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# Hodge–Witt decomposition of relative crystalline cohomology

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#### Abstract

For a smooth and proper scheme over an Artinian local ring with ordinary reduction over the perfect residue field, we prove — under some general assumptions that the relative de Rham–Witt spectral sequence degenerates and the relative crystalline cohomology, equipped with its display structure arising from the Nygaard complexes, has a Hodge–Witt decomposition into a direct sum of (suitably Tate-Twisted) multiplicative displays. As examples, our main results include the cases of abelian schemes, complete intersections, surfaces, varieties of K3 type and some Calabi–Yau *n*-folds.

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## **1** | INTRODUCTION

We fix an Artinian local ring *R* with perfect residue field *k* of characteristic p > 0. Let *X* be a smooth proper scheme over Spec *R*. Under some general assumptions on *X*, we proved in [21] and [11] that the crystalline cohomology  $H^i_{cris}(X/W(R))$  is equipped with the structure of a higher display, with divided Frobenius maps arising from canonical maps on the Nygaard filtration of the relative de Rham–Witt complex  $W\Omega^*_{X/R}$ . Moreover, if the closed fibre  $X_k$  of *X* is an ordinary K3 surface, we proved in [22] that the relative de Rham–Witt spectral sequence

$$E_1^{i,j} = H^j(X, W\Omega^i_{X/R}) \Rightarrow \mathbb{H}^{i+j}(X, W\Omega^{\bullet}_{X/R})$$
(1.0.1)

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degenerates at  $E_1$ , giving rise to a Hodge–Witt decomposition of  $H^2_{cris}(X/W(R))$ , with its display structure, into a direct sum of displays associated to the formal Brauer group, its twisted dual and the étale part of the extended Brauer group.

In this paper, we extend this result and produce new examples of the Hodge–Witt decomposition of relative crystalline cohomology. In the following, let X be a smooth proper scheme over Spec R of relative dimension d satisfying the following assumption.

(A) The closed fibre  $X_k$  has a smooth versal deformation space S, the relative Hodge groups  $R^j f_* \Omega^i_{X/S}$  of the versal family  $f : \mathcal{X} \to S$  are locally free for each i, j, and the relative Hodge-de Rham spectral sequence

$$E_1^{i,j} = R^j f_* \Omega^i_{\mathcal{X}/\mathcal{S}} \Rightarrow \mathbb{R}^{i+j} f_* \Omega^{\bullet}_{\mathcal{X}/\mathcal{S}}$$

degenerates at  $E_1$ .

Remark 1.1. We make some remarks about this condition.

- (a) The assumption (A) ensures that the relative Hodge filtration of the versal family is a filtration by locally direct factors, and commutes with arbitrary base change  $S \rightarrow T$  [17, Corollary 8.3].
- (b) Assumption (A) is satisfied in the case of abelian schemes by [28, Theorem 2.2.1] and [7, 2.1.1], for K3 surfaces by [8, Corollaire 1.2 and Proposition 2.2], and for smooth relative complete intersections by [6, Theorem 1.5] and [13, Theorem 9.4].
- (c) Note that assumption (A) ensures that the deformation X/Spec R admits a compatible system of smooth liftings X<sub>n</sub>/Spec W<sub>n</sub>(R), n ∈ N. Indeed, S = Spf A where A = W(k) [[T<sub>1</sub>,...,T<sub>r</sub>]]. There is a canonical map A → W(A) mapping the T<sub>i</sub> to their Teichmüller representatives in W(A). A homomorphism A → R which produces X as the base change of X induces a homomorphism W(A) → W(R), hence A → R factors through the composition A → W(A) → W(R) → W<sub>n</sub>(R) for each n ∈ N. Then the X<sub>n</sub> := X ×<sub>S</sub> Spec W<sub>n</sub>(R) form a compatible system of smooth liftings of X = X ×<sub>S</sub> Spec R.

By base change, any compatible system of smooth liftings  $X_n/\text{Spec }W_n(R)$ ,  $n \in \mathbb{N}$ , of X/Spec R satisfies the properties:

- (P1) the cohomology  $H^{j}(X_{n}, \Omega^{i}_{X_{n}/W_{n}(R)})$  is a free  $W_{n}(R)$ -module for each i, j;
- (P2) the Hodge-de Rham spectral sequence

$$E_1^{i,j} = H^j(X_n, \Omega^i_{X_n/W_n(R)}) \Rightarrow \mathbb{H}^{i+j}(X_n, \Omega^{\bullet}_{X_n/W_n(R)}) = H^{i+j}_{\mathrm{dR}}(X_n/W_n(R))$$

degenerates at  $E_1$ ;

(d) the assumption (A) implies a Hodge decomposition of the de Rham cohomology of the liftings  $X_n$ , which depends on the choice of splittings of the canonical surjections from the cohomology of the truncated de Rham complex to the Hodge cohomology groups. In Theorem 1.2 and Theorem 1.4, we choose such a splitting and prove a Hodge–Witt decomposition depending on this choice. In Theorem 1.6, we get a canonical Hodge–Witt decomposition since there are canonical splittings coming from the conjugate spectral sequence.

We say that X admits a Hodge–Witt decomposition of  $H^s_{cris}(X/W(R))$  as displays if the relative Hodge–Witt spectral sequence (1.0.1) degenerates at  $E_1$  and if there exists a direct sum decomposition

$$H^{s}_{\operatorname{cris}}(X/W(R)) = \bigoplus_{i+j=s} H^{i}(X, W\Omega^{j}_{X/R})$$
(1.1.1)

on which  $H^s_{cris}(X/W(R))$  is equipped with the display structure arising from the Nygaard complexes, and each  $H^i(X, W\Omega^j_{X/R})$  is equipped with the (-j)-fold Tate twist of a multiplicative display structure induced by the Frobenius *F* on  $W\Omega^j_{X/R}$  such that

$$W(R) \otimes_{F, W(R)} H^{i}(X, W\Omega^{j}_{X/R}) \to H^{i}(X, W\Omega^{j}_{X/R})$$
$$x \otimes m \mapsto xFm$$

is an isomorphism. Note that the crystalline Frobenius on  $H^s_{cris}(X/W(R))$  induces the map  $p^j F$  on  $H^i(X, W\Omega^j_{X/R})$ . In the case R = k a perfect field, the above is the Hodge–Witt or slope decomposition.

Recall from [4, Definition 7.2] and [16, Definition 4.12] that a complete variety *X* over a perfect field *k* of characteristic p > 0 is called ordinary if  $H^i(X, B\Omega_{X/k}^j) = 0$  for all  $i \ge 0$  and j > 0, where  $B\Omega_{X/k}^j = d\Omega_{X/k}^{j-1}$ . This coincides with the usual definition of ordinary for abelian varieties [9, Lemma 6.2], and coincides with having height one formal Brauer group for K3 surfaces [27, Lemma 1.3]. By [16, Théorème 4.13], ordinary varieties admit a slope decomposition. In this article, we consider the generalisation of this to families over *R*. We prove the following:

**Theorem 1.2.** Let  $p \ge 3$ . Suppose that k is algebraically closed. Let X be a smooth projective surface over Spec R satisfying assumption (A) and such that the closed fibre  $X_k$  is ordinary. Suppose moreover that the relative Picard scheme  $\operatorname{Pic}_{X/R}^0$  is smooth (and hence is an abelian scheme). Then X admits a Hodge–Witt decomposition of  $H_{\operatorname{cris}}^{8}(X/W(R))$  for all  $0 \le s \le 4$ .

*Remark* 1.3. For s = 1, the decomposition coincides with the decomposition of the display associated to the *p*-divisible group Pic(X)(p) over *R* of the Picard scheme into connected and étale part (see (4.4.2) and the subsequent discussion). The case s = 3 is obtained by Poincaré/Cartier duality and yields the decomposition of the display associated to the *p*-divisible group of the Albanese scheme Alb<sub>X</sub>. For s = 2, we get the same decomposition as for ordinary K3 surfaces.

The main new examples, apart from abelian surfaces (which are also covered in the next theorem), are smooth complete intersections in a projective space over *R* of dimension 2. They are generically ordinary by [15, Théorème 0.1]. It is well known that the deformations of a complete intersection are unobstructed (see, for example, [13, Theorem 9.4]), and the other conditions in assumption (A) follow by [6, Theorem 1.5]. Moreover for smooth complete intersections the Picard scheme vanishes.

Note that examples where  $\operatorname{Pic}_{X/R}^{0}$  is smooth are provided by [18, Proposition 9.5.19]; if the closed fibre  $X_k$  satisfies in addition that  $H^2(X_k, O_{X_k}) = 0$ , then the relative Picard scheme is smooth. We also point our that under our general assumptions on X the closed fibre of the relative Picard scheme is reduced and therefore an abelian variety. Indeed,  $H^2_{\operatorname{cris}}(X_k/W(k))$  is torsion-free and we conclude by [26, Proposition 2.1].

**Theorem 1.4.** Let A be an abelian scheme over Spec R with ordinary closed fibre, such that dim A = d < p. Then A admits a Hodge–Witt decomposition of  $H^s_{cris}(A/W(R))$  in all degrees  $0 \le s \le 2d$ .

*Remark* 1.5.  $H^s_{\text{cris}}(A/W(R))$  is equipped with the exterior power structure  $\bigwedge^s H^1_{\text{cris}}(A/W(R))$  of the display  $H^1_{\text{cris}}(A/W(R))$ , which is a direct sum

$$H^{1}_{\text{cris}}(A/W(R)) = H^{1}(A, W\mathcal{O}_{A}) \oplus H^{0}(A, W\Omega^{1}_{A/R})$$

of displays according to the connected, respectively, étale part of the *p*-divisible group of the Picard scheme. So we shall derive a canonical isomorphism

$$\bigwedge^{i} H^{1}(A, W\mathcal{O}_{A}) \otimes \bigwedge^{j} H^{0}(A, W\Omega^{1}_{A/R}) \xrightarrow{\simeq} H^{i}(A, W\Omega^{j}_{A/R})$$

of multiplicative displays.

In the final section, we shall prove our most general result on Hodge–Witt decompositions under deformation, with the restriction that *k* is algebraically closed. By an *n*-fold we mean a smooth and proper scheme of dimension *n*, defined over a field. We consider a smooth and proper family of *n*-folds  $f : \mathcal{X} \to S$  over a smooth base  $S \to \text{Spf } W(k)$ , with n < p. Suppose that the following assumption holds.

(B) The relative Hodge sheaves  $R^j f_* \Omega^i_{X/S}$  are locally free for each *i*, *j*, and the relative Hodge–de Rham spectral sequence

$$E_1^{i,j} = R^j f_* \Omega^i_{\mathcal{X}/S} \Rightarrow \mathbb{R}^{i+j} f_* \Omega^{\bullet}_{\mathcal{X}/S}$$

degenerates at  $E_1$ .

Given such a family  $f : \mathcal{X} \to S$  satisfying assumption (B), we prove the following.

**Theorem 1.6.** Suppose that Spec  $k \to S$  is a k-point of S such that the fibre  $X_k := \mathcal{X} \times_S$  Spec k over this point is ordinary. Then for any Artinian local ring R with residue field k, if Spec  $k \to S$  lifts to a morphism Spec  $R \to S$ 



the deformation  $X := \mathcal{X} \times_S \operatorname{Spec} R$  of  $X_k$  admits a Hodge–Witt decomposition of  $H^s_{\operatorname{cris}}(X/W(R))$  as displays in all degrees  $0 \leq s \leq 2n$ .

#### Remark 1.7.

(a) The same argument as in Remark 1.1(c) and (d) yields a compatible system of smooth liftings  $X_n/\text{Spec }W_n(R), n \in \mathbb{N}$ , such that each  $X_n$  satisfies properties (P1) and (P2).

(b) Note that Theorem 1.6 implies results of the same type as Theorems 1.2 and 1.4, by applying Theorem 1.6 in the case that  $f : \mathcal{X} \to S$  is the versal family of  $X_k$ . We have decided to include the separate proofs of Theorems 1.2 and 1.4 because the techniques are different and elucidate different aspects of the theory; for example, the connection with formal groups.

Finally, let us highlight some important examples for which the theorem is applicable, focussing first on the case of Calabi–Yau *n*-folds.

- Let *k* be an algebraically closed field with char(*k*) = p > 0 and consider the Fermat Calabi–Yau *n*-fold  $X = X_{n+2}^n(p)$  given by

$$X_1^{n+2} + X_2^{n+2} + \dots + X_{n+2}^{n+2} = 0$$

in  $\mathbb{P}_{R}^{n+1}$ , such that  $p \equiv 1 \mod n+2$ . In this case,  $X_k$  is ordinary by [29] (see also [30]). Since  $X_k$  is a hypersurface, its deformations are unobstructed, so the deformation space S of  $X_k$  is smooth and we consider the versal family  $f : \mathcal{X} \to S$ . The assumption (B) is satisfied for families of hypersurfaces (indeed they are satisfied for smooth relative complete intersections more generally) [6, Theorem 1.5].

- Another important example is provided by the Dwork pencil of Calabi–Yau *n*-folds. Let  $X = [X_1 : X_2 : ... : X_{n+2}]$  be the homogeneous coordinates of  $\mathbb{P}_R^{n+1}$ . Then the Dwork pencil is the one-parameter family  $V_t$  of Calabi–Yau hypersurfaces in  $\mathbb{P}_R^{n+1}$  over  $t \in \mathbb{P}_R^1$  defined by  $P_t(X) = 0$  where

$$P_t(X) = X_1^{n+2} + X_2^{n+2} + \dots + X_{n+2}^{n+2} - (n+2)tX_1X_2 \cdots X_{n+2}$$

We can consider this family over any ring R such that  $n + 2 \in R^*$ . Let R be Artinian local with algebraically closed residue field k such that char k > n + 2. When we specialise the family to  $\mathbb{P}_k^{n+1}$  under the base change  $R \to k$ , it is shown in [31, Theorem 2.2] that the Dwork family is generically ordinary. Hence we can choose  $t \in R$  such that the variety  $P_{t_0}(X) = 0$  is ordinary, where  $t_0$  is the image of t in k. We may argue in the same way as in first example to see that assumption (B) is satisfied for the versal family of  $X_k$ .

The final example that we point out as of particular interest is the case of varieties of K3 type. Recall from [22, Definition 22] that a smooth and proper scheme  $X_k$  over Spec k of dimension 2d is of K3 type if the lower four rows of the Hodge diamond are of the form

where  $h^{i,j} = \dim_k H^j(X_0, \Omega^i_{X_0/k})$ . Moreover, we require that there exists a  $\sigma \in H^0(X_k, \Omega^2_{X_k/k})$ such that  $\sigma^d \in H^0(X_k, \Omega^{2d}_{X_k/k})$  defines an isomorphism  $\mathcal{O}_{X_k} \xrightarrow{\sim} \Omega^{2d}_{X_k/k}$  and a  $\rho \in H^2(X_k, \mathcal{O}_k)$  such that  $\rho^d$  generates  $H^{2d}(X_k, \mathcal{O}_{X_k})$  and the pairing

$$H^{1}(X_{k}, \Omega^{1}_{X_{k}/k}) \times H^{1}(X_{k}, \Omega^{1}_{X_{k}/k}) \to k, \ \omega_{1} \times \omega_{2} \mapsto \int \omega_{1} \omega_{2} \sigma^{d-1} \rho^{d-1}$$

is perfect. Note that  $\sigma$  induces an isomorphism  $\mathcal{T}_{X_k} := \text{Hom}(\Omega^1_{X_k/k}, \mathcal{O}_{X_k}) \simeq \Omega^1_{X_k/k}$ . It is clear that K3 surfaces are varieties of K3 type. Further examples can be constructed by considering the Hilbert scheme  $X^{[d]}$  of d points on a K3 surface X in characteristic zero. Spreading out  $X^{[d]}$  over a scheme S which is flat and of finite type over Spec  $\mathbb{Z}$  and reducing modulo p gives a variety of K3 type over the residue field for almost all primes p [22, p. 484]. In these examples, the odd Betti numbers vanish by [10, Theorem 0.1].

Let  $X_k$  be a variety of K3 type over Spec k. Since  $H^0(X_k, \mathcal{T}_{X_k}) = H^2(X_k, \mathcal{T}_{X_k}) = 0$ ,  $X_k$  has a universal deformation  $f : \mathcal{X} \to S$  where  $S = \operatorname{Spf} W(k)[\![T_1, ..., T_r]\!]$  and  $r = h^{1,1}$ . Suppose moreover that the odd rows in the Hodge diamond of  $X_k$  vanish (that is,  $H^i(X_k, \Omega^j_{X_k/k}) = 0$  for i + j odd). Then  $f : \mathcal{X} \to S$  satisfies assumption (B) by [12, Proposition 7.5.4] (there is no room for non-zero differentials due to the Hodge number condition). We may then use Theorem 1.6 to conclude that if k is algebraically closed,  $X_k$  is ordinary and dim  $X_k < p$ , then the crystalline cohomology (in all degrees) of any deformation of  $X_k$  over an Artinian local ring R with residue field k admits a Hodge–Witt decomposition as displays.

