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Hybrid predictor-corrector numerical schemes: λ-tracking, sampled-data observers and data assimilation

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Abstract

This paper considers hybrid ordinary differential equation (ODE) solvers for process dynamics constructed by combining standard numerical schemes with standard observers. Specifically, we combine the first-order Euler scheme with a Luenberger observer. The key ideas are to take advantage of available process output information and to switch from the numerical scheme to the process output-driven observer when the numerical scheme alone would produce inadequate results. Within this setup, two tasks emerge: How to choose the observer gain? How to choose the step size in the numerical scheme? Underpinning our approach is a λ tracking-based sampled-data observer that invokes a λ dead zone. This λ tracking observer determines the observer gain and the numerical step-size adaptively. The resulting adaptive hybrid algorithm is a time-stepping numerical scheme. Using a sampled-data observer allows for process measurements to be only available at some discrete times, whilst adaptive tuning allows the gains and sampling times to adjust automatically to each other – rather than both being subjected to designer's choice. Results are illustrated with examples of simulation.

Keywords: Adaptive control; Minimum-phase systems; Robust tracking; Sampled-data control; Hybrid systems;

1. INTRODUCTION

The numerical solution of differential equations is a central tool in the modeling and analysis of dynamic processes. Key to these tools are approximations of derivatives as typified by the first-order Euler approximation Iacchetti et al. [2011]. The idea of combining Euler approximations and observers was introduced in Chapelle et al. [2012] in the context of numerical solution of wave equations. The idea is to take advantage of available process output information to improve the numerical solution. Here, we developed this idea further. Specifically, we switch between the numerical scheme and the observer to make a hybrid numerical scheme – using the numerical scheme when "good enough" and otherwise switching to an observer. The key point is to avoid excessively small (fine scale) numerical step sizes if this can be obviated by using readily available (coarse scale) process information. We used input-to-state stability techniques to establish the stability of the switched system and analyze error convergence for the hybrid numerical scheme. Here we further develop this idea. Specifically, we use an adaptation of the observer gains and sampling period in the hybrid scheme, resulting in numerical algorithms akin to well known predictor-corrector numerical schemes such as Adams-Bashforth Butusov et al. [2020], Lundberg and Poore [1991].

Following Chapelle et al. [2012], we start with the standard, continuous-time Luenberger observer Luenberger [1964]:

$$\begin{aligned} \dot{x}(t) &= Ax(t), \qquad y = Cx(t) \\ \dot{w}(t) &= (A - GC)w(t) + Gy(t)\,, \end{aligned} \tag{1}$$

and apply Euler approximations with step length h to the observer to yield a discrete-time system:

$$\begin{aligned} x_{k+1} &= x_k + hAx_k, \qquad y_k = Cx_k \\ w_{k+1} &= w_k + h(A - GC)w_k + Gy_k\,. \end{aligned} \tag{2}$$

Here, the variables x_k and w_k are numerical approximations of the to-be-computed $x(kh)$ and $w(kh)$. Now replace the numerical x_k and y_k with their actual sampled counterparts $z_k = x(kh)$ and the measured and sampled process output $v_k = y(kh)$ to yield a Euler-based, sampled data observer:

$$\begin{aligned} z_{k+1} &= e^{hA}z_k, \qquad v_k = Cz_k \qquad &\text{(process)} \\ w_{k+1} &= w_k + h(A - GC)w_k + Gv_k\,. \qquad &\text{(observer)} \end{aligned} \tag{3}$$

The observer in (3) is driven by the process samples $v_k = Cz_k = y(kh)$. Crucially, these v_k values are not computed in the numerical scheme, they are samples of the output of the process to-be-computed numerically. So to enact the observer we need to sample the process. In applications,

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process sampling may be costly. Moreover, it might be unnecessary, especially when the Euler scheme is good enough so that these process values can be approximated sufficiently well numerically. This observation suggests an alternative approach whereby we switch between the observer and the standard Euler scheme as follows. Use the standard Euler scheme:

$$x_{k+1} = (I + hA)x_k, \tag{4}$$

When it would work adequately, switch to the sampled-data observer (3) otherwise.

