \left\langle \hat{\boldsymbol{z}} \,,\, \boldsymbol{Q} \left( \boldsymbol{\mathfrak{X}} + \boldsymbol{d} \right) + \boldsymbol{q} \right\rangle \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \\
&= \delta \int_{t_1}^{t_2} \iint -g \left\langle \boldsymbol{\Sigma} \,,\, \boldsymbol{\mathfrak{X}} + \boldsymbol{d} + \boldsymbol{Q}^{-1} \boldsymbol{q} \right\rangle \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \quad \text{with} \quad \boldsymbol{\Sigma} = \boldsymbol{Q}^{-1} \hat{\boldsymbol{z}} \\
&= \int_{t_1}^{t_2} \iint \left\langle \delta \boldsymbol{\mathfrak{X}} \,,\, -g \boldsymbol{\Sigma} \right\rangle \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \iint \left( \left\langle \delta \boldsymbol{x}_2 \,,\, -g \begin{bmatrix} \boldsymbol{e}_3 \cdot \boldsymbol{Q} \boldsymbol{e}_1 \\ \boldsymbol{e}_3 \cdot \boldsymbol{Q} \boldsymbol{e}_2 \end{bmatrix} \right\rangle - \tfrac{1}{2} g \, \boldsymbol{e}_3 \cdot \boldsymbol{Q} \boldsymbol{e}_3 \, \delta h \right) \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \iint \left\langle \delta \boldsymbol{x}_2 \,,\, -g \begin{bmatrix} \boldsymbol{e}_3 \cdot \boldsymbol{Q} \boldsymbol{e}_1 \\ \boldsymbol{e}_3 \cdot \boldsymbol{Q} \boldsymbol{e}_2 \end{bmatrix} - g \underbrace{\begin{bmatrix} \boldsymbol{e}_3 \cdot \boldsymbol{Q} \boldsymbol{e}_3 \, h_x \\ \boldsymbol{e}_3 \cdot \boldsymbol{Q} \boldsymbol{e}_3 \, h_y \end{bmatrix}}_{\text{using (3.3)}} \right\rangle \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \,.
\end{aligned}
\right\}
\tag{C.2}
$$

Taking the variations $\delta \boldsymbol{x}_2$ and $\delta h$ of the mass moment of inertia of the interior shallow water in the action integral (2.15), assuming that $\boldsymbol{\Omega}$ is constant, gives

$$
\left.
\begin{aligned}
&\delta \int_{t_1}^{t_2} \tfrac{1}{2} \left\langle \boldsymbol{\Omega} \,,\, \boldsymbol{I}_f^{SW} \boldsymbol{\Omega} \right\rangle \, \mathrm{d}t = \int_{t_1}^{t_2} \left\langle \boldsymbol{\Omega} \,,\, \delta \boldsymbol{I}_f^{SW} \boldsymbol{\Omega} \right\rangle \, \mathrm{d}t = \int_{t_1}^{t_2} \iint \left\langle \delta \boldsymbol{x}_2 \,, \right. \\
&\left. \begin{bmatrix} \left(x + d_1\right) \left(\Omega_2^2 + \Omega_3^2\right) - \left(y + d_2\right) \Omega_1 \Omega_2 - \left(\tfrac{1}{2}h + d_3\right) \Omega_1 \Omega_3 \\ \left(y + d_2\right) \left(\Omega_1^2 + \Omega_3^2\right) - \left(x + d_1\right) \Omega_1 \Omega_2 - \left(\tfrac{1}{2}h + d_3\right) \Omega_2 \Omega_3 \end{bmatrix} \right\rangle \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \\
&+ \int_{t_1}^{t_2} \iint \left( \tfrac{1}{2} \left( \frac{2}{3} h + d_3 \right) \left(\Omega_1^2 + \Omega_2^2\right) - \tfrac{1}{2} \left(x + d_1\right) \Omega_1 \Omega_3 \right. \\
&\qquad\qquad \left. - \tfrac{1}{2} \left(y + d_2\right) \Omega_2 \Omega_3 \right) \delta h \, \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t = \int_{t_1}^{t_2} \iint \left\langle \delta \boldsymbol{x}_2 \,, \right. \\
&\left. \begin{bmatrix} \left(x + d_1\right) \left(\Omega_2^2 + \Omega_3^2\right) - \left(y + d_2\right) \Omega_1 \Omega_2 - \left(\tfrac{1}{2}h + d_3\right) \Omega_1 \Omega_3 \\ \left(y + d_2\right) \left(\Omega_1^2 + \Omega_3^2\right) - \left(x + d_1\right) \Omega_1 \Omega_2 - \left(\tfrac{1}{2}h + d_3\right) \Omega_2 \Omega_3 \end{bmatrix} \right\rangle \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \\
&+ \int_{t_1}^{t_2} \iint \left\langle \mathrm{d}\boldsymbol{x}_2 \,,\, \frac{1}{h} \boldsymbol{\nabla}_2 \left( h^2 \left[ \tfrac{1}{2} \left( \frac{2}{3} h + d_3 \right) \left(\Omega_1^2 + \Omega_2^2\right) - \tfrac{1}{2} \left(x + d_1\right) \Omega_1 \Omega_3 \right. \right. \right. \\
&\qquad\qquad \left. \left. \left. - \tfrac{1}{2} \left(y + d_2\right) \Omega_2 \Omega_3 \right] \right) \right\rangle \rho \, h_0 \, \mathrm{d}\boldsymbol{a}_2 \, \mathrm{d}t \,,
\end{aligned}
\right\}
\tag{C.3}
$$