#### 2 | INTRODUCTION TO HIGHER DISPLAYS

In this section, we present the main tools developed in [20], [21], [22], [19] and [11] to impose a display structure on relative crystalline cohomology. For a smooth scheme *X* over a ring *R* on which *p* is nilpotent, we defined in [20] the relative de Rham–Witt complex  $W_n \Omega_{X/R}^*$  as an initial object in the category of *F*-*V*-procomplexes, in particular it is equipped with operators  $F: W_n \Omega_{X/R}^* \to W_{n-1} \Omega_{X/R}^*$  and  $V: W_{n-1} \Omega_{X/R}^* \to W_n \Omega_{X/R}^*$  extending the Frobenius and Verschiebung on the Witt vector sheaf  $W_n \mathcal{O}_X$ , and satisfying the standard relations in Cartier theory.  $W_n \Omega_{X/R}^*$  coincides with Deligne–Illusie's de Rham–Witt complex for R = k a perfect field, and its hypercohomology computes the crystalline cohomology of  $X/W_n(R)$ .

Let X be a proper and smooth scheme over Spec R and let  $I_{R,n} = {}^{V}W_{n-1}(R)$  and  $I_{R} = {}^{V}W(R)$ . Then we consider the following variant of  $W_{n}\Omega_{X/R}^{*}$ , denoted by  $N^{r}W_{n}\Omega_{X/R}^{*}$ , for  $r \ge 0$ :

$$W_{n-1}\Omega^0_{X/R[F]} \xrightarrow{d} W_{n-1}\Omega^1_{X/R[F]} \xrightarrow{d} \cdots \xrightarrow{d} W_{n-1}\Omega^{r-1}_{X/R[F]} \xrightarrow{dV} W_n\Omega^r_{X/R} \xrightarrow{d} \cdots$$

This is a complex of  $W_n(R)$ -modules, where  $W_{n-1}\Omega_{X/R[F]}^i$  for i < r denotes  $W_{n-1}\Omega_{X/R}^i$  considered as a  $W_n(R)$ -module via restriction of scalars along  $W_n(R) \xrightarrow{F} W_{n-1}(R)$ . Let  $P_0 := \mathbb{H}^m(X, W\Omega_{X/R}^{\bullet}) = \lim_{n \to \infty} \mathbb{H}^m(X, W_n\Omega_{X/R}^{\bullet})$  and  $P_r := \lim_{n \to \infty} \mathbb{H}^n(X, N^r W_n\Omega_{X/R}^{\bullet})$ . Then there are maps  $F_r : P_r \to P_0$  induced by corresponding divided Frobenius maps

$$\hat{F}_r : N^r W_n \Omega^{\bullet}_{X/R} \to W_{n-1} \Omega^{\bullet}_{X/R}$$

defined as the identity in degrees  $\langle r \rangle$  and as  $p^i F$  in degree r + i, for  $i \ge 0$ . The standard relations between F, V and d imply that  $\hat{F}_r$  is well defined. There are also maps  $\hat{\iota}_r : N^{r+1} W_n \Omega^{\bullet}_{X/R} \to N^r W_n \Omega^{\bullet}_{X/R}$ ,  $\hat{\alpha}_r : I_{R,n} \otimes N^r W_n \Omega^{\bullet}_{X/R} \to N^{r+1}_n W \Omega^{\bullet}_{X/R}$  given explicitly in [21] and [11] that induce three sets of maps.

- (1)  $\dots \to P_{r+1} \xrightarrow{\iota_r} P_r \xrightarrow{\iota_{r-1}} \dots \xrightarrow{\iota_0} P_0$  a chain of W(R)-module homomorphisms.
- (2) W(R)-module homomorphisms  $\alpha_r : I_R \otimes P_r \to P_{r+1}$ .
- (3) Frobenius-linear maps  $F_r : P_r \to P_0$

satisfying the following.

(I) For  $r \ge 1$ 



commutes and the diagonal map  $I_R \otimes P_r \to P_r$  is multiplication. For r = 0, the composition  $I_R \otimes P_0 \xrightarrow{\alpha_0} P_1 \xrightarrow{\iota_0} P_0$  is multiplication.

(II) For  $r \ge 0$ ,

$$F_{r+1} \circ \alpha_r = \tilde{F}_r : I_R \otimes P_r \to P_0$$
$${}^V \xi \otimes x \mapsto \xi F_r x \,.$$

The above data define the structure of a predisplay  $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$  on  $H^n_{\text{cris}}(X/W(R))$  [21, Definition 2.2]. We note that properties (I) and (II) imply the property (III) of a predisplay:

(III)  $F_r(\iota_r(y)) = pF_{r+1}(y)$ 

(see [21, pp. 155–156]), that is the diagram below is commutative



The predisplays appearing in this paper are separated, that is, the map from  $P_{r+1}$  to the fibre product induced by the above diagram is injective [21, Definition 2.3]. A predisplay is of degree *d* (or a *d*-predisplay) if the maps  $\alpha_r$  are surjective for  $r \ge d$  [21, Definition 2.4]. For example, the predisplay structure on  $H^n_{\text{cris}}(X/W(R))$  given above is a predisplay of degree *n*. Note that a (separated) predisplay of degree *d* is uniquely determined by the data

$$P_0, \ldots, P_d, \iota_0, \ldots, \iota_{d-1}, \alpha_0, \ldots, \alpha_{d-1}, F_0, \ldots, F_d$$

(see [21, pp 156-157]).

Now assume that *X* admits a compatible system of liftings  $X_n/W_n(R)$  satisfying the properties (P1) and (P2) in the introduction, and let  $\mathcal{F}^r \Omega^{\bullet}_{X_n/W_n(R)}$  be the following filtered version of the de Rham complex  $\Omega^{\bullet}_{X_n/W_n(R)}$ :

$$I_{R,n} \otimes \mathcal{O}_{X_n} \xrightarrow{pd} I_{R,n} \otimes \Omega^1_{X_n/W_n(R)} \xrightarrow{pd} \cdots \xrightarrow{pd} I_{R,n} \otimes \Omega^{r-1}_{X_n/W_n(R)} \xrightarrow{d} \Omega^r_{X_n/W_n(R)} \xrightarrow{d} \cdots$$

As one of the main results in [19], used in [22] and [11], we recall

**Theorem 2.1.** For r < p, the complexes  $\mathcal{F}^r \Omega^{\bullet}_{X_n/W_n(R)}$  and  $N^r W_n \Omega^{\bullet}_{X/R}$  are isomorphic in the derived category of  $W_n(R)$ -modules.

One might view this theorem as a filtered version of the comparison between de Rham-Witt cohomology and the de Rham cohomology of a lifting.

Next we point out that the complexes  $\mathcal{F}^r \Omega^{\bullet}_{X_n/W_n(R)}$  possess — under the assumption (A) — very nice properties: the  $E_1$ -hypercohomology spectral sequence associated to  $\mathcal{F}^r \Omega^{\bullet}_{X_n/W_n(R)}$  degenerates at  $E_1$  (compare [21], Proposition 3.2 and the properties following Proposition 3.1). The theorem yields a description of the  $\mathbb{H}^m(X, \mathcal{F}^r \Omega^{\bullet}_{X_n/W_n(R)})$  in terms of de Rham cohomology:

$$P_r \cong I_{R,n}L_0 \oplus I_{R,n}L_1 \oplus \cdots \oplus I_{R,n}L_{r-1} \oplus L_r \oplus \cdots \oplus L_m,$$

where  $L_i := H^{m-i}(X_n, \Omega^i_{X_n/W_n(R)})$ , indeed we have the following lemma.

**Lemma 2.2.** Let X be a proper and smooth scheme over Spec R satisfying assumption (A). Let  $X_n/\text{Spec }W_n(R)$ ,  $n \in \mathbb{N}$ , be a compatible system of smooth and proper liftings of X/R (which exists by Remark 1.1(c)). Then the  $E_1$ -hypercohomology spectral sequence associated to  $\mathcal{F}^r \Omega^{\bullet}_{X_n/W_n(R)}$  degenerates at  $E_1$ . A choice of splitting of the Hodge filtration on  $H^m_{dR}(X_n/W_n(R))$  (see Remark 1.1(d)) induces a direct sum decomposition for  $m \ge r$ 

$$\begin{split} \mathbb{H}^{m}(X_{n},\mathcal{F}^{r}\Omega^{\bullet}_{X_{n}/W_{n}(R)}) \simeq &I_{R,n} \otimes H^{m}(X_{n},\mathcal{O}_{X_{n}}) \oplus I_{R,n} \otimes H^{m-1}(X_{n},\Omega^{1}_{X_{n}/W_{n}(R)}) \oplus \cdots \\ & \oplus I_{R,n} \otimes H^{m-(r-1)}(X_{n},\Omega^{r-1}_{X_{n}/W_{n}(R)}) \oplus H^{m-r}(X_{n},\Omega^{r}_{X_{n}/W_{n}(R)}) \\ & \oplus \cdots \oplus H^{0}(X_{n},\Omega^{m}_{X_{n}/W_{n}(R)}). \end{split}$$

*Proof.* Consider the complex  $\Omega_{p,r,X_n}^{\bullet}$ 

$$\mathcal{O}_{X_n} \xrightarrow{pd} \Omega^1_{X_n/W_n(R)} \xrightarrow{pd} \cdots \xrightarrow{pd} \Omega^{r-1}_{X_n/W_n(R)} \xrightarrow{d} \Omega^r_{X_n/W_n(R)} \xrightarrow{d} \cdots \,.$$

By [21, Propositions 3.1 & 3.2], the hypercohomology spectral sequence of this complex degenerates. For each  $j \ge 0$ , we claim that we can choose splittings of the canonical map  $\mathbb{H}^m(X_n, \Omega_{p,r,X_n}^{\ge j}) \twoheadrightarrow H^{m-j}(X_n, \Omega_{X_n/W_n(R)}^j)$  which are compatible with the chosen splittings  $\mathbb{H}^m(X_n, \Omega_{X_n/W_n(R)}^{\ge j}) \twoheadrightarrow H^{m-j}(X_n, \Omega_{X_n/W_n(R)}^j)$  of the Hodge filtration under the map

 $\mathbb{H}^m(X_n,\Omega^{\scriptscriptstyle\bullet}_{p,r,X_n})\to\mathbb{H}^m(X_n,\Omega^{\scriptscriptstyle\bullet}_{X_n/W_n(R)}) \text{ induced by the morphism of complexes}$ 

We leave the proof of this claim until the end of the proof. Accepting the claim, we get a direct sum decomposition

$$\mathbb{H}^{m}(X_{n},\Omega_{p,r,X_{n}}^{\bullet})\simeq\bigoplus_{j\geqslant 0}H^{m-j}(X_{n},\Omega_{X_{n}/W_{n}(R)}^{j}).$$

We have an exact sequence of complexes

$$\mathcal{F}^{r}\Omega^{\bullet}_{X_{n}/W_{n}(R)} \to \Omega^{\bullet}_{p,r,X_{n}} \to \sigma_{\leq r-1}\Omega^{\bullet}_{p,r,X}.$$

The hypercohomology spectral sequence of the complex  $\sigma_{\leq r-1}\Omega_{p,r,X}^* = \sigma_{\leq r-1}\Omega_{p,r,X_1}^*$  degenerates and leads again to a direct sum decomposition

$$\mathbb{H}^{m}(X,\sigma_{\leqslant r-1}\Omega^{\bullet}_{p,r,X})\simeq \bigoplus_{j=0}^{r-1}H^{m-j}(X,\Omega^{j}_{X/R}).$$

Evidently the above exact sequence remains exact after truncation:

$$\mathcal{F}^{r}\Omega_{X_{n}/W_{n}(R)}^{\geq j} \to \Omega_{p,r,X_{n}}^{\geq j} \to \sigma_{\leq r-1}\Omega_{p,r,X}^{\geq j}.$$

Taking cohomology, we get a short exact sequence

$$0 \to \mathbb{H}^{m}(X_{n}, \mathcal{F}^{r}\Omega_{X_{n}/W_{n}(\mathbb{R})}^{\geq j}) \to \mathbb{H}^{m}(X_{n}, \Omega_{p,r,X_{n}}^{\geq j}) \to \mathbb{H}^{m}(X, \sigma_{\leq r-1}\Omega_{p,r,X}^{\geq j}) \to 0.$$

Indeed,  $\mathbb{H}^m(X_n, \Omega_{p,r,X_n}^{\geq j})$  is a direct sum of Hodge cohomology groups of  $X_n/W_n(R)$  and  $\mathbb{H}^m(X, \sigma_{\leq r-1}\Omega_{p,r,X}^{\geq j})$  is a direct sum of Hodge cohomology groups of X/R. Thus the map  $\mathbb{H}^i(X_n, \Omega_{p,r,X_n}^{\geq j}) \to \mathbb{H}^i(X, \sigma_{\leq r-1}\Omega_{p,r,X}^{\geq j})$  is surjective in each cohomological degree *i* since Hodge cohomology commutes with base change [17, Corollary 8.3].

We have a canonical map

$$I_{R,n}H^{m-j}(X_n,\Omega^j_{X_n/W_n(R)}) \to H^{m-j}(X_n,\Omega^j_{X_n/W_n(R)}) \to \mathbb{H}^m(X_n,\Omega^{\geq j}_{p,r,X_n})$$

induced by the splitting of  $\mathbb{H}^m(X_n, \Omega_{p,r,X_n}^{\geq j}) \to H^{m-j}(X_n, \Omega_{X_n/W_n(R)}^j)$  and the composite with  $\mathbb{H}^m(X_n, \Omega_{p,r,X_n}^{\geq j}) \to \mathbb{H}^m(X_n, \sigma_{\leq r-1}\Omega_{p,r,X}^{\geq j})$  is the zero map. Hence we get an induced map  $I_{R,n}H^{m-j}(X_n, \Omega_{X_n/W_n(R)}^j) \to \mathbb{H}^m(X_n, \mathcal{F}^r\Omega_{X_n/W_n(R)}^{\geq j})$  which splits the surjection  $\mathbb{H}^m(X_n, \mathcal{F}^r\Omega_{X_n/W_n(R)}^{\geq j}) \to I_{R,n}H^{m-j}(X_n, \Omega_{X_n/W_n(R)}^j)$ . The degeneracy of the hypercohomology

spectral sequence associated to  $\mathcal{F}^r \Omega^{\bullet}_{X_n/W_n(R)}$  (again by [21, Propositions 3.1 & 3.2]) then yields the direct sum decomposition for  $\mathbb{H}^m(\mathcal{F}^r \Omega^{\bullet}_{X_n/W_n(R)})$ .

To see the claim that we may choose compatible splittings, consider the situation for the versal family  $\mathcal{X}$  over the versal deformation space Spf A. Consider the morphism of complexes  $\Omega_{p,r,\mathcal{X}}^* \to \tilde{\Omega}_{p,r,\mathcal{X}}^*$  given by

since multiplication by p is injective, this is an isomorphism of complexes.