The error $e_k = z_k - w_k$ satisfies

$$\begin{aligned} e_{k+1} &= e^{hA}z_k - (I + h(A - GC))w_k - hGCz_k \\ &= \underbrace{(I + h(A - GC))}_{\text{stable}}\, e_k + \underbrace{\mathcal{O}(h^2)z_k}_{\text{small}} . \end{aligned} \tag{5}$$

Here, the $\mathcal{O}(h^2)z_k$ arises from the quadratic truncation error in the Taylor expansion in h of $e^{hA}z_k$. So, when the Luenberger observer is active, propagated errors are small with error bounds controllable via the feedback matrix $I + h(A - GC)$, the choice of G, and the relative smallness of h. Previous studies have shown how to choose G to minimise propagation errors. But in this setting h, the sampling period, is fixed and sufficiently small. While we find that h is much larger than needed for the stability of Euler, it could still be quite conservative. Now in standard numerical schemes, such conservatism can be overcome by adapting the sampling period, i.e., step size, such as in Adams-Bashforth Butcher [2016], Choi and Laub [1990], Kreiss and Ortiz [2014], Peinado et al. [2010].

So, can we use adaptive techniques here for this Hybrid Euler-observer-based scheme? So, in this case, both G and h would change with time:

$$G = G_k; \qquad h = h_k.$$

Let, $t_{k+1} = t_k + h_k$ and so $z_k = z(t_k)$ with error $e_k = z(t_k) - w_k$. Then, the observer error satisfies:

$$e_{k+1} = (I + h_k(A - G_kC))e_k + \mathcal{O}\left(h_k^2\right) z(t_k)\,. \tag{6}$$

The error dynamics take the form of a stable system with state matrix $I + h_k(A - G_kC)$ driven by a bounded disturbance – leading to input-to-state stability considerations. In this paper, we explore the use of adaptive control – choosing the gain and sampling period adaptively to produce a hybrid "predictor-corrector" numerical scheme. Because of the input-to-state stability setting, the adaptive strategies are based on dead zones and so-called λ trackers.

The paper is organized as follows: In Section 2, we formulate the sampled-data observer, the associated Euler scheme, and the hybrid switching. In Section 3, we study the related λ-tracking-based sampled-data observer, and then we develop the hybrid predictor-corrector numerical scheme with the Lyapunov-based switching and adaptive dead zone. Section 4 contains illustrative simulations, and Section 5 provides a conclusion.

2. HYBRID OBSERVER-BASED NUMERICAL SCHEMES

2.1 A Lyapunov function-based switching condition

Here, we use a Lyapunov-based switching criterion, recalled briefly here as follows:

- In the Luenberger observer part of (3), choose G so that $I + h(A - GC)$ is Schur stable, true, for example, if $A - GC$ is Hurwitz stable and h is small enough, Mareels [1984]. Notably, the size of h can be significantly larger than the size of h needed for $I+hA$ to be Schur stable.
- We then choose P so that $V(x) = w^TPw$ is a discrete-time Lyapunov function for $I+h(A-GC)$. This is the case, for example, if P is a continuous-time Lyapunov function for $A - GC$ and h is small enough.
- Then, we use the Euler scheme so long as the "energy" V decreases along the Euler scheme. More specifically, for a chosen $\gamma < 1$, we use Euler if
$$(w_k)^T(I + hA)^TP(I + hA)w_k \le \gamma\, w_k^TPw_k, \tag{7}$$
but switch to the Luenberger observer otherwise.

Here, P is obtained by solving the discrete-time Lyapunov equation

$$(I + h(A - GC))^TP(I + h(A - GC)) - P = -Q \tag{8}$$

for some chosen $Q > 0$, for example $Q = I$, and h is small enough so that $I + h(A - GC)$ is Schur stable, or G is suitably chosen.