which leads to (3.16).

# D Proof of (4.4)

The variational derivatives in (4.2) defining the canonical momenta $\mathcal{P}$ for the shallow-water Lagrangian (2.15) are taken using a *mass-weighted* inner product in the Lagrangian particle-path setting as follows:

$$\delta\mathscr{L}_{SW/\mathsf{GN}} = \iint \left( \left\langle \frac{\delta\widehat{\mathscr{L}}}{\delta\mathfrak{X}}\,,\,\delta\mathfrak{X} \right\rangle + \left\langle \frac{\delta\widehat{\mathscr{L}}}{\delta\dot{\mathfrak{X}}}\,,\,\delta\dot{\mathfrak{X}} \right\rangle \right) \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\,. \tag{D.1}$$

The first component in (D.1) reads

$$\begin{cases} \displaystyle\iint \left\langle \frac{\delta\widehat{\mathscr{L}}}{\delta\mathfrak{X}}\,,\,\delta\mathfrak{X} \right\rangle \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2 = \iint \left( \left\langle \frac{\delta\widehat{\mathscr{L}}}{\delta\boldsymbol{x}_2}\,,\,\delta\boldsymbol{x}_2 \right\rangle + \frac{\delta\widehat{\mathscr{L}}}{\delta h}\,\delta h \right) \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2 \\ \displaystyle = \iint \left\langle \frac{\delta\widehat{\mathscr{L}}}{\delta\boldsymbol{x}_2} + \frac{1}{h}\boldsymbol{\nabla}_2\Big(h^2\frac{\delta\widehat{\mathscr{L}}}{\delta h}\Big)\,,\,\delta\boldsymbol{x}_2 \right\rangle \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2 \quad \text{(using (3.3))}\,, \end{cases} \tag{D.2}$$

and the second component in (D.1) takes the form

$$\iint \left\langle \frac{\delta\widehat{\mathscr{L}}}{\delta\dot{\mathfrak{X}}}\,,\,\delta\dot{\mathfrak{X}} \right\rangle \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2 = \iint \left( \left\langle \frac{\delta\widehat{\mathscr{L}}}{\delta\dot{\boldsymbol{x}}_2}\,,\,\delta\dot{\boldsymbol{x}}_2 \right\rangle + \frac{\delta\widehat{\mathscr{L}}}{\delta\dot{h}}\,\delta\dot{h} \right) \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\,. \tag{D.3}$$