The hypercohomology spectral sequence of  $\tilde{\Omega}_{p,r,\mathcal{X}}^{\bullet}$  also degenerates under our assumptions (using [21, Propositions 3.1 and 3.2] again). Hence, using the above isomorphism of complexes, to construct compatible splittings for  $\mathbb{H}^m(\mathcal{X}, \Omega_{p,r,\mathcal{X}}^{\geq j}) \twoheadrightarrow H^{m-j}(\mathcal{X}, \Omega_{\mathcal{X}/A}^j)$  and  $\mathbb{H}^m(\mathcal{X}, \Omega_{\mathcal{X}/A}^{\geq j}) \twoheadrightarrow H^{m-j}(\mathcal{X}, \Omega_{\mathcal{X}/A}^j)$  it is equivalent to construct compatible splittings for  $\mathbb{H}^m(\mathcal{X}, \Omega_{\mathcal{X}/A}^{\geq j}) \twoheadrightarrow H^{m-j}(\mathcal{X}, p_{\mathcal{X}/A}^j)$  and  $\mathbb{H}^m(\mathcal{X}, \Omega_{\mathcal{X}/A}^{\geq j}) \twoheadrightarrow H^{m-j}(\mathcal{X}, p^{[r-j-1]}\Omega_{\mathcal{X}/A}^j)$  and  $\mathbb{H}^m(\mathcal{X}, \Omega_{\mathcal{X}/A}^{\geq j}) \twoheadrightarrow H^{m-j}(\mathcal{X}, p_{\mathcal{X}/A}^j)$ , where

$$p^{[r-j-1]} := \begin{cases} p^{r-j-1} & j < r \\ 1 & j \ge r \end{cases}.$$

(Compatible means with respect to the map  $\tilde{\Omega}_{p,r,\mathcal{X}}^{\geq j} \to \Omega_{\mathcal{X}/A}^{\geq j}$  induced by the canonical inclusion.) Since  $\Omega_{\mathcal{X}/A}^{j} \cong p^{[r-j-1]}\Omega_{\mathcal{X}/A}^{j}$ , we have

$$H^{m-j}(\mathcal{X}, p^{[r-j-1]}\Omega^j_{\mathcal{X}/A}) = p^{[r-j-1]}H^{m-j}(\mathcal{X}, \Omega^j_{\mathcal{X}/A}).$$

Choose a splitting  $s_1 : H^{m-j}(\mathcal{X}, \Omega^j_{\mathcal{X}/A}) \to \mathbb{H}^m(\mathcal{X}, \Omega^{\geq j}_{\mathcal{X}/A})$ . Then the image of the injective map

$$s_0: p^{[r-j-1]}H^{m-j}(\mathcal{X}, \Omega^j_{\mathcal{X}/A}) = H^{m-j}(\mathcal{X}, p^{[r-j-1]}\Omega^j_{\mathcal{X}/A}) \to H^{m-j}(\mathcal{X}, \Omega^j_{\mathcal{X}/A}) \xrightarrow{s_1} \mathbb{H}^m(\mathcal{X}, \Omega^{\geq j}_{\mathcal{X}/A})$$

is  $p^{[r-j-1]}$ im $(s_1)$ . The map  $\mathbb{H}^*(\mathcal{X}, \tilde{\Omega}_{p,r,\mathcal{X}}^{\geq j}) \to \mathbb{H}^*(\mathcal{X}, \Omega_{\mathcal{X}/A}^{\geq j})$  induced by  $\tilde{\Omega}_{p,r,\mathcal{X}}^{\geq j} \to \Omega_{\mathcal{X}/A}^{\geq j}$  is injective because it is injective on the level of Hodge cohomology groups and the hypercohomology spectral sequences degenerate. Observe that

$$\mathbb{H}^{*}(\mathcal{X}, \Omega_{\mathcal{X}/A}^{\geq j})/\mathbb{H}^{*}(\mathcal{X}, \tilde{\Omega}_{p,r,\mathcal{X}}^{\geq j}) \cong \mathbb{H}^{*}\left(\Omega_{\mathcal{X}/A}^{\geq j}/\tilde{\Omega}_{p,r,\mathcal{X}}^{\geq j}\right)$$

is killed by  $p^{[r-j-1]}$  because the cohomology of all entries of the quotient complex is killed by  $p^{[r-j-1]}$ . Hence the image of  $p^{[r-j-1]}im(s_1)$  in this cohomology vanishes, so the injective map  $s_0$  factors uniquely through  $\mathbb{H}^{m-j}(\mathcal{X}, \tilde{\Omega}_{p,r,\mathcal{X}}^{\geq j})$  and defines a splitting which is compatible with  $s_1$ , by definition.

Hence we have a compatible pair of splittings, also denoted  $s_0$  and  $s_1$ , for the surjections  $\mathbb{H}^m(\mathcal{X}, \Omega_{p,r,\mathcal{X}}^{\geq j}) \twoheadrightarrow H^{m-j}(\mathcal{X}, \Omega_{\mathcal{X}/A}^j)$  and  $\mathbb{H}^m(\mathcal{X}, \Omega_{\mathcal{X}/A}^{\geq j}) \twoheadrightarrow H^{m-j}(\mathcal{X}, \Omega_{\mathcal{X}/A}^j)$ .

Now consider a map  $A \to W_n(R)$  which defines  $X_n$  as a base change from the versal family  $\mathcal{X}$ . Since Hodge and de Rham cohomology commute with base change, we get a compatible pair of splittings for  $\mathbb{H}^m(X_n, \Omega_{p,r,X_n}^{\geq j}) \twoheadrightarrow H^{m-j}(X_n, \Omega_{X_n/W_n(R)}^j)$  and  $\mathbb{H}^m(X_n, \Omega_{X_n/W_n(R)}^{\geq j}) \twoheadrightarrow H^{m-j}(X_n, \Omega_{X_n/W_n(R)}^j)$ , as desired.  $\Box$ 

*Remark* 2.3. Note that we did not use the full strength of the hypotheses in Lemma 2.2. Rather, the essential assumption is that the compatible system of smooth proper liftings  $X_n/W_n(R)$  arises as the base change of a smooth proper morphism of formal schemes  $\mathcal{X} \to S$  along a compatible system of morphisms Spec  $W_n(R) \to S$  ( $n \in \mathbb{N}$ ) that the Hodge and de Rham cohomology of the  $X_n/W_n(R)$  are the base change of the Hodge and de Rham cohomology of  $\mathcal{X}/S$  along these morphisms Spec  $W_n(R) \to S$ , and that multiplication by p is injective on S.

Now pass to the projective limit. One defines Frobenius-linear maps  $\Phi_r : L_r \to P_0$  by  $\Phi_r := F_r|_{L_r}$ , where  $L_r = H^{m-r}(\mathcal{X}, \Omega^r_{\mathcal{X}/W(R)})$  and  $\mathcal{X} = \underset{n}{\lim} X_n$ . It is shown in [11, Theorem 1.1 a)], [21, Theorem 5.7], that for m < p, the map

$$\bigoplus_{i=0}^{m} \Phi_i \ : \ P_0 = \bigoplus_{i=0}^{m} L_i \to P_0 = \bigoplus_{i=0}^{m} L_i$$

is a Frobenius-linear isomorphism, and hence we have

**Theorem 2.4.** Let m < p. The predisplay  $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$  is a display; it is isomorphic to a display of degree m given by standard data (see [21, p. 149] and [11, Appendix]).

*Remark* 2.5. In order to see that the maps  $\iota_r$  and  $\alpha_r$  defined via the Nygaard complexes agree with the maps defined on the standard data (diagrams (12) and (13) in [21, p. 158]), one may assume that  $X_n$  is a Witt-lift of X (Witt-lifts always exist locally. In the absence of a global Witt-lift, one uses a simplicial argument as in the proof of [19, Theorem 1.2]). Then the morphism of complexes defining the quasi-isomorphism  $\mathcal{F}^r \Omega^*_{X_n/W_n(R)} \to N^r W_n \Omega^*_{X/R}$  is explicitly given in [21, Corollary 4.3]. It is then easy to write down the morphisms of complexes  $\mathcal{F}^{r+1}\Omega^*_{X_n/W_n(R)} \to \mathcal{F}^r \Omega^*_{X_n/W_n(R)}$ and  $I_{R,n} \otimes \mathcal{F}^r \Omega^*_{X_n/W_n(R)} \to \mathcal{F}^{r+1}\Omega^*_{X_n/W_n(R)}$  which correspond, respectively, to  $\hat{\iota}_r$  and  $\hat{\alpha}_r$ .

In our results on the Hodge–Witt decomposition, it turns out that the display defined on  $H^n_{cris}(X/W(R))$  is a direct sum of twisted multiplicative displays. We recall the definitions.

A 3*n*-display  $(P, Q, F, V^{-1})$  as defined in [32, Definition 1] gives rise to a (pre-)display of degree 1 with  $P_0 = P$ ,  $P_1 = Q$ ,  $F_0 = F$  and  $F_1 = V^{-1}$ . (Note that, by definition,  $I_R P \subset Q$ , there exists a direct sum decomposition of W(R)-modules  $P = L \oplus T$  with  $Q = L \oplus I_R T$  and  $V^{-1}$  is an <sup>*F*</sup>-linear isomorphism).

In the 'degenerate' case  $(P, Q, F_0, F_1)$  with  $Q = I_R P$  and  $F_1$  bijective, we call  $(P, Q, F_0, F_1)$  a multiplicative display (see [24, § 6]). A special example is the unit display  $P_0 = W(R)$ ,  $P_1 = I_R$ ,  $F_0 = F$  the Frobenius on W(R) and  $F_1 = V^{-1}$ . A multiplicative display (and hence a unit display) has degree 0. We can extend a multiplicative display to a display  $(P_i, \iota_i, \alpha_i, F_i)$  by setting

$$P_i = \begin{cases} P & \text{for } i = 0\\ I_R P & \text{for } i \ge 1 \end{cases},$$

$$\iota_{i} = \begin{cases} I_{R}P \hookrightarrow P & \text{for } i = 1\\ I_{R}P \xrightarrow{p} I_{R}P & \text{for } i \ge 2, \end{cases}$$
$$\alpha_{i} = \begin{cases} I_{R}P \xrightarrow{\text{id}} I_{R}P & \text{for } i \ge 2, \\ I_{R} \otimes I_{R}P \rightarrow I_{R}P; \ ^{V}\xi \otimes ^{V}\mu x \mapsto ^{V}(\xi\mu)x & \text{for } i \ge 1, \end{cases}$$

and

 $F_i = F_1$  for all  $i \ge 1$ .

For  $i \ge 1$ ,  $\alpha_i$  is called Verjüngung [21, pp 155-156]. Since the Verjüngung is surjective, an extension of a multiplicative display as above is still of degree 0.

For a multiplicative display  $(P_0, P_1, F, F_1) = P$ , we can define the (-1)-fold Tate twist as a 1-display by the data

$$\mathcal{P}(-1) = (P'_i, \iota'_i, \alpha'_i, F'_i),$$

where for  $i \ge 1$   $P'_i = P_{i-1}$ ,  $\iota'_i = \iota_{i-1}$ ,  $\alpha'_i = \alpha_{i-1}$  and  $F'_i = F_{i-1}$ ,  $P'_0 = P_0 = P'_1$ ,  $F'_0 = pF_0$ ,  $\iota'_0 = \mathrm{id}_{P_0}$ and  $\alpha'_0 = I_R \otimes P_0 \to P_0$  is the multiplication map. The underlying data

$$\mathcal{P}^{\text{et}} = (P = P_0, Q = P_0, pF_0, F_0)$$

form an étale display (for the definition of étale displays see also [24, § 6]). Note that by definition the map  $F'_2$  in  $\mathcal{P}(-1)$  is bijective. We can iterate the construction to define the (-n)-fold Tate twist  $\mathcal{P}(-n)$  for any  $n \ge 0$ . It is a display of degree n.

The Hodge–Witt decomposition of  $H^m_{cris}(X/W(R))$  yields an alternative description as display given by standard data. Since the Frobenius on the de Rham–Witt complex  $W\Omega^*_{X/R}$  (the crystalline Frobenius) is defined by  $p^i F$  on  $W\Omega^i_{X/R}$  where F is the Frobenius on  $W\Omega^i_{X/R}$ , we first prove that  $(H^{m-i}(X, W\Omega^i_{X/R}), F)$  is a multiplicative display, and then consider the (-i)-fold Tate twist of it as in the above construction of Tate twists. In order to identify the display  $H^m_{cris}(X/W(R))$  as a direct sum of displays given by the standard data  $H^{m-i}(X, W\Omega^i_{X/R})$ , we tacitly apply the (-i)-fold Tate twist to  $P_0 = H^{m-i}(X, W\Omega^i_{X/R})$  (a display of degree i). In the cases where we prove the Hodge–Witt decomposition, we will not explicitly mention this extension again, but only state the Hodge–Witt decomposition as a direct sum decomposition of Tate-twisted multiplicative displays.

## 3 | AN INDUCTION ARGUMENT FOR COMPARING HODGE WITH HODGE-WITT COHOMOLOGY

In this section, we will prepare an inductive argument used in the proofs of the main theorems which allows — under the assumptions of those theorems — a construction of a comparison map between Hodge and Hodge–Witt cohomology, and to derive the Hodge–Witt decomposition from the Hodge decomposition. As in Lemma 2.2, consider a smooth proper scheme X/Spec R satisfying assumption (A) (or the weaker but more complicated to state condition in Remark 2.3), and we consider a compatible system  $X_n/\text{Spec }W_n(R)$  of smooth liftings of X. Fix splittings  $H^i(X_{\bullet}, \Omega^j_{X_{\bullet}/W_{\bullet}(R)}) \to \mathbb{H}^{i+j}(X_{\bullet}, \Omega^{\geq j}_{X_{\bullet}/W_{\bullet}(R)})$  of the Hodge filtration, and corresponding

splittings  $I_R H^i(X_{\bullet}, \Omega^j_{X_{\bullet}/W_{\bullet}(R)}) \to \mathbb{H}^{i+j}(X_{\bullet}, \mathcal{F}^r \Omega^{\geq j}_{X_{\bullet}/W_{\bullet}(R)})$  as in Section 2. For j = 0, we consider the composite maps

$$\beta_0 : H^i(X_{\bullet}, \mathcal{O}_{X_{\bullet}}) \to H^i_{\mathrm{dR}}(X_{\bullet}/W_{\bullet}(R)) \cong \mathbb{H}^i(X, W_{\bullet}\Omega^{\bullet}_{X/R}) \to H^i(X, W_{\bullet}\mathcal{O}_X)$$
(3.0.1)

and

$${}^{F}\beta_{0}: I_{R}H^{i}(X_{\bullet}, \mathcal{O}_{X_{\bullet}}) \to \mathbb{H}^{i}(\mathcal{F}^{1}\Omega^{\bullet}_{X_{\bullet}/W_{\bullet}(R)}) \cong \mathbb{H}^{i}(X, N^{1}W_{\bullet}\Omega^{\bullet}_{X/R}) \to H^{i}(X, W_{\bullet}\mathcal{O}_{X}).$$
(3.0.2)

We will see below that  $V \circ {}^F \beta_0$  is the restriction of  $\beta_0$  to  $I_R H^i(X_{\bullet}, \mathcal{O}_{X_{\bullet}})$ , and in particular  $\beta_0(I_R H^i(X_{\bullet}, \mathcal{O}_{X_{\bullet}})) \subseteq V H^i(X, W_{\bullet} \mathcal{O}_X)$  (just apply the proof starting from formula (3.0.9) to the case r = 0 where the induction hypothesis is not yet needed). In the theorems, it turns out that  $\beta_0$  is an isomorphism of  $W_{\bullet}(R)$ -modules,  ${}^F \beta_0$  is bijective and  $\beta_0(I_R H^i(X_{\bullet}, \mathcal{O}_{X_{\bullet}})) = V H^i(X, W_{\bullet} \mathcal{O}_X)$ . This property will be the start of the induction. Now we formulate the induction hypothesis:

Let r > 0. For all  $s < r, i \ge 0$ , we consider the composite map

$$\mathbb{H}^{i}(X_{\bullet}, \sigma_{\leq s}\Omega^{\bullet}_{X_{\bullet}/W_{\bullet}(R)}) \to H^{i}_{\mathrm{dR}}(X_{\bullet}/W_{\bullet}(R)) \cong \mathbb{H}^{i}(X, W_{\bullet}\Omega^{\bullet}_{X/R}) \to \mathbb{H}^{i}(X, \sigma_{\leq s}W_{\bullet}\Omega^{\bullet}_{X/R})$$

where the first map is induced by the splitting of the Hodge filtration. We assume that these maps are isomorphisms of  $W_{\bullet}(R)$ -modules for all  $s < r, i \ge 0$ . In particular, we have isomorphisms

$$\beta_{s}: H^{i-s}(X_{\bullet}, \Omega^{s}_{X_{\bullet}/W_{\bullet}(R)}) \to H^{i-s}(X, W_{\bullet}\Omega^{s}_{X/R})$$
(3.0.3)

for all *s* < *r*, *i*  $\ge$  *s*. Moreover, we assume that the composite maps for *i*  $\ge$  0

$$\mathbb{H}^{i}(X_{\bullet}, \sigma_{\leqslant s}\mathcal{F}^{r}\Omega_{X_{\bullet}/W_{\bullet}(R)}^{\bullet}) \to \mathbb{H}^{i}(X_{\bullet}, \mathcal{F}^{r}\Omega_{X_{\bullet}/W_{\bullet}(R)}^{\bullet}) \cong \mathbb{H}^{i}(X, N^{r}W_{\bullet}\Omega_{X/R}^{\bullet}) \to \mathbb{H}^{i}(X, \sigma_{\leqslant s}N^{r}W_{\bullet}\Omega_{X/R}^{\bullet})$$

are isomorphisms for all s < r,  $i \ge 0$ , where the first map arises again from the splitting of the Hodge filtration on  $\mathbb{H}^{i}(X_{\bullet}, \mathcal{F}^{r}\Omega^{\bullet}_{X_{\bullet}/W(R)})$ . In particular, the induced maps  $(i \ge s)$ 

$${}^{F}\beta_{s}: I_{R}H^{i-s}(X_{\bullet}, \Omega^{s}_{X_{\bullet}/W_{\bullet}(R)}) \to H^{i-s}(X, W_{\bullet}\Omega^{s}_{X/R})$$

$$(3.0.4)$$

are bijective.

From the induction hypothesis, we can construct the map  $\beta_r$  as the composite map

$$\beta_r : H^{i-r}(X_{\bullet}, \Omega^r_{X_{\bullet}/W_{\bullet}(R)}) \to H^i_{\mathrm{dR}}(X_{\bullet}/W_{\bullet}(R)) \cong \mathbb{H}^i(X, W_{\bullet}\Omega^{\bullet}_{X/R})$$

$$\to \mathbb{H}^i(X, W_{\bullet}\Omega^{\geqslant r}_{X/R}) \to H^{i-r}(X, W_{\bullet}\Omega^r_{X/R}),$$
(3.0.5)

where the first maps come from the splitting of the Hodge filtration, and the second map is induced by the splitting of  $\mathbb{H}^{i}(X, W_{\bullet}\Omega^{\bullet}_{X/R}) \to \mathbb{H}^{i}(X, \sigma_{\leq r-1}W_{\bullet}\Omega^{\bullet}_{X/R})$  which exists by our induction hypothesis. Similarly, we construct the map  ${}^{F}\beta_{r}$  from the induction hypothesis as the composite map

$${}^{F}\beta_{r}: I_{R}H^{i-r}(X_{\bullet}, \Omega^{r}_{X_{\bullet}/W_{\bullet}(R)}) \to \mathbb{H}^{i}(X_{\bullet}, \mathcal{F}^{r+1}\Omega^{\bullet}_{X_{\bullet}/W_{\bullet}(R)}) \cong \mathbb{H}^{i}(X, N^{r+1}W_{\bullet}\Omega^{\bullet}_{X/R})$$

$$\xrightarrow{\kappa} \mathbb{H}^{i}(X, N^{r+1}W_{\bullet}\Omega^{\geq r}_{X/R}) \to H^{i-r}(X, W_{\bullet}\Omega^{r}_{X/R}),$$
(3.0.6)

where the first map comes from the splitting of the Hodge filtration, and the map  $\kappa$  is induced by the splitting of  $\mathbb{H}^{i}(X, N^{r}W_{\bullet}\Omega^{\bullet}_{X/R}) \to \mathbb{H}^{i}(X, \sigma_{\leq r-1}N^{r}W_{\bullet}\Omega^{\bullet}_{X/R})$ , which exists by the induction hypothesis, composed with the canonical map  $\mathbb{H}^{i}(X, N^{r}W_{\bullet}\Omega^{\bullet}_{X/R}) \to \mathbb{H}^{i}(X, N^{r+1}W_{\bullet}\Omega^{\bullet}_{X/R})$  given on the level of complexes by the identity in degrees  $\langle r, by F$  in degree r (here we use FdV = d), and multiplication by V(1) in degrees  $\rangle r$  (here we use that VF = V(1)).

We claim that  $V \circ {}^F \beta_r$  is the restriction of  $\beta_r$  to  $I_R H^{i-r}(X_{\bullet}, \Omega^r_{X_{\bullet}/W_{\bullet}(R)})$ . To see this, we consider the situation at the beginning of the proof of [19, Theorem 2.1]: assume there exists a compatible system of closed embeddings  $i_n : X_n \hookrightarrow Z_n$  such that  $Z_n$  is a smooth Witt lift of  $Z = Z_n \times_{W_n(R)} R$ . Let  $D_n$  be the PD-envelope of the embedding  $i_n$  and let  $\mathcal{J}_n$  be the divided power ideal. By [3, Theorem 7.2], we have quasi-isomorphisms

$$\Omega_{X_n/W_n(R)}^{\geq r} \simeq \left( \mathcal{J}_n^{[r]} \to \mathcal{J}_n^{[r-1]} \Omega_{D_n/W_n(R)}^1 \to \cdots \to \Omega_{D_n/W_n(R)}^r \xrightarrow{d} \cdots \right).$$

Since all entries are locally free  $W_n(R)$ -modules, we get, after tensoring with R, the corresponding quasi-isomorphisms for  $X = X_1$ . Combining both quasi-isomorphisms, we obtain a quasi-isomorphism

$$I_{R,n}\Omega_{X_n/W_n(R)}^{\geq r} := \left(I_{R,n}\Omega_{X_n/W_n(R)}^r \xrightarrow{d} I_{R,n}\Omega_{X_n/W_n(R)}^{r+1} \xrightarrow{d} \cdots\right)[-r]$$

$$\simeq \left(I_{R,n}\mathcal{J}_n^{[r]} \to I_{R,n}\mathcal{J}_n^{[r-1]}\Omega_{D_n/W_n(R)}^1 \to \cdots \to I_{R,n}\Omega_{D_n/W_n(R)}^r \xrightarrow{d} \cdots\right).$$
(3.0.7)

The Witt lift comes with a canonical map  $\mathcal{O}_{Z_n} \xrightarrow{\mathfrak{S}} W_n \mathcal{O}_X$  inducing  $\mathcal{O}_{D_n} \xrightarrow{\mathfrak{S}} W_n \mathcal{O}_X$  and likewise  $\Omega_{D_n/W_n(R)}^l \xrightarrow{\mathfrak{S}} W_n \Omega_{X/R}^l$ . We have divided Frobenius maps

$$F'_{k+1}: I_{R,n}\mathcal{J}_n^{[k]}\Omega_{D_n/W_n(R)}^{r-k} \to W_{n-1}\Omega_{X/R}^{r-k}$$

defined by (see notation in [19, p 1869])

$$F'_{k+1}({}^V\xi\underline{x}^{[\underline{m}]}\omega) = \xi F_k(\underline{x}^{[\underline{m}]}\omega),$$

where  $F_k$  is given in [19, p 1869]. In particular, we have divided Frobenius maps

$$F_1': I_{R,n}\Omega_{D_n/W_n(R)}^r \to W_{n-1}\Omega_{X/R}^r$$

given by

$$F_1'(^V \xi \omega) = \xi F(\mathfrak{S}\omega).$$

Since the diagrams in [19, (1.6.2), (1.6.4)] commute, we get a morphism of complexes, denoted by  $F_1$ :

$$\begin{array}{lll}
\mathcal{K}_{D_n/W_n(R)}(r) : & I_{R,n}\mathcal{J}_n^{[r]} \xrightarrow{d} I_{R,n}\mathcal{J}_n^{[r-1]}\Omega_{D_n/W_n(R)}^1 \xrightarrow{d} \cdots \xrightarrow{d} I_{R,n}\Omega_{D_n/W_n(R)}^r \xrightarrow{d} \Omega_{D_n/W_n(R)}^{r+1} \xrightarrow{d} \cdots \\ & \downarrow^{F'_{r+1}} & \downarrow^{F'_r} & \downarrow^{F'_1} & \downarrow^{\mathfrak{S}} \\
N^{r+1}W_n\Omega_{X/R}^{\bullet} : & W_{n-1}\mathcal{O}_X \xrightarrow{d} W_{n-1}\Omega_{X/R}^1 \xrightarrow{d} \cdots \xrightarrow{d} W_{n-1}\Omega_{X/R}^r \xrightarrow{dV} W_n\Omega_{X/R}^{r+1} \xrightarrow{d} \cdots \\
\end{array}$$
(3.0.8)

We can now identify the map  ${}^{F}\beta_{r}$  as the composite map

$$I_{R}H^{i-r}(X_{\bullet},\Omega^{r}_{X_{\bullet}/W_{\bullet}(R)}) \to \mathbb{H}^{i}(X_{\bullet},I_{R}\Omega^{\geqslant r}_{X_{\bullet}/W_{\bullet}(R)}) \to \mathbb{H}^{i}(D_{\bullet},\mathcal{K}_{D_{\bullet}/W_{\bullet}(R)}(r)) \xrightarrow{F_{1}} \mathbb{H}^{i}(X,N^{r+1}W_{\bullet}\Omega^{\bullet}_{X/R})$$
$$\xrightarrow{\kappa} \mathbb{H}^{i}(X,N^{r+1}W_{\bullet}\Omega^{\geqslant r}_{X/R}) \to H^{i-r}(X,W_{\bullet}\Omega^{r}_{X/R}).$$
(3.0.9)

Indeed, we now check that the map (3.0.9) is the same as the map (3.0.6). Recall that the quasiisomorphism  $\mathcal{F}^{r+1}\Omega^{\bullet}_{X_{\bullet}/W_{\bullet}(R)} \cong N^{r+1}W_{\bullet}\Omega^{\bullet}_{X/R}$  in [19, Theorem 1.2] is given by the diagram (see [19, (1.7)])

$$\mathcal{F}^{r+1}\Omega^{\bullet}_{X_{\bullet}/W_{\bullet}(R)} \xleftarrow{\sim} \operatorname{Fil}^{r+1}\Omega^{\bullet}_{D_{\bullet}/W_{\bullet}(R)} \xrightarrow{\sim} N^{r+1}W_{\bullet}\Omega^{\bullet}_{X/R},$$

where  $\operatorname{Fil}^{r+1}\Omega^{\bullet}_{D_{\bullet}/W_{\bullet}(R)}$  is the complex defined on [19, p. 1868]. Since the first map and the final two maps in the compositions (3.0.6) and (3.0.9) are the same, to see that (3.0.6) and (3.0.9) are the same it suffices to see that the following diagram commutes

where the left vertical arrow is the inclusion of complexes, and the middle vertical arrow is the obvious map  $(\mathcal{F}^{r+1}\Omega^*_{X,/W,(R)})$  is defined as a degree-wise direct sum of terms which include those of  $\mathcal{K}_{D,/W,(R)}(r)$ ). The left square of (3.0.10) clearly commutes, and the right triangle commutes since the construction of the morphism  $\operatorname{Fil}^{r+1}\Omega^*_{D,/W,(R)} \xrightarrow{\sim} N^{r+1}W_*\Omega^*_{X/R}$  is in terms of the maps  $F'_{k+1}$  used to define  $F_1$  - see [19, pp. 1869-1870] (note that the maps we call  $F'_{k+1}$  are called  $F_{k+1}$  in [19]).

From the explicit construction of  ${}^{F}\beta_{r}$  in (3.0.9), it is clear that  $V \circ {}^{F}\beta_{r}$  is the restriction of  $\beta_{r}$  to  $I_{R}H^{i-r}(X_{*}, \Omega^{r}_{X_{*}/W_{*}(R)})$ . Indeed, we have a commutative square

where the lower map is given by  $p^{r-i}V$  in degrees  $i \le r$ , and by the identity in degrees > r (see (3.0.11)). Taking cohomology and using (3.0.7) gives the middle square of the following diagram

Going from the top left to the bottom right via the left-hand side gives  $V \circ {}^F \beta_r$ , whilst going via right-hand side gives  $\beta_r$ .

In the absence of a global Witt lift, we proceed by simplicial methods as in [19, Theorem 1.2]. We omit the details here.

In the proofs of the theorems we will carry out — in each case — the induction step and prove that  $\beta_r$  is an isomorphism. We consider the following morphisms of complexes  $N^r W_* \Omega^*_{X/R} \to W_* \Omega^*_{X/R}$  and  $\mathcal{F}^r \Omega^*_{X_*/W_*(R)} \to \Omega^*_{X_*/W_*(R)}$  given as follows:

(it is a morphism of complexes because Vd = pdV) and

where the vertical map  $\times p^i$  is the inclusion of  $I_R \Omega^j_{X_*/W_*(R)}$  into  $\Omega^j_{X_*/W_*(R)}$  multiplied by  $p^i$ .

The construction of the comparison isomorphism  $\mathcal{F}^r\Omega^{\bullet}_{X, /W, (R)} \to N^rW_{\bullet}\Omega^{\bullet}_{X/R}$  via the map  $\Sigma$ : Fil<sup>*r*</sup> $\Omega^{\bullet}_{D, /W, (R)} \to N^rW_{\bullet}\Omega^{\bullet}_{X/R}$  in [19, (1.5)] and the comparison between  $\Omega^{\bullet}_{X, /W, (R)}$  and  $W_{\bullet}\Omega^{\bullet}_{X/R}$  show that the induced homomorphisms on hypercohomology

$$\lambda_1 : \mathbb{H}^*(X, N^r W_{\bullet} \Omega^{\bullet}_{X/R}) \to \mathbb{H}^*(X, W_{\bullet} \Omega^{\bullet}_{X/R}) = H^*_{\mathrm{cris}}(X/W_{\bullet}(R))$$

and

$$\lambda_2 : \mathbb{H}^*(X_{\bullet}, \mathcal{F}^r\Omega^{\bullet}_{X_{\bullet}/W_{\bullet}(R)}) \to \mathbb{H}^*(X_{\bullet}, \Omega^{\bullet}_{X_{\bullet}/W_{\bullet}(R)}) = H^*_{\mathrm{dR}}(X_{\bullet}/W_{\bullet}(R))$$

agree. Then we shall use the following argument in various proofs by induction later on:

Assume the maps  $\beta_s : H^i(X_{\bullet}, \Omega^s_{X_{\bullet}/W_{\bullet}(R)}) \to H^i(X, W_{\bullet}\Omega^s_{X/R})$  are isomorphisms, the map *V* is injective on  $H^i(X, W_{\bullet}\Omega^s_{X/R})$  and hence the maps  ${}^F\beta_s$  are bijective for s < r. Then truncation induces a commutative diagram

with horizontal isomorphisms. Since the hypercohomology spectral sequences of  $\mathcal{F}^{r+1}\Omega^{\bullet}_{X_{\bullet}/W_{\bullet}(R)}$ and  $\Omega^{\bullet}_{X_{\bullet}/W_{\bullet}(R)}$  degenerate, the vertical maps in (3.0.12) are injective and the cokernels are isomorphic to  $H^{i-r}(X, \Omega^{r}_{X/R})$ .

#### 4 | HODGE-WITT COHOMOLOGY AS MULTIPLICATIVE DISPLAYS

In this section, we derive the proof of Theorem 1.2. It relies on the following more general proposition which holds in all examples.

**Proposition 4.1.** Fix a pair (i, j),  $0 \le i, j \le d = \dim X$ , where X is any of the schemes in Theorems 1.2, 1.4 and 1.6.

(i) There is an exact sequence induced by the action of V on  $W\Omega_{X/R}^{j}$ 

$$0 \to H^{i}(X, W\Omega^{j}_{X/R}) \xrightarrow{V} H^{i}(X, W\Omega^{j}_{X/R}) \to H^{i}(X, \Omega^{j}_{X/R}) \to 0.$$

(ii) Let  $\mathfrak{X}$  be the ind-scheme over the ind-scheme Spec  $W_{\bullet}(R)$  arising from the compatible family of liftings  $X_n$ . Then there exists a multiplicative display  $\mathcal{P} = (P, Q = I_R P, F, F_1)$  over R and a homomorphism  $P \xrightarrow{\varsigma} H^i(X, W\Omega^j_{X/R})$  compatible with the action of Frobenius, where F on the right is induced by the Frobenius on  $W\Omega^j_{X/R}$ , such that  $\varsigma(I_R P) \subset VH^i(X, W\Omega^j_{X/R})$  and the induced map

$$\overline{\varsigma} : P/I_R P \to H^i(X, W\Omega^j_{X/R}) / \operatorname{im} V \cong H^i(X, \Omega^j_{X/R})$$

is an isomorphism of free R-modules. (iii) The map ς is an isomorphism.

*Remark* 4.2. The Frobenius-equivariant map  $\varsigma$  in (*ii*) fits into a commutative diagram

Indeed,

$$V(\varsigma(F_1({}^V\xi x))) = V(\varsigma(\xi F x)) = V\xi F\varsigma(x) = {}^V\xi\varsigma(x) = \varsigma({}^V\xi x).$$

So  $\varsigma(I_R P) \subset VH^i(X, W\Omega^j_{X/R})$  and  $\overline{\varsigma}$  is well defined. Note that  $F_1$  is a bijection because F is — by definition — an F-linear isomorphism in a multiplicative display.

In many cases, *P* is isomorphic to  $H^i(\mathfrak{X}, \Omega^j_{\mathfrak{X}/W(R)})$ .

In this section, we will prove Proposition 4.1 for surfaces and the case  $i \ge 0$ , j = 0 for abelian schemes.

**Lemma 4.3.** Assume that properties (i) and (ii) in Proposition 4.1 hold for a fixed pair (i, j). Then property (iii) holds.

*Proof.* The proof is identical to the corresponding section in the proof of [22, Lemma 47]. For completeness, we include the argument. The property  $\varsigma(I_R P) \subset VH^i(X, W\Omega^j_{X/R})$  implies  $\varsigma(I_n P) \subset$ 

 $V^n H^i(X, W\Omega^j_{X/R})$  for  $I_n := {}^{V^n} W(R)$ . Then one has a commutative diagram

where  $F_1$  is bijective again.

We claim that the maps

$$\overline{\varsigma} : \frac{I_n P}{I_{n+1} P} \to \frac{V^n H^i(X, W\Omega^j_{X/R})}{V^{n+1} H^i(X, W\Omega^j_{X/R})}$$
(4.3.1)

are surjective. For n = 0, this is property (*ii*). Let  $m \in H^i(X, W\Omega^j_{X/R})$ . Find, by induction, elements  $x \in I_{n-1}P$  and  $m_1 \in H^i(X, W\Omega^j_{X/R})$  such that  $V^{n-1}m = \varsigma(x) + V^nm_1$ . We write  $x = F_1y$  for  $y \in I_nP$ . Then

$$V^{n}m = V\varsigma(F_{1}y) + V^{n+1}m_{1} = \varsigma(y) + V^{n+1}m_{1},$$

which ends the induction and proves the claim.