But the second component in (D.3) reads

$$\begin{cases} \displaystyle\iint \frac{\delta\widehat{\mathscr{L}}}{\delta\dot{h}}\,\delta\dot{h}\,\rho\, h_0\, \mathrm{d}\boldsymbol{a}_2 = \iint h\,\frac{\delta\widehat{\mathscr{L}}}{\delta\dot{h}}\,\delta\dot{h}\,\rho\, \mathrm{d}\boldsymbol{x}_2 \\ \displaystyle = \iint h\,\frac{\delta\widehat{\mathscr{L}}}{\delta\dot{h}}\,\overbrace{\Big(-\dot{h}\boldsymbol{\nabla}_2\boldsymbol{\cdot}\delta\boldsymbol{x}_2 - h\boldsymbol{\nabla}_2\boldsymbol{\cdot}\delta\dot{\boldsymbol{x}}_2\Big)}^{=\,\delta\dot{h}}\,\rho\, \mathrm{d}\boldsymbol{x}_2 \\ \displaystyle = \underbrace{\iint \left( \left\langle \boldsymbol{\nabla}_2\Big(h\dot{h}\,\frac{\delta\widehat{\mathscr{L}}}{\delta\dot{h}}\Big)\,,\,\delta\boldsymbol{x}_2 \right\rangle + \left\langle \boldsymbol{\nabla}_2\Big(h^2\,\frac{\delta\widehat{\mathscr{L}}}{\delta\dot{h}}\Big)\,,\,\delta\dot{\boldsymbol{x}}_2 \right\rangle \right) \rho\, \mathrm{d}\boldsymbol{x}_2} \\ \quad\hookrightarrow \begin{cases} \text{integrating by parts and imposing the boundary conditions} \\ \delta x = \delta\dot{x} = 0 \text{ at } x = 0, L_1 \text{ and } \delta y = \delta\dot{y} = 0 \text{ at } y = 0, L_2 \end{cases} \\ \displaystyle = \iint \left( \left\langle \frac{1}{h}\boldsymbol{\nabla}_2\Big(h\dot{h}\,\frac{\delta\widehat{\mathscr{L}}}{\delta\dot{h}}\Big)\,,\,\delta\boldsymbol{x}_2 \right\rangle + \left\langle \frac{1}{h}\boldsymbol{\nabla}_2\Big(h^2\,\frac{\delta\widehat{\mathscr{L}}}{\delta\dot{h}}\Big)\,,\,\delta\dot{\boldsymbol{x}}_2 \right\rangle \right) \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\,, \end{cases} \tag{D.4}$$

and hence (D.3) modifies to

$$\begin{cases} \displaystyle\iint \left\langle \frac{\delta\widehat{\mathscr{L}}}{\delta\dot{\mathfrak{X}}}\,,\,\delta\dot{\mathfrak{X}} \right\rangle \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2 = \iint \left( \left\langle \frac{1}{h}\boldsymbol{\nabla}_2\Big(h\dot{h}\,\frac{\delta\widehat{\mathscr{L}}}{\delta\dot{h}}\Big)\,,\,\delta\boldsymbol{x}_2 \right\rangle \right. \\ \displaystyle\qquad\qquad \left. + \left\langle \frac{\delta\widehat{\mathscr{L}}}{\delta\dot{\boldsymbol{x}}_2} + \frac{1}{h}\boldsymbol{\nabla}_2\Big(h^2\,\frac{\delta\widehat{\mathscr{L}}}{\delta\dot{h}}\Big)\,,\,\delta\dot{\boldsymbol{x}}_2 \right\rangle \right) \rho\, h_0\, \mathrm{d}\boldsymbol{a}_2\,. \end{cases} \tag{D.5}$$

Now substituting (D.2) and (D.5) into (D.1) recovers (4.4).

# E Derivation of the generalized Whitham equations (5.25), (5.26) and (5.27) for fluid sloshing in three-dimensional rotating and translating coordinates

Hamilton's variational principle (5.23), for the generalized Whitham equations, for the variations $\delta h$ and $\delta\Phi$ takes the form