We know that  $H^i(X, W\Omega^j_{X/R})$  is V-adically separated [22, Lemma 39]. Therefore, the surjectivity of 4.3.1 implies that  $P \xrightarrow{\varsigma} H^i(X, W\Omega^j_{X/R})$  is surjective and  $H^i(X, W\Omega^j_{X/R})$  is V-adically complete [5, § 2.8, Théorème 1]. Since V is injective by property (i),  $H^i(X, W\Omega^j_{X/R})$  is a reduced Cartier module.

Now consider (4.3.1) as a homomorphism of  $W_{n+1}(R)$ -modules. We claim that both sides of (4.3.1) are isomorphic as  $W_{n+1}(R)$ -modules and are Noetherian. Since a surjective endomorphism of Noetherian modules is an isomorphism, this implies that (4.3.1) is an isomorphism and therefore  $\varsigma$  is an isomorphism as well. It therefore suffices to prove the claim:

Since

$$F_1 : I_n P \to I_{n-1} P$$
$$V^n \xi x \mapsto V^{n-1} \xi F x$$

is a bijection, we get a bijection

$$F_1$$
:  $I_n P/I_{n+1}P \rightarrow I_{n-1}P/I_nP$ 

Let  $P/I_1P_{[F^n]}$  denote the  $W_{n+1}(R)$ -module given by restriction of scalars along  $F^n$ :  $W_{n+1}(R) \rightarrow R$ . Iterating  $F_1$ , we get an isomorphism of  $W_{n+1}(R)$ -modules

$$F_1^n : I_n P/I_{n+1} P \to P/I_1 P_{[F^n]}.$$

Since *R* is *F*-finite,  $P/I_1P_{[F^n]}$  is a Noetherian  $W_{n+1}(R)$ -module.

Now for the reduced Cartier module  $M = H^i(X, W\Omega^j_{X/R})$  we obtain, analogously, the isomorphism

$$V^n: M/VM_{[F^n]} \xrightarrow{\simeq} V^n M/V^{n+1}M.$$

Hence properties (i) and (ii) show the claim.

**Lemma 4.4.** Proposition 4.1 holds for j = 0,  $i \ge 0$  if X is an abelian scheme or a smooth proper surface with geometrically connected fibres, ordinary closed fibre and k is algebraically closed.

*Proof.* For i = j = 0, the exact sequence in question reads

$$0 \to W(R) \xrightarrow{V} W(R) \to R \to 0$$

and there is nothing to prove.  $H^0(X, W\mathcal{O}_X)$  is isomorphic to the multiplicative (unit) display.

Now let X = A be an abelian scheme. Then  $H^1(A, W\mathcal{O}_A)$  is the reduced Cartier module of the formal *p*-divisible group  $\widehat{\text{Pic}}_{A/R}$  with tangent space  $H^1(A, \mathcal{O}_A)$ , and we have an exact sequence

$$0 \to H^1(A, W\mathcal{O}_A) \xrightarrow{V} H^1(A, W\mathcal{O}_A) \to H^1(A, \mathcal{O}_A) \to 0.$$

Since the closed fibre is ordinary, the associated display is multiplicative, so

$$(P, Q, F, F_1) = (H^1(A, W\mathcal{O}_A), I_R H^1(A, W\mathcal{O}_A), F, F_1)$$

The Grothendieck–Messing crystal  $\mathbb{D}(\widehat{\operatorname{Pic}}_{A/R})$  evaluated at  $W(R) \to R$  yields

$$\mathbb{D}(\widehat{\mathrm{Pic}}_{A/R})_{W(R)} = \mathbb{D}(\widehat{\mathbb{G}}^d_{m/R})_{W(R)} = H^1(A, W\mathcal{O}_A) = H^1(\mathcal{A}, \mathcal{O}_A),$$

where  $\mathcal{A}$  is a lifting as ind-scheme over Spec  $W_{\bullet}(R)$ .

For  $i \ge 2$ , we have a commutative diagram

$$\bigwedge^{i} H^{1}(A, W\mathcal{O}_{A}) \longrightarrow H^{i}(A, W\mathcal{O}_{A}) \longrightarrow H^{i}(A, \mathcal{O}_{A})$$

$$\xrightarrow{i} H^{1}(A, \mathcal{O}_{A}) \xrightarrow{\simeq} H^{i}(A, \mathcal{O}_{A})$$

The right arrow  $H^i(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \to H^i(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$  is a surjection because de Rham cohomology and Hodge cohomology commute with base change [17, Corollary 8.3] and we have a commutative diagram

Hence the exact sequence

$$0 \to W\mathcal{O}_A \xrightarrow{V} W\mathcal{O}_A \to \mathcal{O}_A \to 0$$

induces exact sequences for all i

$$0 \to H^{i}(A, W\mathcal{O}_{A}) \xrightarrow{V} H^{i}(A, W\mathcal{O}_{A}) \to H^{i}(A, \mathcal{O}_{A}) \to 0 \,.$$

Now,  $P = \bigwedge^{i} H^{1}(A, W\mathcal{O}_{A})$  carries the exterior power structure of the multiplicative display  $H^{1}(A, W\mathcal{O}_{A})$  and hence is again multiplicative:

$$\mathcal{P} = (P, I_R P, F, F_1),$$

where *F* is defined by  $\bigwedge^i F$  on *P*. The above diagram shows that the *F*-equivariant map  $\varsigma : P \to H^i(A, W\mathcal{O}_A)$  induces an isomorphism

$$\overline{\varsigma}$$
:  $P/I_R P \xrightarrow{-} H^i(A, W\mathcal{O}_A)/\mathrm{im}V \cong H^i(A, \mathcal{O}_A)$ .

We conclude property (*iii*) by Lemma 4.3. Hence the lemma holds for abelian schemes.

Now let *X* be smooth projective scheme over Spec *R* of dimension 2 satisfying assumption(A) with ordinary closed fibre. Recall that

$$H^{0}(X_{k}, B\Omega^{1}_{X_{k}/k}) \cong \ker \left(C : H^{0}\left(X_{k}, \left(\Omega^{1}_{X_{k}/k}\right)_{d=0}\right) \to H^{0}(X_{k}, \Omega^{1}_{X_{k}/k})\right)$$
$$\cong H^{1}((X_{k})_{\text{fppf}}, \alpha_{p})$$
$$\cong \operatorname{Hom}_{k-\operatorname{grp}}(\alpha_{p}, \operatorname{Pic}_{X_{k}})$$

by applying [14, (2.1.11)], [25, Proposition 4.14] and [25, Proposition 4.16] in turn (recall that the Cartier dual of  $\alpha_p$  is itself). Since  $X_k$  is ordinary, we conclude that the above groups are trivial. In particular,  $(\operatorname{Pic}_{X_k}^0)_{\text{red}}$  is ordinary. The formal Picard group  $\widehat{Pic}_{X_k/k}$  of  $X_k$  is the formal completion of  $(\operatorname{Pic}_{X_k}^0)_{\text{red}}$  along the zero section [1, Remark (1.9)(ii)], so we conclude that  $\widehat{Pic}_{X_k/k}$  is multiplicative. Since k is algebraically closed, we therefore have  $\widehat{Pic}_{X_k/k} \simeq \widehat{\mathbb{G}}_m^g$ . By rigidity of  $\widehat{\mathbb{G}}_m$ , we see that  $\widehat{Pic}_{X/R} \simeq \widehat{\mathbb{G}}_{m/R}^g$ . In analogy to the case of abelian schemes, the Grothendieck–Messing crystal  $\mathbb{D}(\widehat{Pic}_{X/R})$  evaluated at  $W(R) \to R$  yields

$$\mathbb{D}(\widehat{\operatorname{Pic}}_{X/R})_{W(R)} = \mathbb{D}(\widehat{\mathbb{G}}_{m/R}^g)_{W(R)} = H^1(X, W\mathcal{O}_X) \cong H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \operatorname{Lie}\widehat{\operatorname{Pic}}_{\mathcal{X}/W(R)},$$

where  $\mathcal{X}$  is a lifting as ind-scheme over Spec  $W_{\bullet}(R)$ . We have a commutative diagram

$$H^{1}_{d\mathbb{R}}(\mathcal{X}/W(\mathbb{R})) \longrightarrow H^{1}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \longrightarrow H^{1}(X, \mathcal{O}_{X})$$

$$\simeq \downarrow$$

$$H^{1}_{cris}(X/W(\mathbb{R})) \longrightarrow H^{1}(X, W\mathcal{O}_{X})$$

where the map  $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$  is surjective because de Rham cohomology and Hodge cohomology commute with base change. Hence the sequence

$$0 \to H^1(X, W\mathcal{O}_X) \xrightarrow{V} H^1(X, W\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to 0$$

is exact. From the long exact cohomology sequence associated to

$$0 \to W\mathcal{O}_X \xrightarrow{V} W\mathcal{O}_X \to \mathcal{O}_X \to 0$$

we get that

$$0 \to H^2(X, W\mathcal{O}_X) \xrightarrow{V} H^2(X, W\mathcal{O}_X) \to H^2(X, \mathcal{O}_X) \to 0$$

is exact as well, taking into account that  $H^i(X, W\mathcal{O}_X) = 0$  for i > 2 since X is a surface.

Under the one-to-one correspondence of formal *p*-divisible groups and displays in [32, Theorem 103], the display associated to  $\widehat{\text{Pic}}_{X/R} = \widehat{\mathbb{G}}_{m/R}^g$  is  $P = \mathbb{D}(\widehat{\text{Pic}}_{X/R})_{W(R)}$  equipped with a multiplicative display structure, so  $\mathcal{P} = (P, Q = I_R P, F, F_1)$ . It is obvious that the isomorphism  $P \cong H^1(X, W\mathcal{O}_X)$  maps  $I_R P$  to  $VH^1(X, W\mathcal{O}_X)$  and induces an isomorphism

$$P/I_R P \cong H^1(X, W\mathcal{O}_X)/VH^1(X, W\mathcal{O}_X) \cong H^1(X, \mathcal{O}_X),$$

which is the tangent space of the display. Hence Proposition 4.1 holds for j = 0, i = 1 and X a surface.

By [1, Corollary 3.3], the Cartier module of the formal Brauer group  $\widehat{Br}_{X_k/k}$  is  $H^2(X_k, W\mathcal{O}_{X_k})$ . Under assumption (A), the crystalline cohomology of  $X_k$  is torsion-free. Since  $X_k$  is ordinary, the Newton and Hodge polygons of  $X_k$  coincide [4, Proposition 7.3]. Since  $H^2(X_k, W\mathcal{O}_{X_k})$  is the slope 0 part of the crystalline cohomology, its rank (that is, the height of  $\widehat{Br}_{X_k/k}$ ) equals the dimension of  $H^2(X, \mathcal{O}_X)$ , and hence  $\widehat{Br}_{X_k/k}$  is multiplicative. Since k is algebraically closed, we see that  $\widehat{Br}_{X_k/k} \simeq \widehat{G}_m^h$ , and therefore  $\widehat{Br}_{X/R} \simeq \widehat{G}_{m/R}^h$  by rigidity. The Grothendieck–Messing crystal  $\mathbb{D}(\widehat{Br}_{X/R})$  evaluated at  $W(R) \to R$  yields, in analogy with  $\widehat{Pic}_{X/R}$ ,

$$\mathbb{D}(\widehat{\mathrm{Br}}_{X/R})_{W(R)} = \mathbb{D}(\widehat{\mathbb{G}}_{m/R}^{h})_{W(R)} = H^{2}(X, W\mathcal{O}_{X}) \cong H^{2}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \operatorname{Lie} \widehat{\mathrm{Br}}_{\mathcal{X}/W(R)}$$

with  $I_R H^2(X, W\mathcal{O}_X) = VH^2(X, W\mathcal{O}_X)$ . Since

$$H^{2}(X, W\mathcal{O}_{X})/VH^{2}(X, W\mathcal{O}_{X}) \cong H^{2}(X, \mathcal{O}_{X})$$

which is the tangent space of  $\widehat{Br}_{X/R}$  we see that  $(H^2(X, W\mathcal{O}_X), VH^2(X, W\mathcal{O}_X), F, F_1 = V^{-1})$  defines a multiplicative display structure and is the display associated to the formal Brauer group. This finishes the proof of Lemma 4.4.

In the following, we will prove Proposition 4.1 for i = 0, j = 1 for surfaces and abelian schemes. Using the quasi-isomorphism  $N^2 W \Omega^{\bullet}_{X/R} \cong \mathcal{F}^2 \Omega^{\bullet}_{\mathfrak{X}/W(R)}$  and the fact that the  $E_1$ -hypercohomology spectral sequence associated to  $\mathcal{F}^2 \Omega^{\bullet}_{\mathfrak{X}/W(R)}$  degenerates, we compute the cohomology of the Nygaard complex (compare [22, Remark 42]):

 $- \mathbb{H}^{0}(N^{2}W\Omega_{X/R}^{\bullet}) \cong I_{R}H^{0}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = I_{R}W(R),$ 

$$\begin{split} &- \mathbb{H}^{1}(N^{2}W\Omega_{X/R}^{\bullet}) \cong I_{R}H^{1}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) \oplus I_{R}H^{0}(\mathfrak{X},\Omega_{\mathfrak{X}/W(R)}^{1}), \\ &- \mathbb{H}^{2}(N^{2}W\Omega_{X/R}^{\bullet}) \cong I_{R}H^{2}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) \oplus I_{R}H^{1}(\mathfrak{X},\Omega_{\mathfrak{X}/W(R)}^{1}) \oplus H^{0}(\mathfrak{X},\Omega_{\mathfrak{X}/W(R)}^{2}), \\ &- \mathbb{H}^{3}(N^{2}W\Omega_{X/R}^{\bullet}) \cong I_{R}H^{3}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) \oplus I_{R}H^{2}(\mathfrak{X},\Omega_{\mathfrak{X}/W(R)}^{1}) \oplus H^{1}(\mathfrak{X},\Omega_{\mathfrak{X}/W(R)}^{2}) \\ &- \mathbb{H}^{0}(\mathfrak{X},\Omega_{\mathfrak{X}/W(R)}^{3}), \\ &- \mathbb{H}^{4}(N^{2}W\Omega_{X/R}^{\bullet}) \cong I_{R}H^{4}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) \oplus I_{R}H^{3}(\mathfrak{X},\Omega_{\mathfrak{X}/W(R)}^{1}) \oplus H^{2}(\mathfrak{X},\Omega_{\mathfrak{X}/W(R)}^{2}) \\ &\oplus H^{1}(\mathfrak{X},\Omega_{\mathfrak{X}/W(R)}^{3}) \oplus H^{0}(\mathfrak{X},\Omega_{\mathfrak{X}/W(R)}^{4}). \end{split}$$

Since the map  $\partial$  :  $H^0(X, W\mathcal{O}_X) \cong W(R) \to \mathbb{H}^0(W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R})$  is induced by the differential *d* which vanishes on W(R),  $\partial$  is the zero map. Then we have the following commutative diagram

We will see below that  $H^1(X, W\mathcal{O}_X) = \mathbb{D}(\widehat{\text{Pic}}_{X/R})$  is a direct summand of  $H^1_{\text{cris}}(X/W(R))$ , hence the upper right arrow is a surjection. It is easy to see that the composite map

$$I_R H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to \mathbb{H}^1(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \cong \mathbb{H}^1(N^2W\Omega^{\bullet}_{X/R}) \to H^1(X, W\mathcal{O}_X) \xrightarrow{V} VH^1(X, W\mathcal{O}_X)$$

can be identified with the isomorphism  $I_R H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \xrightarrow{\sim} V H^1(X, W\mathcal{O}_X)$  constructed earlier. The lower right arrow in the diagram can then be identified with the map  $\mathbb{H}^1(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \to I_R H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}).$ 

Since *V* is injective on  $H^1(X, W\mathcal{O}_X)$  and  $VH^1(X, W\mathcal{O}_X) \cong I_R H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ , the lower right arrow is surjective too. The left vertical arrow can be identified with the map (compare (3.0.12))

$$\mathbb{H}^{0}(I_{R}\Omega^{1}_{\mathfrak{X}/W(R)} \xrightarrow{d} \Omega^{2}_{\mathfrak{X}/W(R)}) = I_{R}H^{0}(\mathfrak{X},\Omega^{1}_{\mathfrak{X}}) \to H^{0}(\mathfrak{X},\Omega^{1}_{\mathfrak{X}}),$$

which is injective and has cokernel  $H^0(X, \Omega^1_{X/R})$ . Then the commutative diagram

together with the injectivity of the map (V, id) in [22, Lemma 44] shows that V is injective on  $H^0(X, W\Omega^1_{X/R})$  and the cokernel is  $H^0(X, \Omega^1_{X/R})$ .