$$\left\{\begin{aligned}
&\delta\mathscr{L}_{\mathsf{GW}}\left(h,\Phi\right) = \iiint\left(\frac{1}{2}\left(h\nabla_2^2\Phi\right)^2\delta h + \tfrac{1}{2}h\big(\Omega_2\Phi_x - \Omega_1\Phi_y\big)\delta h - \tfrac{1}{2}h\begin{bmatrix}\Omega_2\\-\Omega_1\end{bmatrix}\boldsymbol{\cdot}\boldsymbol{R}_2\,\delta h\right.\\
&+h\left(\boldsymbol{R}_2 - \begin{bmatrix}\Omega_2\left(\tfrac{1}{2}h + d_3\right) - \Omega_3\left(y+d_2\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_1\\ \Omega_3\left(x+d_1\right) - \Omega_1\left(\tfrac{1}{2}h+d_3\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_2\end{bmatrix}\right)\boldsymbol{\cdot}\delta\boldsymbol{R}_2\\
&+\left(\tfrac{1}{2}h^2\boldsymbol{\nabla}_2\boldsymbol{\cdot}\delta\boldsymbol{R}_2 + h\boldsymbol{\nabla}_2\boldsymbol{\cdot}\boldsymbol{R}_2\,\delta h - h\nabla_2^2\Phi\,\delta h\right)\left(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_3\right)\\
&+\Big[\tfrac{1}{2}h\big(\Omega_2\dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_1 - \Omega_1\dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_2 - g\boldsymbol{e}_3\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_3\big) - \Phi_t - \tfrac{1}{2}\boldsymbol{\nabla}_2\Phi\boldsymbol{\cdot}\boldsymbol{\nabla}_2\Phi + \tfrac{1}{2}\boldsymbol{R}_2\boldsymbol{\cdot}\boldsymbol{R}_2\\
&-\left(\boldsymbol{R}_2 - \boldsymbol{\nabla}_2\Phi\right)\boldsymbol{\cdot}\begin{bmatrix}\Omega_2\left(\tfrac{1}{2}h + d_3\right) - \Omega_3\left(y+d_2\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_1\\ \Omega_3\left(x+d_1\right) - \Omega_1\left(\tfrac{1}{2}h+d_3\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_2\end{bmatrix} + \tfrac{1}{2}\dot{\boldsymbol{q}}\boldsymbol{\cdot}\dot{\boldsymbol{q}}\\
&+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{\Omega}\times\left(\boldsymbol{\mathfrak{X}} + \boldsymbol{d}\right) - g\left(\boldsymbol{Q}\left(\boldsymbol{\mathfrak{X}}+\boldsymbol{d}\right) + \boldsymbol{q}\right)\boldsymbol{\cdot}\hat{\boldsymbol{z}} + \tfrac{1}{2}\left(x+d_1\right)^2\left(\Omega_2^2 + \Omega_3^2\right)\\
&+\tfrac{1}{2}\left(y+d_2\right)^2\left(\Omega_1^2+\Omega_3^2\right) + \tfrac{1}{2}\left(h+d_3\right)^2\left(\Omega_1^2+\Omega_2^2\right) - \left(x+d_1\right)\left(y+d_2\right)\Omega_1\Omega_2\\
&-\left(x+d_1\right)\left(h+d_3\right)\Omega_1\Omega_3 - \left(y+d_2\right)\left(h+d_3\right)\Omega_2\Omega_3\Big]\delta h - h\boldsymbol{\nabla}_2\Phi\boldsymbol{\cdot}\boldsymbol{\nabla}_2\delta\Phi - h\,\delta\Phi_t\\
&+\frac{1}{3}h^3\nabla_2^2\Phi\nabla_2^2\delta\Phi + h\boldsymbol{\nabla}_2\delta\Phi\boldsymbol{\cdot}\begin{bmatrix}\Omega_2\left(\tfrac{1}{2}h + d_3\right) - \Omega_3\left(y+d_2\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_1\\ \Omega_3\left(x+d_1\right) - \Omega_1\left(\tfrac{1}{2}h+d_3\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_2\end{bmatrix}\\
&\left.-\tfrac{1}{2}h^2\,\boldsymbol{\nabla}_2\boldsymbol{\cdot}\boldsymbol{\nabla}_2\delta\Phi\left(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_3\right)\right)\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t = 0\,,
\end{aligned}\right. \tag{E.1}$$

where

$$\delta\boldsymbol{R}_2 = \big(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_3\big)\boldsymbol{\nabla}_2\delta h\,. \tag{E.2}$$

Substituting for $\boldsymbol{R}_2$ and $\delta\boldsymbol{R}_2$ in the second line and the first term in the third line of (E.1) gives

$$\left\{\begin{aligned}
&\iiint h\left(\boldsymbol{R}_2 - \begin{bmatrix}\Omega_2\left(\tfrac{1}{2}h + d_3\right) - \Omega_3\left(y+d_2\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_1\\ \Omega_3\left(x+d_1\right) - \Omega_1\left(\tfrac{1}{2}h+d_3\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_2\end{bmatrix}\right)\boldsymbol{\cdot}\delta\boldsymbol{R}_2\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\\
&+\iiint \tfrac{1}{2}h^2\boldsymbol{\nabla}_2\boldsymbol{\cdot}\delta\boldsymbol{R}_2\left(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_3\right)\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\\
&= \iiint\left(\big(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_3\big)^2 h\,\boldsymbol{\nabla}_2 h\boldsymbol{\cdot}\boldsymbol{\nabla}_2\delta h\right.\\
&\qquad -h^2\big(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_3\big)\begin{bmatrix}\Omega_2\\-\Omega_1\end{bmatrix}\boldsymbol{\cdot}\boldsymbol{\nabla}_2\delta h\\
&\qquad \left.+\tfrac{1}{2}h^2\big(\Omega_1\left(y+d_2\right) - \Omega_2\left(x+d_1\right) + \dot{\boldsymbol{q}}\boldsymbol{\cdot}\boldsymbol{Q}\boldsymbol{e}_3\big)^2\,\boldsymbol{\nabla}_2\boldsymbol{\cdot}\boldsymbol{\nabla}_2\delta h\right)\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\,.
\end{aligned}\right. \tag{E.3}$$