Let us now derive the Hodge–Witt decomposition of  $H^1_{\text{cris}}(X/W(R))$ . We point out that under our assumptions the Picard scheme  $\text{Pic}^0_{X/R}$  has reduced fibres and is an abelian scheme. Let  $\text{Alb}_{X/R}$  be its dual; it is again an abelian scheme and there is a morphism  $X \to \text{Alb}_{X/R}$  induced by the relative Poincaré bundle on  $X \times_R \operatorname{Pic}^0_{X/R}$  (see [18, p. 289]). Consider the induced map

$$H^1_{\operatorname{cris}}(\operatorname{Alb}_{X/R}/W(R)) \to H^1_{\operatorname{cris}}(X/W(R)).$$

The first crystalline cohomology of an abelian scheme is the Dieudonné crystal of the *p*-divisible group of the dual abelian scheme by [23, Chapter 2] (see also [2, Théoremè 2.5.6]), so we have a direct sum decomposition

$$H^{1}_{\operatorname{cris}}(\operatorname{Alb}_{X/R}/W(R)) \simeq \mathbb{D}(\operatorname{Pic}^{0}_{X/R}(p)) = \mathbb{D}(\operatorname{Pic}^{0}_{X/R}) \oplus \mathbb{D}(\operatorname{Pic}^{0}_{X/R}(p)^{\operatorname{\acute{e}t}})$$
(4.4.2)

into a direct sum of Dieudonné modules associated to the connected and étale part of the *p*-divisible group associated to  $(Alb_{X/R}) = (Pic_{X/R}^0)^{\vee}$ . The induced map

$$\mathbb{D}(\widehat{\operatorname{Pic}}^0_{X/R}) \to H^1_{\operatorname{cris}}(X/W(R)) \to H^1(X, W\mathcal{O}_X)$$

is a map of Dieudonné modules. The induced map

$$\mathbb{D}(\widehat{\operatorname{Pic}}_{X/R}^{0})/I_{R}\mathbb{D}(\widehat{\operatorname{Pic}}_{X/R}^{0}) \to H^{1}(X, W\mathcal{O}_{X})/\operatorname{im} V \simeq H^{1}(X, \mathcal{O}_{X})$$

is a homomorphism of free *R*-modules of rank = dim  $\operatorname{Pic}_{X/R}^0$ , and it is an isomorphism because it is an isomorphism after base change along  $R \to k$  since  $X_k$  is ordinary and using [14, Remarque 3.11.2]. Hence  $\mathbb{D}(\operatorname{Pic}_{X/R}^0) \to H^1(X, W\mathcal{O}_X)$  is an isomorphism by Lemma 4.3. Since  $\mathbb{D}(\operatorname{Pic}_{X/R}^0)$ carries the structure of a multiplicative display (because  $\operatorname{Pic}_{X/R}^0$  is multiplicative, since  $X_k$  is ordinary), the isomorphism imposes the structure of a multiplicative display on  $H^1(X, W\mathcal{O}_X)$ . It identifies  $H^1(X, W\mathcal{O}_X)$  as a direct summand of  $H^1_{\operatorname{cris}}(X/W(R))$ .

Now consider the induced map

$$\eta : \mathcal{P}^{\text{\acute{e}t}} = \mathbb{D}(\operatorname{Pic}^{0}_{X/R}(p)^{\text{\acute{e}t}}) \to \mathbb{H}^{0}(X, W\Omega^{\geq 1}_{X/R}) \to H^{0}(X, W\Omega^{1}_{X/R})$$

which is compatible with the Frobenius on the left and pF on  $H^0(X, W\Omega^1_{X/R})$ . We already have a commutative diagram

$$\mathcal{P}^{\mathrm{\acute{e}t}} \xrightarrow{\eta} H^0(X, W\Omega^1_{X/R}) \ \downarrow F \ \mathcal{P}^{\mathrm{\acute{e}t}} \xrightarrow{\eta} H^0(X, W\Omega^1_{X/R}).$$

Since  $\mathcal{P}^{\text{ét}}$  is an étale display,  $F_1$  is defined on the the whole of  $\mathcal{P}^{\text{ét}}$ . On the versal deformation of X, the diagram commutes because it commutes after multiplication by p and p is injective on the

versal deformation. Then the diagram

with  $F_2({}^V \xi x) = \xi F_1 x$  commutes, hence we get an induced map of free *R*-modules

$$\overline{\eta} : \mathcal{P}^{\text{\'et}}/I_R \mathcal{P}^{\text{\'et}} \to H^0(X, W\Omega^1_{X/R}) / \text{im } V \cong H^0(X, \Omega^1_{X/R})$$

of rank = dim  $\operatorname{Pic}_{X/R}^0$ . It is enough to show that  $\overline{\eta}$  is surjective to show that it is an isomorphism. We show this after base change along  $R \to k$ . Over k it is known that  $\mathbb{D}(\operatorname{Pic}_{X_k}^0(p)^{\text{ét}})$  is the slope p-part of in  $H^1_{\operatorname{cris}}(X_k/W(k))$ , hence is isomorphic to  $H^0(X_k, W\Omega^1_{X_k/k})$  because we are in the ordinary case. Since the map  $H^0(X_k, W\Omega^1_{X_k/k}) \to H^0(X_k, \Omega^1_{X_k/k})$  is surjective,  $\overline{\eta}$  is surjective and hence an isomorphism. Applying Lemma 4.3 shows that  $\eta : \mathbb{D}(\operatorname{Pic}_{X/R}^0(p)^{\text{ét}}) \to H^0(X, W\Omega^1_{X/R})$  is an isomorphism. One consequence of the case i = 0, j = 1 is that  $\mathbb{H}^0(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) = H^0(X, W\Omega^1_{X/R})$  in diagram (4.4.1). Indeed, the isomorphism  $\eta$  induces a surjection  $\mathbb{H}^0(X, W\Omega^{\geq 1}_{X/R}) \to H^0(X, W\Omega^1_{X/R})$  so the map  $\mathbb{H}^0(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) \to H^0(X, W\Omega^1_{X/R})$  is surjective too, and hence the identity. The commutativity of the diagram

implies the isomorphism  $H^0(X, W\Omega^1_{X/R}) \cong H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/W(R)})$ . (Note that  $H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$  is the tangent space of  $\widehat{\operatorname{Pic}}^0_{\mathfrak{X}/W(R)}$ , hence is isomorphic to the value of the Dieudonné crystal of  $\widehat{\operatorname{Pic}}^0_{\mathfrak{X}/W(R)}$  at W(R), which is  $H^1(X, W\mathcal{O}_X)$  by rigidity of  $\widehat{\operatorname{Pic}}$ ). The decomposition (4.4.2) then reflects the Hodge–Witt decomposition in degree one. We have isomorphisms

$$H^{1}_{\operatorname{cris}}(\operatorname{Alb}_{X/R}/W(R)) \simeq H^{1}_{\operatorname{cris}}(X/W(R) \simeq \mathbb{D}(\widehat{\operatorname{Pic}}^{0}_{X/R}) \oplus \mathbb{D}(\operatorname{Pic}^{0}_{X/R}(p)^{\operatorname{\acute{e}t}}).$$

The display structure on  $H^1_{\text{cris}}(X/W(R))$  arising from the Nygaard complex  $N^1W\Omega^*_{X/R}$  has been analysed in [20, 3.4]. We have  $P = H^1_{\text{cris}}(X/W(R)), Q = \ker(H^1_{\text{cris}}(X/W(R)) \to H^1(X, \mathcal{O}_X), F$  is the crystalline Frobenius, and  $F_1$  is defined on the Nygaard filtration  $Q \cong \mathbb{H}^1(X, N^1W\Omega^*_{X/R})$ . We see then that

$$Q = I_R \mathbb{D}(\widehat{\operatorname{Pic}}_{X/R}^0) \oplus \mathbb{D}(\operatorname{Pic}_{X/R}^0(p)^{\text{ét}})$$

because  $H^1(X, \mathcal{O}_X)$  is the tangent space of  $\widehat{\operatorname{Pic}}^0_{X/R}$  and we have an exact sequence

$$0 \to I_R \mathbb{D}(\widehat{\operatorname{Pic}}^0_{X/R}) \to \mathbb{D}(\widehat{\operatorname{Pic}}^0_{X/R}) \to H^1(X, \mathcal{O}_X) \to 0.$$

On  $I_R \mathbb{D}(\widehat{\text{Pic}}_{X/R}^0)$  the map  $F_1$  is defined by  $F_1({}^V \xi \alpha) = \xi F \alpha$ . Since  $\mathbb{D}(\text{Pic}_{X/R}^0(p)^{\text{ét}})$  is an étale display,  $F_1$  is defined on  $\mathbb{D}(\text{Pic}_{X/R}^0(p)^{\text{ét}})$  given that for a display of an étale group one has 'P = Q'. We conclude that the above decomposition is a direct sum of displays.

Before we can finish the proof of Theorem 1.2, we must prove Proposition 4.1 in the cases  $i = 1, 2, j = 1, \dim X = 2$ . For i = 1, the proof is very similar as for [22, Lemma 46].

Then diagram (92) in [22] holds verbatim for general smooth projective surfaces:

$$H^{0}(X, W\Omega^{2}) \longrightarrow \mathbb{H}^{1}(X, W\Omega^{1} \xrightarrow{d} W\Omega^{2}) \longrightarrow H^{1}(X, W\Omega^{1}) \longrightarrow H^{1}(X, W\Omega^{2})$$

$$\uparrow^{a} \qquad \uparrow^{a} \qquad \uparrow^{v} \qquad \uparrow^{a} \qquad \uparrow^{v} \qquad \uparrow^{a} \qquad (4.4.3)$$

$$H^{0}(X, W\Omega^{2}) \longrightarrow \mathbb{H}^{1}(X, W\Omega^{1} \xrightarrow{dV} W\Omega^{2}) \longrightarrow H^{1}(X, W\Omega^{1}) \longrightarrow H^{1}(X, W\Omega^{2})$$

As in cohomological degree 1, one sees that the composite map

$$I_R H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to \mathbb{H}^2(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \cong \mathbb{H}^2(N^2W\Omega^{\bullet}_{X/R}) \to H^2(X, W\mathcal{O}_X) \xrightarrow{V} VH^2(X, W\mathcal{O}_X)$$

agrees with the isomorphism constructed earlier. Hence  $\mathbb{H}^2(N^2W\Omega^{\bullet}_{X/R}) \to H^2(X, W\mathcal{O}_X)$  is surjective. Then we have a commutative diagram of isomorphisms

and the right vertical map can be identified under the isomorphism between crystalline and de Rham cohomology with the injection

$$I_{R}H^{2}(\boldsymbol{\mathfrak{X}},\Omega_{\boldsymbol{\mathfrak{X}}/W(R)}^{1})\oplus H^{1}(\boldsymbol{\mathfrak{X}},\Omega_{\boldsymbol{\mathfrak{X}}/W(R)}^{2})\to H^{2}(\boldsymbol{\mathfrak{X}},\Omega_{\boldsymbol{\mathfrak{X}}/W(R)}^{1})\oplus H^{1}(\boldsymbol{\mathfrak{X}},\Omega_{\boldsymbol{\mathfrak{X}}/W(R)}^{2}),$$

hence (*V*, id) is injective in the above diagram. This fact together with the injectivity of  $\hat{\alpha}$  in diagram (4.4.3) imply that *V* is injective on  $H^1(X, W\Omega^1_{X/R})$  and coker $V \cong H^1(X, \Omega^1_{X/R})$ . To finish the proof of Proposition 4.1 in the case i = 1, j = 1, we can apply [22, Lemma 47]. Note that we also have  $(P, Q, F, F_1) = \mathbb{D}(\Phi^{\acute{e}t}_{X/R})$ , the display of the étale part of the extended formal Brauer group  $\Phi_{X/R}$ , and  $\mathbb{D}(\Phi^{\acute{e}t}_{X/R}) \xrightarrow{\sim} H^1(X, W\Omega^1_{X/R})$  where the map is given by the composite map

$$\mathbb{D}(\Phi^{\acute{e}t}_{X/R}) \to \mathbb{H}^2(X, W\Omega^{\ge 1}_{X/R}) \to H^1(X, W\Omega^1_{X/R})\,.$$

[22, Lemma 47] implies that  $\mathbb{H}^2(X, W\Omega_{X/R}^{\geq 1}) \to H^1(X, W\Omega_{X/R}^1)$  is surjective, hence we get a commutative diagram

$$\begin{aligned} H^{1}(X, W\Omega^{2}) & \longleftrightarrow \mathbb{H}^{2}(X, W\Omega^{1} \xrightarrow{a} W\Omega^{2}) \longrightarrow H^{2}(X, W\Omega^{1}) \longrightarrow H^{2}(X, W\Omega^{2}) \longrightarrow \mathbb{H}^{3}(X, W\Omega^{1} \xrightarrow{a} W\Omega^{2}) \\ & \uparrow \\ & \uparrow \\ H^{1}(X, W\Omega^{2}) & \hookrightarrow \mathbb{H}^{2}(X, W\Omega^{1} \xrightarrow{dV} W\Omega^{2}) \longrightarrow H^{2}(X, W\Omega^{1}) \longrightarrow H^{2}(X, W\Omega^{2}) \longrightarrow \mathbb{H}^{3}(X, W\Omega^{1} \xrightarrow{dV} W\Omega^{2}) \end{aligned}$$

We have already seen that (V, id) is injective in cohomological degree 2; a similar argument shows that (V, id) is injective in cohomological degree 3 as well. Then the above diagram implies that V is injective on  $H^2(X, W\Omega^1_{X/R})$  and its cokernel is  $H^2(X, \Omega^1_{X/R})$ .

Let  $Alb(p)^{0}(1)$  be the (twisted) connected component of the *p*-divisible group associated to the Albanese scheme. It is known that this is the Cartier dual of the étale *p*-divisible group  $\operatorname{Pic}_{X/R}^{0}(p)^{\text{ét}}$ . Let  $\mathbb{D}(Alb(p)^{0})(1)$  be the associated display. Under the Poincaré duality pairing

$$H^1_{\operatorname{cris}}(X/W(R)) \times H^3_{\operatorname{cris}}(X/W(R))(2) \to W(R),$$

the dual of the map

$$\mathbb{D}(\operatorname{Pic}^{0}_{X/R}(p)^{\text{\'et}}) \to H^{1}_{\operatorname{cris}}(X/W(R))$$

is the map

$$H^3_{\text{cris}}(X/W(R))(2) \to \mathbb{D}(\mathcal{A}lb(p)^0)(1)$$

hence  $\mathbb{D}(Alb(p)^0)(-1)$  is a direct summand of

$$H^{3}_{\operatorname{cris}}(X/W(R)) \cong \mathbb{H}^{2}(X, W\Omega^{1}_{X/R} \xrightarrow{d} W\Omega^{2}_{X/R}).$$

It induces a map

$$\mathbb{D}(\mathcal{A}lb(p)^0)(-1) \to H^2(X, W\Omega^1_{X/R}),$$

where the Frobenius on the right is induced by the crystalline Frobenius, hence by pF, where F is the Frobenius map on  $W\Omega^1_{X/R}$ . By an analogous argument as for  $H^i(X, W\Omega^1_{X/R})$  (i = 0, 1), we get a homomorphism

$$\varsigma : \mathbb{D}(\mathcal{A}lb(p)^0) \to H^2(X, W\Omega^1_{X/P})$$

where  $\mathbb{D}(\mathcal{A}lb(p)^0)$  is the multiplicative display associated to  $\mathcal{A}lb(p)^0$  and the Frobenius on  $H^2(X, W\Omega^1_{X/R})$  is the one induced by *F* on  $W\Omega^1_{X/R}$ . The induced map

$$\mathbb{D}(\mathcal{A}lb(p)^0)/I_R\mathbb{D}(\mathcal{A}lb(p)^0) \to H^2(X, W\Omega^1_{X/R})/\mathrm{im}\, V \cong H^2(X, \Omega^1_{X/R})$$

is an isomorphism because it is so after base change along  $R \rightarrow k$ , since we are in the ordinary case. Hence  $\varsigma$  is an isomorphism by Lemma 4.3.

Now we can complete the proof of Theorem 1.2:

Under the duality of  $H_{\text{cris}}^1$  and  $H_{\text{cris}}^3$ , we get a direct summand decomposition

$$H^3_{\operatorname{cris}}(X/W(R)) = H^2(X, W\Omega^1_{X/R}) \oplus \mathbb{D}(\mathcal{A}lb(p)^{\operatorname{\acute{e}t}})(-1)$$

$$(4.4.4)$$

$$= \mathbb{D}(\mathcal{A}lb(p)^{0})(-1) \oplus \mathbb{D}(\mathcal{A}lb(p)^{\acute{e}t}(-1))$$
(4.4.5)

of Dieudonné modules, where  $Alb(p)^{\acute{e}t}(1)$  is the Cartier dual of  $\operatorname{Pic}^{0}_{X/R}(p)^{0}$ . Since the cup product of  $H^{1}(X, W\mathcal{O}_{X})$  with  $H^{2}(X, W\Omega^{1}_{X/R})$  vanishes, we get an induced map

$$\mathbb{D}(\mathcal{A}lb(p)^{\text{\'et}})(-1) \to H^1(X, W\Omega^2_{X/R}),$$

where the Frobenius on the right is induced by  $p^2 F$ . It is clear that this map is an isomorphism too.