With Green's first identity, we may write

$$\begin{cases} \iiint \Big( \big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)^2 h\,\boldsymbol{\nabla}_2 h \\ \qquad -h^2\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)\begin{bmatrix}\Omega_2\\-\Omega_1\end{bmatrix}\Big)\cdot\boldsymbol{\nabla}_2\delta h\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t \\ = \iiint \Big(4h\,\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)\,(\Omega_2 h_x - \Omega_1 h_y) \\ \qquad -\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)^2\,\big(\boldsymbol{\nabla}_2 h\cdot\boldsymbol{\nabla}_2 h + h\,\nabla_2^2 h\big) \\ \qquad -\left(\Omega_1^2+\Omega_2^2\right)h^2\Big)\,\delta h\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t \\ + \iint h\,\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)^2\,\boldsymbol{\nabla}_2 h\cdot\boldsymbol{n}_b\,\delta h\,\rho\,\mathrm{d}s\,\mathrm{d}t \\ - \iint h^2\,\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)\begin{bmatrix}\Omega_2\\-\Omega_1\end{bmatrix}\cdot\boldsymbol{n}_b\,\delta h\,\rho\,\mathrm{d}s\,\mathrm{d}t\,, \end{cases} \tag{E.4}$$

and

$$\begin{cases} \iiint \tfrac{1}{2}h^2\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)^2\,\boldsymbol{\nabla}_2\cdot\boldsymbol{\nabla}_2\delta h\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t \\ = \iiint \Big( -4h\,\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)\,(\Omega_2 h_x - \Omega_1 h_y) \\ \qquad +\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)^2\,\big(\boldsymbol{\nabla}_2 h\cdot\boldsymbol{\nabla}_2 h + h\,\nabla_2^2 h\big) \\ \qquad +\left(\Omega_1^2+\Omega_2^2\right)h^2\Big)\,\delta h\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t \\ - \iint h\,\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)^2\,\boldsymbol{\nabla}_2 h\cdot\boldsymbol{n}_b\,\delta h\,\rho\,\mathrm{d}s\,\mathrm{d}t \\ + \iint h^2\,\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)\begin{bmatrix}\Omega_2\\-\Omega_1\end{bmatrix}\cdot\boldsymbol{n}_b\,\delta h\,\rho\,\mathrm{d}s\,\mathrm{d}t \\ + \iint \tfrac{1}{2}h^2\,\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)^2\,\boldsymbol{\nabla}_2\delta h\cdot\boldsymbol{n}_b\,\rho\,\mathrm{d}s\,\mathrm{d}t\,, \end{cases} \tag{E.5}$$

where $\boldsymbol{n}_b$ is the outward-pointing normal to the boundary of the rigid-body relative to the body frame $\boldsymbol{x}_b$. Now, substituting (E.4) and (E.5) in (E.3), and taking into account that $\boldsymbol{\nabla}_2\delta h\cdot\boldsymbol{n}_b = 0$ along the boundaries of the rigid-body, i.e. applying symmetric boundary conditions for the wave height $h\,(x,y,t)$ for the interior fluid sloshing, the right-hand side in (E.3) vanishes. Hence, after substituting for $\boldsymbol{R}_2$, the variational principle (E.1) for the variation $\delta h$ takes the form

$$\begin{cases} \iiint \Big\langle h\boldsymbol{w}_2\,,\,\boldsymbol{\nabla}_2\Big(\tfrac{1}{2}\left(h\nabla_2^2\Phi\right)^2 - \tfrac{1}{2}\boldsymbol{\nabla}_2\Phi\cdot\boldsymbol{\nabla}_2\Phi - \Phi_t \\ \qquad +2h\,(-\Omega_2 h_x + \Omega_1 h_y)\,\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big) \\ \qquad +\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)^2\,\big(\tfrac{1}{2}\boldsymbol{\nabla}_2 h\cdot\boldsymbol{\nabla}_2 h + h\nabla_2^2 h\big) \\ \qquad -h\big(\Omega_1\,(y+d_2) - \Omega_2\,(x+d_1) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)\nabla_2^2\Phi \\ \qquad +\big(\Omega_2\,(h+d_3) - \Omega_3\,(y+d_2) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\big)\Phi_x \\ \qquad +\big(\Omega_3\,(x+d_1) - \Omega_1\,(h+d_3) + \dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\big)\Phi_y + \end{cases}$$

$$
\left\{\begin{aligned}
&+\tfrac{1}{2}\left(\Omega_1^2+\Omega_2^2\right)h^2+\tfrac{1}{2}\left(\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)^2+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)\big)\\
&+\tfrac{1}{2}\Omega_2^2\left(x+d_1\right)^2+\tfrac{1}{2}\Omega_1^2\left(y+d_2\right)^2-\left(x+d_1\right)\left(y+d_2\right)\Omega_1\Omega_2\\
&-g\left(\boldsymbol{Q}\left(\widehat{\mathfrak{X}}+\boldsymbol{d}\right)+\boldsymbol{q}\right)\cdot\hat{\boldsymbol{z}}\bigg)\bigg\rangle\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t=0\,,
\end{aligned}\right. \tag{E.6}
$$

in which $\delta h$ is replaced with $-\boldsymbol{\nabla}_2\cdot\left(h\boldsymbol{w}_2\right)$ from (3.27), the resulting expression is integrated by parts, and the boundary conditions $w_1\left(0,y,t\right)=w_1\left(L_1,y,t\right)=0$ and $w_2\left(x,0,t\right)=w_2\left(x,L_2,t\right)=0$ are imposed. Now, since $\boldsymbol{w}_2$ is arbitrary, it can be inferred that

$$
\left\{\begin{aligned}
&\boldsymbol{\nabla}_2\Big(\Phi_t+\tfrac{1}{2}\boldsymbol{\nabla}_2\Phi\cdot\boldsymbol{\nabla}_2\Phi-\tfrac{1}{2}\left(h\nabla_2^2\Phi\right)^2-\big(\Omega_2\left(h+d_3\right)-\Omega_3\left(y+d_2\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\big)\Phi_x\\
&\qquad-\big(\Omega_3\left(x+d_1\right)-\Omega_1\left(h+d_3\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\big)\Phi_y\\
&\qquad+h\left(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)\nabla_2^2\Phi\\
&\qquad-2h\left(\Omega_1h_y-\Omega_2h_x\right)\left(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)\\
&\qquad-\big(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)^2\left(\tfrac{1}{2}\boldsymbol{\nabla}_2h\cdot\boldsymbol{\nabla}_2h+h\nabla_2^2h\right)\\
&\qquad-\tfrac{1}{2}\Omega_1^2\left(\left(y+d_2\right)^2+h^2\right)-\tfrac{1}{2}\Omega_2^2\left(\left(x+d_1\right)^2+h^2\right)\\
&\qquad+\Omega_1\Omega_2\left(x+d_1\right)\left(y+d_2\right)-\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)\big)\\
&\qquad-\tfrac{1}{2}\left(\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)^2+g\left(\boldsymbol{Q}\left(\widehat{\mathfrak{X}}+\boldsymbol{d}\right)+\boldsymbol{q}\right)\cdot\hat{\boldsymbol{z}}\Big)=\boldsymbol{0}\,.
\end{aligned}\right. \tag{E.7}
$$

Integrating this equation in space gives

$$
\left\{\begin{aligned}
&\Phi_t+\tfrac{1}{2}\boldsymbol{\nabla}_2\Phi\cdot\boldsymbol{\nabla}_2\Phi-\tfrac{1}{2}\left(h\nabla_2^2\Phi\right)^2-\big(\Omega_2\left(h+d_3\right)-\Omega_3\left(y+d_2\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\big)\Phi_x\\
&-\big(\Omega_3\left(x+d_1\right)-\Omega_1\left(h+d_3\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\big)\Phi_y\\
&+h\left(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)\nabla_2^2\Phi\\
&-2h\left(\Omega_1h_y-\Omega_2h_x\right)\left(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)\\
&-\big(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)^2\left(\tfrac{1}{2}\boldsymbol{\nabla}_2h\cdot\boldsymbol{\nabla}_2h+h\nabla_2^2h\right)\\
&-\tfrac{1}{2}\Omega_1^2\left(\left(y+d_2\right)^2+h^2\right)-\tfrac{1}{2}\Omega_2^2\left(\left(x+d_1\right)^2+h^2\right)\\
&+\Omega_1\Omega_2\left(x+d_1\right)\left(y+d_2\right)-\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)\big)\\
&-\tfrac{1}{2}\left(\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\right)^2+g\left(\boldsymbol{Q}\left(\widehat{\mathfrak{X}}+\boldsymbol{d}\right)+\boldsymbol{q}\right)\cdot\hat{\boldsymbol{z}}=\mathsf{B}\left(t\right)\,,
\end{aligned}\right. \tag{E.8}
$$