For  $H_{cris}^2$ , we can follow the argument in the proof of [22, Theorem 40] to get the Hodge–Witt decomposition

$$H^2_{\mathrm{cris}}(X/W(R)) = H^2(X, W\mathcal{O}_X) \oplus H^1(X, W\Omega^1_{X/R}) \oplus H^0(X, W\Omega^2_{X/R})$$

into a direct sum of displays associated to  $\widehat{Br}_{X/R}$ ,  $\Phi_{X/R}^{\acute{e}t}$  and the twisted dual  $\widehat{Br}_{X/R}^*$ (-1). This finishes the proof of Theorem 1.2.

## 5 HODGE-WITT DECOMPOSITION FOR ABELIAN SCHEMES

For abelian schemes, we reformulate Proposition 4.1 as follows:

**Proposition 5.1.** Let A be an abelian scheme over Spec R, with  $d = \dim A < p$ , such that the closed fibre is ordinary. Fix a pair (i, j) with  $0 \le i, j \le d$ . Then we have:

(i) there is an exact sequence

$$0 \to H^{i}(A, W\Omega^{j}_{A/R}) \xrightarrow{V} H^{i}(A, W\Omega^{j}_{A/R}) \to H^{i}(A, \Omega^{j}_{A/R}) \to 0$$

induced by the action of V on  $W\Omega^{j}_{A/R}$ ;

(ii) we have canonical isomorphisms

$$\bigwedge^{j}(H^{0}(A, W\Omega^{1}_{A/R})) \otimes \bigwedge^{i}(H^{1}(A, W\mathcal{O}_{A})) \xrightarrow{\varsigma} H^{i}(A, W\Omega^{j}_{A/R})$$

compatible with the Frobenius action  $\bigwedge^{j} F \otimes \bigwedge^{i} F$  on the left and the Frobenius induced by F on  $W\Omega^{j}_{A/R}$  on the right, such that there are isomorphisms induced by the maps  $\beta_{r}$  in Section 3

$$H^{i}(A, W\Omega^{j}_{A/R}) \cong H^{i}(\mathcal{A}, \Omega^{j}_{\mathcal{A}/W(R)})$$

with isomorphisms

$$VH^{i}(A, W\Omega^{j}_{A/R}) \cong I_{R}H^{i}(\mathcal{A}, \Omega^{j}_{\mathcal{A}/W(R)})$$

In particular,

$$(P,Q,F,F_1) = (H^i(A,W\Omega^j_{A/R}), VH^i(A,W\Omega^j_{A/R}), F, V^{-1})$$

is a multiplicative display.

*Proof.* We prove this by induction on j, the case j = 0 having already been covered. Assume that the proposition holds for all j < r and all i. Consider the exact sequence

$$0 \longrightarrow \mathbb{H}^{i-1}(A, N^{r+1}W\Omega_{A/R}^{\geq 1}) \longrightarrow \mathbb{H}^{i}(A, N^{r+1}W\Omega_{A/R}^{\bullet}) \longrightarrow H^{i}(A, W\mathcal{O}_{A}) \longrightarrow 0$$

$$\uparrow \cong \qquad \cong \qquad \cong \left[ {}^{F}\beta_{0} \\ \mathbb{H}^{i}(\mathcal{A}, \mathcal{F}^{r+1}\Omega_{\mathcal{A}/W(R)}^{\bullet}) \longrightarrow I_{R}H^{i}(\mathcal{A}, \mathcal{O}_{A}) \right]$$

where the last map is surjective because the  $E_1$ -hypercohomology spectral sequence associated to  $\mathcal{F}^{r+1}\Omega^{\bullet}_{\mathcal{A}/W(R)}$  degenerates. The same argument applies to the previous cohomological degree i-1, hence the first map is injective, and we have

$$\mathbb{H}^{i-1}(A, N^{r+1}W\Omega_{A/R}^{\geq 1}) \cong \mathbb{H}^{i-1}(\mathcal{A}, \mathcal{F}^{r+1}\Omega_{\mathcal{A}/W(R)}^{\geq 1})$$

By induction, one proves that for all  $s \leq r$ , we have

$$\mathbb{H}^{i-s}(A, N^{r+1}W\Omega_{A/R}^{\geq s}) \cong \mathbb{H}^{i-s}(\mathcal{A}, \mathcal{F}^{r+1}\Omega_{\mathcal{A}/W(R)}^{\geq s}).$$

Indeed, we have an exact sequence

and hence we get

$$\mathbb{H}^{i-(s+1)}(A, N^{r+1}W\Omega_{A/R}^{\geq s+1}) \cong \mathbb{H}^{i-(s+1)}(\mathcal{A}, \mathcal{F}^{r+1}\Omega_{A/W(R)}^{\geq s+1}).$$

We conclude under this isomorphism the map

$$\mathbb{H}^{i}(A, N^{r+1}W\Omega_{A/R}^{\geqslant r}) \xrightarrow{(V, \mathrm{id})} \mathbb{H}^{i}(A, W\Omega_{A/R}^{\geqslant r})$$

corresponds to

$$\mathbb{H}^{i}(\mathcal{A},\mathcal{F}^{r+1}\Omega_{\mathcal{A}/W(R)}^{\geqslant r})\to\mathbb{H}^{i}(\mathcal{A},\Omega_{\mathcal{A}/W(R)}^{\geqslant r})$$

and hence is injective and the cokernel is isomorphic to  $H^i(A, \Omega^r_{A/R})$  (see (3.0.12)).

Now consider the diagram

The injectivity of (V, id) in cohomological degrees *i* and *i* + 1 implies that *V* is injective on  $H^i(A, W\Omega^r_{A/R})$  and has cokernel  $H^i(A, \Omega^r_{A/R})$  as desired.

For the second part of the proposition, we have a commutative diagram (compare the cases j = 0 and i = 0, j = 1)

$$\begin{split} & \bigwedge^{j} H^{0}(A, W\Omega^{1}_{A/R}) \otimes \bigwedge^{i} H^{1}(A, W\mathcal{O}_{A}) \overset{\varsigma}{\longrightarrow} H^{i}(A, W\Omega^{j}_{A/R}) \longrightarrow H^{i}(A, \Omega^{j}_{A/R}) \\ & \simeq & & \\ & \swarrow & & \\ & \bigwedge^{j} H^{0}(\mathcal{A}, \Omega^{1}_{\mathcal{A}/W(R)}) \otimes \bigwedge^{i} H^{1}(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \overset{\simeq}{\longrightarrow} H^{i}(\mathcal{A}, \Omega^{j}_{\mathcal{A}}), \end{split}$$

where the horizontal maps are cup products in cohomology.

We define  $(P, Q = I_R P, F, F_1)$  to be the multiplicative display given by setting  $P = \bigwedge^j H^0(A, W\Omega^1_{A/R}) \otimes \bigwedge^i H^1(A, W\mathcal{O}_A)$ . Since  $V \circ \varsigma \circ F_1 = \varsigma|_{I_R P}$ , we get the induced homomorphism of free *R*-modules of rank  $h^{j,i}$ 

$$\overline{\varsigma}$$
 :  $P/I_R P \to H^i(A, W\Omega^j_{A/R}) / \operatorname{im} V \cong H^i(A, \Omega^j_{A/R}),$ 

which coincides with the cup product map

$$\bigwedge^{j} H^{0}(A, \Omega^{1}_{A/R}) \otimes \bigwedge^{i} H^{1}(A, \mathcal{O}_{A}) \to H^{i}(A, \Omega^{j}_{A/R}).$$

Since for abelian varieties we have  $H^s_{dR}(A/R) = \bigwedge^s H^1_{dR}(A/R)$ , the map  $\overline{\varsigma}$  is an isomorphism. Therefore,  $\varsigma$  is an isomorphism by Lemma 4.3.

Since the constructions of the maps  $\beta_r$  and  ${}^F\beta_r$  in Section 3 are compatible with taking tensor products of complexes, the isomorphism  $H^i(\mathcal{A}, \Omega^j_{\mathcal{A}/W(R)}) \xrightarrow{\sim} H^i(\mathcal{A}, W\Omega^j_{\mathcal{A}/R})$  obtained above coincides with the canonical map  $\beta_j$ .

This finishes the proof of the Hodge–Witt decomposition of  $H^s_{cris}(A/W(R))$ , that is, Theorem 1.4 holds. Note that the isomorphism

$$\beta_j : H^i(\mathcal{A}, \Omega^j_{\mathcal{A}/W(R)}) \to \mathbb{H}^i(\mathcal{A}, \Omega^{\geq j}_{\mathcal{A}/W(R)}) \to \mathbb{H}^i(\mathcal{A}, W\Omega^{\geq j}_{\mathcal{A}/R}) \to H^i(\mathcal{A}, W\Omega^j_{\mathcal{A}/R})$$

splits the surjection  $\mathbb{H}^{i}(A, W\Omega_{A/R}^{\geq j}) \twoheadrightarrow H^{i}(A, W\Omega_{A/R}^{j})$  for all *j* and hence induces the Hodge–Witt decomposition on crystalline cohomology.

#### 6 | HODGE-WITT DECOMPOSITION FOR *n*-FOLDS

In this section, *k* is an algebraically closed field of characteristic p > 0. Let *S* be a smooth formal scheme over Spf W(k), and let  $f : \mathcal{X} \to S$  be a smooth and proper family of *n*-folds, where n < p. Suppose that the condition (B) from the introduction is satisfied. In this section, we shall prove the following theorem (Theorem 1.6 from the introduction):

**Theorem 6.1.** Let  $X_k := \mathcal{X} \times_S \text{Spec } k$  be the fibre over an ordinary k-point  $\text{Spec } k \to S$ . Then for any commutative diagram



where R is an Artinian local ring with residue field k, the deformation  $X := \mathcal{X} \times_S \text{Spec } R$  of  $X_k$  admits a Hodge–Witt decomposition of  $H^s_{\text{cris}}(X/W(R))$  as displays in all degrees  $0 \le s \le 2n$ .

*Remark* 6.2. Since our techniques are crystalline in nature, the smoothness of S seems to be indispensable in our approach. It would be interesting to understand relative Hodge–Witt decompositions for varieties with obstructed deformations.

Before giving the proof of Theorem 6.1, we shall recall the theory of ordinary Hodge *F*-crystals from [7]. Let  $A = W(k)[[T_1, ..., T_r]]$  and  $A_0 = k[[T_1, ..., T_r]]$ , for some  $r \ge 0$ . A *crystal over*  $A_0$  is a finitely generated free *A*-module *H* together with an integrable and topologically nilpotent connection

$$\nabla : H \to H \otimes_A \Omega^1_{A/W(k)}$$

A ring endomorphism  $\phi : A \to A$  which restricts to the Frobenius on W(k) is called a *lift of Frobenius* if it reduces modulo p to the Frobenius endomorphism  $\sigma : A_0 \to A_0$  which sends x to  $x^p$ . A crystal  $(H, \nabla)$  over  $A_0$  is called an *F*-crystal over  $A_0$  if for every lift of Frobenius  $\phi : A \to A$ , there is a given A-module homomorphism  $F(\phi) : \phi^*H \to H$  which is horizontal for  $\nabla$ , that is, the square

commutes, and such that  $F(\phi) \otimes \mathbb{Q}_p$  is an isomorphism. For any two liftings of Frobenius  $\phi, \psi$ :  $A \to A$ , we also require that  $F(\psi) \circ \chi(\phi, \psi) = F(\phi)$ , where  $\chi(\phi, \psi) : \phi^*H \xrightarrow{\sim} \psi^*H$  is the usual isomorphism of *A*-modules coming from parallel transport with respect to the connection  $\nabla$ :

$$\chi(\phi,\psi): \phi^*H \to \psi^*H$$
$$x \mapsto \sum_{m_1,\dots,m_r \ge 0} \prod_{j=1}^r \frac{p^{m_j}}{m_j} \left(\frac{\phi(T_j) - \psi(T_j)}{p}\right)^{m_j} \psi^*(D_1^{m_1} \circ \cdots \circ D_n^{m_n}(x)).$$

Here the operator  $D_j : H \to H$  denotes  $(\frac{d}{dT_j} \otimes 1) \circ \nabla$ . An *F*-crystal  $(H, \nabla, F)$  is said to be a *unit F*-crystal if  $F(\phi)$  is an isomorphism for some (hence any) lift of Frobenius  $\phi : A \to A$ .

Let *H* be an *F*-crystal over *A* (we henceforth drop the  $\nabla$  and *F* from the notation), and write  $H_0 := H \otimes_A A_0$ . Given a lift of Frobenius  $\phi : H \to H$ , define a decreasing filtration Fil<sup>\*</sup> $H_0$  and

an increasing filtration  $\operatorname{Fil}^{*}H_{0}$  of  $H_{0}$  by  $A_{0}$ -submodules as follows:

Fil<sup>*i*</sup>
$$H_0 := \{x \in H_0 : \exists y \in H \text{ with } y \equiv x \mod p \text{ and } F(\phi)\phi^*y \in p^iH\}$$
  
Fil<sub>*i*</sub> $H_0 := \{x \in H_0 : \exists y \in H \text{ with } y \equiv x \mod p \text{ and } p^iy \in \operatorname{im} F(\phi)\}.$ 

These are the *Hodge* and *conjugate filtrations* of  $H_0$ , respectively. It is clear that they are finite, separated and exhaustive, and that they are independent of the choice of lift of Frobenius  $\phi$  [7, § 1.3]. Let  $\nabla_0 := \nabla \mod p$  be the connection on  $H_0$  induced by  $\nabla$ . Then the Hodge filtration satisfies Griffiths transversality

$$\nabla_0 \operatorname{Fil}^i H_0 \subset \operatorname{Fil}^{i-1} H_0 \otimes_{A_0} \Omega^1_{A_0/k}$$

and the conjugate filtration is horizontal for  $\nabla_0$ 

$$\nabla_0 \operatorname{Fil}_i H_0 \subset \operatorname{Fil}_i H_0 \otimes_{A_0} \Omega^1_{A_0/k}$$

An *F*-crystal *H* over  $A_0$  is called *ordinary* if the graded  $A_0$ -module gr<sup>•</sup> $H_0$  associated to the Hodge filtration (equivalently the graded  $A_0$ -module gr<sub>•</sub> $H_0$  associated to the conjugate filtration) is free, and the Hodge and conjugate filtrations are opposite, that is if

$$H_0 = \operatorname{Fil}_i H_0 \oplus \operatorname{Fil}^{i+1} H_0$$

for every *i*. It is shown in [7, Proposition 1.3.2] that *H* is ordinary if and only there exists a (unique) increasing filtration  $U_{\bullet}$  (the *conjugate filtration*) of *H* by sub-*F*-crystals such that

$$U_i \otimes_A A_0 = \operatorname{Fil}_i H_0$$

for every *i*, and such that

$$\operatorname{gr}_{i}H := U_{i}/U_{i-1} \simeq V_{i}(-i)$$

is the (-i)-fold Tate twist of a unit *F*-crystal.

A Hodge F-crystal over  $A_0$  is an F-crystal H over  $A_0$  together with a finite decreasing filtration Fil<sup>\*</sup>H (the Hodge filtration) of H by free A-submodules which lifts the Hodge filtration on  $H_0$ , and satisfies Griffiths transversality:

$$\operatorname{Fil}^{i} H \otimes_{A} A_{0} = \operatorname{Fil}^{i} H_{0}, \qquad \qquad \nabla \operatorname{Fil}^{i} H \subset \operatorname{Fil}^{i-1} H \otimes_{A} \Omega^{1}_{A/W(k)}.$$

A Hodge *F*-crystal  $(H, \text{Fil}^{+}H)$  is said to be *ordinary* if its underlying *F*-crystal *H* is ordinary. It is shown in [7, Proposition 1.3.6] that the conjugate and Hodge filtrations of an ordinary Hodge *F*-crystal  $(H, \text{Fil}^{+}H)$  are opposite, that is

$$H = U_i \oplus \operatorname{Fil}^{i+1} H$$

for every *i*, and one has a Hodge decomposition [7, (1.3.6.1)]

$$H = \bigoplus_i H^i$$

with  $H^i = U_i \cap \operatorname{Fil}^i H$ .

*Remark* 6.3. The induced inclusions  $H^i \subset U_i/U_{i-1}$  and  $H^i \subset \operatorname{Fil}^i H/\operatorname{Fil}^{i+1} H$  must then be isomorphisms, so

$$H^i \cong U_i/U_{i-1} \cong \operatorname{gr}^i H$$
.

We now prove Theorem 6.1:

*Proof.* After possibly taking the formal completion of *S* at a closed point, we may and do assume that *S* is S = Spf A where  $A = W(k)[[T_1, ..., T_r]]$ . Let  $\mathcal{X}_0 := \mathcal{X} \times_A A_0$ .

The Gauss–Manin connection of the family  $f : \mathcal{X} \to S$ 

$$\nabla : H^s_{\mathrm{dR}}(\mathcal{X}/A) \to H^s_{\mathrm{dR}}(\mathcal{X}/A) \otimes_A \Omega^1_{A/W(k)}$$

gives the crystalline cohomology  $H^s_{cris}(\mathcal{X}_0/A) \simeq H^s_{dR}(\mathcal{X}/A)$ , together with its crystalline Frobenius, the structure of an *F*-crystal over  $A_0$ . The Hodge filtration Fil<sup>\*</sup> $H^s_{dR}(\mathcal{X}/A)$  satisfies Griffiths transversality and thus the pair  $(H^s_{dR}(\mathcal{X}/A), Fil^*H^s_{dR}(\mathcal{X}/A))$  is a Hodge *F*-crystal over  $A_0$ . Let  $e_0 : A_0 \to k$  denote the augmentation map. Since the closed fibre  $X_k$  is ordinary, the Newton and Hodge polygons of  $e_0^*H^s_{dR}(\mathcal{X}/A)) \simeq H^s_{cris}(X_k/W(k))$  coincide. Therefore,  $(H^s_{dR}(\mathcal{X}/A), Fil^*H^s_{dR}(\mathcal{X}/A))$  is an ordinary Hodge *F*-crystal by [7, Proposition 1.3.2], and hence we have the conjugate filtration  $U_{\bullet}$  of  $H^s_{cris}(\mathcal{X}_0/A)$  lifting the conjugate filtration Fil\_ $H^s_{cris}(\mathcal{X}_0/A_0)$ , and such that  $U_i/U_{i-1} \simeq V_i(-i)$  is the (-i)-fold Tate twist of a unit *F*-crystal. Moreover, the filtration  $U_{\bullet}$  is opposite to the Hodge filtration Fil^\*H^s\_{dR}(\mathcal{X}/A).

Now we evaluate this filtration of F-crystals on W(A) and get a filtration of W(A)-modules

$$0 \subset U_{0W(A)} \subset U_{1W(A)} \subset \dots \subset U_{nW(A)} = H^s_{cris}(\mathcal{X}_0/W(A))$$

Note that the Frobenius on W(A) is a lifting of the Frobenius on  $A_0 = W(A)/\langle {}^V W(A), p \rangle$ . Then  $U_{iW(A)}/U_{i-1W(A)}$  is a free W(A)-module such that the crystalline Frobenius induces the (-i)-fold Tate twist of the multiplicative (unit-) crystal evaluated at W(A) on which F acts as an F-linear isomorphism. If U is a unit F-crystal over  $A_0$  and we evaluate it at W(A), then F:  $W(A) \otimes_{\sigma, W(A)} U_{W(A)} \to U_{W(A)}$  is an isomorphism. By definition,

$$(P_0 = U_{W(A)}, P_1 = {}^V W(A) P_0, F, F_1 : P_1 \to P_0)$$

with  $F_1$ :  ${}^V \xi \alpha \mapsto \xi F \alpha$  is then a multiplicative display. If U is the Tate-twist of a unit F-crystal, then  $U_{W(A)}$  is the Tate-twist of a multiplicative display. We can now take the base change of F-crystals, respectively, of displays, with respect to the map  $A \to R$  to get a filtration

$$0 \subset U_{0W(R)} \subset U_{1W(R)} \subset \dots \subset U_{nW(R)} = H^s_{cris}(X/W(R))$$

of *F*-crystals evaluated at W(R), such that the successive quotients are Tate-twists of multiplicative displays.

We already know that (by Remark 6.3)

$$U_{0W(R)} \simeq H^{s}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$$

and we claim that the composite map

$$\varsigma : U_{0W(R)} \to H^s_{cris}(X/W(R)) \to H^s(X, W\mathcal{O}_X)$$

is an isomorphism. Note that the diagram



commutes by construction. To see the claim, let  $\mathfrak{X}$  denote the ind-scheme over Spec  $W_{\bullet}(R)$  arising from the compatible family of liftings  $X_n$ /Spec  $W_n(R)$  constructed using the canonical map  $A \rightarrow W(A) \rightarrow W_n(R)$  as in Remark 1.7(a). Then for each *i*, we have maps

$$H^{i}_{\mathrm{cris}}(X/W(R)) \cong H^{i}_{\mathrm{dR}}(\mathfrak{X}/W(R)) \twoheadrightarrow H^{i}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \twoheadrightarrow H^{i}(X, \mathcal{O}_{X}),$$

where the final arrow is surjective because de Rham cohomology and Hodge cohomology commute with base change. The composition factors through the map  $H^i(X, W\mathcal{O}_X) \to H^i(X, \mathcal{O}_X)$ induced by  $W\mathcal{O}_X \to \mathcal{O}_X$ . Hence the cohomology sequence coming from the short exact sequence

$$0 \to W\mathcal{O}_X \xrightarrow{V} W\mathcal{O}_X \to \mathcal{O}_X \to 0$$

splits into short exact sequences

$$0 \to H^{i}(X, W\mathcal{O}_{X}) \xrightarrow{V} H^{i}(X, W\mathcal{O}_{X}) \to H^{i}(X, \mathcal{O}_{X}) \to 0$$

for each *i*. In particular, we see that the map  $\varsigma$  reduces modulo  $I_R$  to the map

$$\overline{\varsigma}$$
:  $U_{0W(R)}/I_R U_{0W(R)} \to H^s(X, W\mathcal{O}_X)/\operatorname{im} V \simeq H^s(X, \mathcal{O}_X)$ .

As in the previous sections, we see that  $\overline{\zeta}$  is an isomorphism by reducing to the case R = k, where  $H^s(X_k, W\mathcal{O}_{X_k})$  is the slope 0 part in  $H^s_{\text{cris}}(X_k/W(k))$ . By Lemma 4.3, we conclude that  $\zeta$  is an isomorphism, hence  $\zeta$  imposes a multiplicative display structure on  $H^s(X, W\mathcal{O}_X)$ . It is clear that under the isomorphism  $U_0 \cong H^s(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ , the composite map

$$H^{s}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to H^{s}_{dR}(\mathfrak{X}/W(R)) \simeq H^{s}_{cris}(X/W(R)) \to H^{s}(X, W\mathcal{O}_{X})$$

agrees with the map  $\beta_s$ .

Since  $H^{s}(X, W\mathcal{O}_{X})$  is a direct summand of  $H^{s}_{cris}(X/W(R))$ , we get an induced map

$$\varsigma: P_1 = U_{1W(R)}/U_{0W(R)} \to \mathbb{H}^s(X, 0 \to W\Omega^1_{X/R} \xrightarrow{d} \cdots \xrightarrow{d} W\Omega^s_{X/R}) \to H^{s-1}(X, W\Omega^1_{X/R})$$

The diagram

is commutative because it already commutes when X is replaced with the versal deformation  $\mathcal{X}$ . The F on the left, which is the untwist of the crystalline Frobenius, is an F-linear isomorphism because it is defined on a multiplicative (unit-) display.

We claim that we have exact sequences

$$0 \to H^{i}(X, W\Omega^{1}_{X/R}) \xrightarrow{V} H^{i}(X, W\Omega^{1}_{X/R}) \to H^{i}(X, \Omega^{1}_{X/R}) \to 0$$

induced by the action of V on  $W\Omega^1_{X/R}$ , for all *i*. Indeed, consider the commutative diagram

$$0 \longrightarrow \mathbb{H}^{i}(X, N^{2}W\Omega_{X/R}^{\geq 1}) \longrightarrow \mathbb{H}^{i+1}(X, N^{2}W\Omega_{X/R}^{\bullet}) \longrightarrow H^{i+1}(X, W\mathcal{O}_{X}) \longrightarrow 0$$

$$\uparrow \cong \qquad \cong \uparrow^{F}\beta_{0}$$

$$\mathbb{H}^{i+1}(\mathfrak{X}, \mathcal{F}^{2}\Omega_{\mathfrak{X}/W(R)}^{\bullet}) \longrightarrow I_{R}H^{i+1}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$$

(The maps  ${}^{F}\beta_{r}$  exist in this setting - see Remark 2.3 and Section 3). The last map is surjective because the  $E_1$ -hypercohomology spectral sequence associated to  $\mathcal{F}^2\Omega^*_{\mathcal{X}/W(R)}$  degenerates. Applying the same argument in cohomological degree *i*, we see that the first map is injective. Therefore,

$$\mathbb{H}^{i}(X, N^{2}W\Omega_{X/R}^{\geq 1}) \cong \mathbb{H}^{i}(\mathfrak{X}, \mathcal{F}^{2}\Omega_{\mathfrak{X}/W(R)}^{\geq 1}).$$

Hence the composite map

$$\mathbb{H}^{i}(X, N^{2}W\Omega_{X/R}^{\geq 1}) \xrightarrow{(V, \mathrm{id})} \mathbb{H}^{i}(X, W\Omega_{X/R}^{\geq 1}) \to \mathbb{H}^{i}(X, W\Omega_{X/R}^{\bullet})$$

can be identified under the comparison between de Rham–Witt and de Rham cohomology with the composite map (see (3.0.12))

$$\mathbb{H}^{i}(\boldsymbol{\mathfrak{X}}, \mathcal{F}^{2}W\Omega_{\boldsymbol{\mathfrak{X}}/W(R)}^{\geq 1}) \to \mathbb{H}^{i}(\boldsymbol{\mathfrak{X}}, \Omega_{\boldsymbol{\mathfrak{X}}/W(R)}^{\geq 1}) \to H^{i}_{\mathrm{dR}}(\boldsymbol{\mathfrak{X}}/W(R)),$$

which is injective due to the degeneracy of the  $E_1$ -hypercohomology spectral sequences. We conclude that the map

$$\mathbb{H}^{i}(X, N^{2}W\Omega_{X/R}^{\geq 1}) \xrightarrow{(V, \mathrm{id})} \mathbb{H}^{i}(X, W\Omega_{X/R}^{\geq 1})$$

is injective and has cokernel  $H^i(X, \Omega^1_{X/R})$ . Now consider the diagram

$$\begin{split} \mathbb{H}^{i}(W\Omega^{\geq 2}) & \longrightarrow \mathbb{H}^{i}(W\Omega^{\geq 1}) & \longrightarrow H^{i}(W\Omega^{1}) & \longrightarrow \mathbb{H}^{i+1}(W\Omega^{\geq 2}) & \longrightarrow \\ & = \left[ \operatorname{id} \qquad \qquad \uparrow (V, \operatorname{id}) \qquad \qquad \downarrow V \qquad = \left[ \operatorname{id} \qquad \qquad \\ \mathbb{H}^{i}(W\Omega^{\geq 2}) & \longrightarrow \mathbb{H}^{i}(N^{2}W\Omega^{\geq 1}) & \longrightarrow H^{i}(W\Omega^{1}) & \longrightarrow \mathbb{H}^{i+1}(W\Omega^{\geq 2}) & \longrightarrow \\ \end{split}$$

The injectivity of (V, id) in cohomological degrees *i* and *i* + 1 implies that *V* is injective on  $H^i(X, W\Omega^1_{X/R})$  and has cokernel  $H^i(X, \Omega^1_{X/R})$  as desired.

As before, we see that the induced map

$$\overline{\varsigma}$$
:  $P_1/I_RP_1 \to H^{s-1}(X, W\Omega^1_{X/R})/\operatorname{im} V \simeq H^{s-1}(X, \Omega^1_{X/R})$ 

is an isomorphism by reducing to the case R = k, where  $H^{s-1}(X_k, W\Omega^1_{X_k/k})$  is the slope 1 part in  $H^s_{\text{cris}}(X_k/W(k))$ . By Lemma 4.3, we conclude that  $\varsigma$  is an isomorphism and that  $P_1 \cong H^{s-1}(\mathfrak{X}, \Omega^1_{\mathfrak{X}/W(R)})$  (see Remark 6.3). It is clear that the map  $\zeta$  agrees with the map

$$\beta_1 : H^{s-1}(\mathfrak{X}, \Omega^1_{\mathfrak{X}/W(R)}) \to H^{s-1}(X, W\Omega^1_{X/R})$$

and

$${}^{F}\beta_{1}: I_{R}H^{s-1}(\boldsymbol{\mathfrak{X}}, \Omega^{1}_{\boldsymbol{\mathfrak{X}}/W(R)}) \to H^{s-1}(X, W\Omega^{1}_{X/R})$$
  
$$z \mapsto F(\beta_{1}(z))$$

is a bijection.

Now consider the composite map

$$\varsigma: P_2 = U_{2W(R)}/U_{1W(R)} \to \mathbb{H}^s(X, 0 \to 0 \to W\Omega^2_{X/R} \xrightarrow{d} \cdots \xrightarrow{d} W\Omega^s_{X/R}) \to H^{s-2}(X, W\Omega^2_{X/R}).$$

The same argument as before shows that the square

is commutative, where F on the left is again the untwist of the crystalline Frobenius.

We claim that we have exact sequences

$$0 \to H^{i}(X, W\Omega^{2}_{X/R}) \xrightarrow{V} H^{i}(X, W\Omega^{2}_{X/R}) \to H^{i}(X, \Omega^{2}_{X/R}) \to 0$$

induced by the action of V on  $W\Omega^2_{X/R}$ , for all *i*. We proceed in a similar manner as before: Consider the commutative diagram

The last map is surjective because the  $E_1$ -hypercohomology spectral sequence associated to  $\mathcal{F}^3\Omega^{\bullet}_{\mathfrak{X}/W(R)}$  degenerates, and the same argument in cohomological degree *i*, shows the first map

is injective. Therefore

$$\mathbb{H}^{i}(X, N^{3}W\Omega_{X/R}^{\geq 1}) \cong \mathbb{H}^{i}(\mathfrak{X}, \mathcal{F}^{3}\Omega_{\mathfrak{X}/W(R)}^{\geq 1})$$

for each *i*.

Considering the commutative diagram

then shows that

$$\mathbb{H}^{i}(X, N^{3}W\Omega_{X/R}^{\geq 2}) \cong \mathbb{H}^{i}(\mathfrak{X}, \mathcal{F}^{3}\Omega_{\mathfrak{X}/W(R)}^{\geq 2})$$

for each *i*, as well. Finally, the diagram

$$\begin{split} \mathbb{H}^{i}(W\Omega^{\geq 3}) & \longrightarrow \mathbb{H}^{i}(W\Omega^{\geq 2}) & \longrightarrow H^{i}(W\Omega^{2}) & \longrightarrow \mathbb{H}^{i+1}(W\Omega^{\geq 3}) & \longrightarrow \\ & = \left[ id & & \uparrow (V, id) & & \uparrow V & = \uparrow id \\ & \mathbb{H}^{i}(W\Omega^{\geq 3}) & \longrightarrow \mathbb{H}^{i}(N^{3}W\Omega^{\geq 2}) & \longrightarrow H^{i}(W\Omega^{2}) & \longrightarrow \mathbb{H}^{i+1}(W\Omega^{\geq 3}) & \longrightarrow \end{split}$$

and injectivity of (V, id) in cohomological degrees i and i + 1 (compare (3.0.12)) implies that V is injective on  $H^i(X, W\Omega^2_{X/R})$  and has cokernel  $H^i(X, \Omega^2_{X/R})$  as desired.

By the same argument as before, we see that the induced map

$$\overline{\varsigma} : P_2/I_R P_2 \to H^{s-2}(X, W\Omega^2_{X/R}) / \operatorname{im} V \simeq H^{s-2}(X, \Omega^2_{X/R})$$

is an isomorphism by reducing to the case R = k, where  $H^{s-2}(X_k, W\Omega_{X_k/k}^2)$  is the slope 2 part in  $H^s_{\text{cris}}(X_k/W(k))$ . By Lemma 4.3, we conclude that  $\varsigma$  is an isomorphism and that  $P_2 \cong H^{s-2}(\mathfrak{X}, \Omega_{\mathfrak{X}/W(R)}^2)$  (by Remark 6.3).

By an induction argument as in the case of abelian schemes, one derives an exact sequence for all i and j

$$0 \to H^{i}(X, W\Omega^{j}_{X/R}) \xrightarrow{V} H^{i}(X, W\Omega^{j}_{X/R}) \to H^{i}(X, \Omega^{j}_{X/R}) \to 0$$

and that the map

$$P_i := U_{iW(R)} / U_{i-1W(R)} \simeq H^{s-i}(\boldsymbol{\mathfrak{X}}, \Omega^i_{\boldsymbol{\mathfrak{X}}}) \xrightarrow{\varsigma} H^{s-i}(X, W\Omega^i_{X/R})$$

is an isomorphism, and hence imposes a multiplicative display structure on  $H^{s-i}(X, W\Omega^i_{X/R})$  and identifies this Hodge–Witt cohomology group as a direct summand in crystalline cohomology. This concludes the proof of Theorem 6.1.

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