where $\mathsf{B}\left(t\right)$ can be absorbed into $\Phi\left(x,y,t\right)$, which recovers the evolution equation (5.25).

To derive a second evolutionary equation which results from $\delta\Phi$ variation terms in the variational principle (E.1), we first note that

$$
\iiint -h\,\delta\Phi_t\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t=\iiint h_t\,\delta\Phi\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\,, \tag{E.9}
$$

with fixed endpoints $\delta\Phi\left(t_1\right)=\delta\Phi\left(t_2\right)=0$. With Green's first identity, we may write

$$
\left.\begin{aligned}
\iiint -h\,\boldsymbol{\nabla}_2\Phi\cdot\boldsymbol{\nabla}_2\delta\Phi\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t \;=\;& \iiint\left(\boldsymbol{\nabla}_2h\cdot\boldsymbol{\nabla}_2\Phi+h\,\nabla_2^2\Phi\right)\delta\Phi\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\\
&-\iint h\boldsymbol{\nabla}_2\Phi\cdot\boldsymbol{n}_b\,\delta\Phi\,\rho\,\mathrm{d}s\,\mathrm{d}t\,,
\end{aligned}\right\} \tag{E.10}
$$

and

$$\left\{\begin{aligned}&\iiint h\begin{bmatrix}\Omega_2\left(\frac{1}{2}h+d_3\right)-\Omega_3\left(y+d_2\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\\\Omega_3\left(x+d_1\right)-\Omega_1\left(\frac{1}{2}h+d_3\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\end{bmatrix}\cdot\boldsymbol{\nabla}_2\delta\Phi\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\\&=\iiint\Big(-\big(\Omega_2\left(h+d_3\right)-\Omega_3\left(y+d_2\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\big)h_x\\&\qquad\qquad-\big(\Omega_3\left(x+d_1\right)-\Omega_1\left(h+d_3\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\big)h_y\Big)\,\delta\Phi\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\\&+\iint h\begin{bmatrix}\Omega_2\left(\frac{1}{2}h+d_3\right)-\Omega_3\left(y+d_2\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\\\Omega_3\left(x+d_1\right)-\Omega_1\left(\frac{1}{2}h+d_3\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\end{bmatrix}\cdot\boldsymbol{n}_b\,\delta\Phi\,\rho\,\mathrm{d}s\,\mathrm{d}t\,.\end{aligned}\right.\tag{E.11}$$

With Green's second identity, we may write

$$\left\{\begin{aligned}&\iiint\frac{1}{3}h^3\,\nabla_2^2\Phi\,\nabla_2^2\delta\Phi\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t=\iiint\nabla_2^2\left(\frac{1}{3}h^3\,\nabla_2^2\Phi\right)\,\delta\Phi\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\\&+\iint\frac{1}{3}h^3\,\nabla_2^2\Phi\,\boldsymbol{\nabla}_2\delta\Phi\cdot\boldsymbol{n}_b\,\rho\,\mathrm{d}s\,\mathrm{d}t-\iint\frac{1}{3}\boldsymbol{\nabla}_2\left(h^3\nabla_2^2\Phi\right)\cdot\boldsymbol{n}_b\,\delta\Phi\,\rho\,\mathrm{d}s\,\mathrm{d}t\,,\end{aligned}\right.\tag{E.12}$$

and

$$\left\{\begin{aligned}&\iiint-\tfrac{1}{2}h^2\,\boldsymbol{\nabla}_2\cdot\boldsymbol{\nabla}_2\delta\Phi\,\big(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\\&=\iiint\Big(-\big(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)\,\big(\boldsymbol{\nabla}_2h\cdot\boldsymbol{\nabla}_2h+h\nabla_2^2h\big)\\&\qquad\qquad+2\Omega_2hh_x-2\Omega_1hh_y\Big)\,\delta\Phi\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t\\&+\iint h\begin{bmatrix}\big(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)h_x-\frac{1}{2}\Omega_2h\\\big(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)h_y+\frac{1}{2}\Omega_1h\end{bmatrix}\cdot\boldsymbol{n}_b\,\delta\Phi\,\rho\,\mathrm{d}s\,\mathrm{d}t\\&-\iint\tfrac{1}{2}\,h^2\big(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)\,\boldsymbol{\nabla}_2\delta\Phi\cdot\boldsymbol{n}_b\,\rho\,\mathrm{d}s\,\mathrm{d}t\,.\end{aligned}\right.\tag{E.13}$$

Taking into account the shallow water scaling (5.22), if we only retain terms of order one in (5.19), we obtain

$$\boldsymbol{u}_2=\boldsymbol{\nabla}_2\Phi+\begin{bmatrix}\Omega_3\left(y+d_2\right)-\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\\-\Omega_3\left(x+d_1\right)-\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\end{bmatrix}\,.\tag{E.14}$$

Hence, from the boundary conditions $u\left(0,y,t\right)=u\left(L_1,y,t\right)=0$ and $v\left(x,0,t\right)=v\left(x,L_2,t\right)=0$ and the approximation (E.14), it can be inferred that $\boldsymbol{\nabla}_2\delta\Phi\cdot\boldsymbol{n}_b=0$ along the boundaries, i.e. on the wetted surface $\mathsf{S}\left(t\right)$, and so the second term on the right-hand side of (E.12), and the last line in (E.13) vanishes. Now, substituting (E.9), (E.10), (E.11), (E.12) and (E.13) in (E.1), the variational principle (E.1) for the variation $\delta\Phi$ takes the form

$$\left\{\begin{aligned}\delta\mathscr{L}_{\mathsf{GW}}=&\iiint\Big(h_t+\boldsymbol{\nabla}_2\cdot\left(h\,\boldsymbol{\nabla}_2\Phi\right)+\nabla_2^2\left(\frac{1}{3}h^3\nabla_2^2\Phi\right)\\&-\big(\Omega_1\left(y+d_2\right)-\Omega_2\left(x+d_1\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_3\big)\big(\boldsymbol{\nabla}_2h\cdot\boldsymbol{\nabla}_2h+h\nabla_2^2h\big)\\&+\left(\Omega_2\left(h-d_3\right)+\Omega_3\left(y+d_2\right)-\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_1\right)h_x\\&-\left(\Omega_3\left(x+d_1\right)+\Omega_1\left(h-d_3\right)+\dot{\boldsymbol{q}}\cdot\boldsymbol{Q}\boldsymbol{e}_2\right)h_y\Big)\,\delta\Phi\,\rho\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}t+\end{aligned}\right.$$

$$
\begin{cases}
+\iint \Bigg( - h\, \boldsymbol{\nabla}_2\Phi - \boldsymbol{\nabla}_2 \left( \frac{1}{3} h^3 \nabla_2^2 \Phi \right) \\
\qquad + h \begin{bmatrix} \Omega_2 \left( \frac{1}{2}h + d_3 \right) - \Omega_3 \left( y + d_2 \right) + \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_1 \\ \Omega_3 \left( x + d_1 \right) - \Omega_1 \left( \frac{1}{2}h + d_3 \right) + \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_2 \end{bmatrix} \\
\qquad + h \begin{bmatrix} \big( \Omega_1 \left( y + d_2 \right) - \Omega_2 \left( x + d_1 \right) + \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3 \big) h_x - \frac{1}{2}\Omega_2 h \\ \big( \Omega_1 \left( y + d_2 \right) - \Omega_2 \left( x + d_1 \right) + \dot{\boldsymbol{q}} \cdot \boldsymbol{Q}\boldsymbol{e}_3 \big) h_y + \frac{1}{2}\Omega_1 h \end{bmatrix} \Bigg) \cdot \boldsymbol{n}_b\, \delta\Phi\, \rho\, \mathrm{d}s\, \mathrm{d}t = 0\,.
\end{cases}
\tag{E.15}
$$

From (E.15), it can be concluded that invariance of $\mathscr{L}_{\mathsf{GW}}$ with respect to a variation in the velocity potential $\Phi$ yields the evolution equation (5.26). Moreover, the invariance of $\mathscr{L}_{\mathsf{GW}}$ with respect to a variation in the velocity potential $\Phi$ along the wetted surface $\mathsf{S}\,(t)$ recovers the rigid-wall boundary conditions (5.27).

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