2.2 The observer switching scheme

This leads to the switched scheme:

$$\left.\begin{aligned} &\text{If (7) holds, } w_{k+1} = (I + hA)w_k\,. \\ &\text{If (7) fails, } \quad w_{k+1} = w_k + h(A - GC)w_k + hGv_k\,. \end{aligned}\right\} \tag{9}$$

Note that in (9), the observer is driven by $v_k = Cz_k = y(kh)$. We emphasize that $y(kh)$ is not computed numerically, it is simply the process output $y(t)$ sampled at $t = kh$. For stability analysis purposes we note that

$$v_k = Cz_k \qquad \text{with} \quad z_{k+1} = e^{hA}z_k\,.$$

Theorem 1. ***[Stability of the hybrid scheme]*** *Assume that $\gamma < 1$. Then, the switched scheme* (9) *is input-to-state stable. Specifically,*

$$V(w_{k+1}) \le \alpha V(w_k) + \mathcal{O}(h^2), \tag{10}$$

where

$$\alpha = \max\{\gamma, \beta\}\,,$$

with $\beta < 1$.

Remark 1. *The input-to-state stability described by* (10) *can be used to prove convergence of the scheme.*

Remark 2. *The choices of fixed G and h are interwoven and, using various estimation techniques, may be conservative. So it would be advantageous if these parameters internal to the numerical scheme could be found adaptively, in analogy to the way step sizes are adjusted in adaptive numerical schemes like Adams-Bashforth.*

3. HYBRID OBSERVER-BASED ADAPTIVE NUMERICAL SCHEMES

In the spirit of adaptive step size numerical schemes like Adams-Bashforth, in this section we consider how to adjust, that is adapt, the gain G and sampling periods h in (9). We appeal to adaptive control approaches based on so-called λ-trackers. We divide the section into two parts.

First, we consider a sampled-data observer without switching to the Euler scheme. Then, we combine the resulting λ-tracking observer, through switching, with the numerical scheme as in Bullinger and Allgöwer [2005], Ilchmann and Ryan [1994], Ilchmann and Townley [1999a,b].

3.1 Sampled-data λ-tracking observer

We focus on the observer error equation (6). We choose $G = g_k B$ and still with variable sampling period $h = h_k$. Then we have an observer error

$$e_{k+1} = (I + h_k(A - g_k BC))e_k + \mathcal{O}\left(h_k^2\right) z(t_k)\,. \tag{11}$$

We choose B so that (A, B, C) is the minimum phase, with a known "sign" of the high-frequency gain – say CB has only right half plane eigenvalues, Owens [1996]. We can now appeal to adaptive control techniques to adjust the gain g_k and the sampling period h_k to guarantee convergence of the error, but based only on process output information. Because of the already discussed input-to-state stability conditions, we invoke λ-tracking style dead-zones in the adaptation.

Let, $\theta_k = Ce(t_k) = y(t_k) - Cw_k$. Note that θ_k is determined by the state of the numerical scheme and the process output – both available for implementation of the numerical scheme.

The gain g_k and sampling period h_k are generated by:

$$\begin{aligned} h_k &= \frac{1}{g_k \log g_k}, \quad t_{k+1} = t_k + h_k, \quad k \in \mathbb{N}_0, \\ g_{k+1} &= g_k + \gamma\delta(\theta_k) g_k h_k \, \|\theta_k\|^2, \\ \delta(\theta) &= \begin{cases} 1 & \text{if } \|\theta\| \geq \lambda \\ 0 & \text{if } \|\theta\| < \lambda \end{cases} \end{aligned} \tag{12}$$

with $t_0 = 0$, $g_0 > 1$. Then

$$\lim_{k\to\infty} g_k = g_\infty < \infty\,,$$

$$\lim_{k\to\infty} h_k = h_\infty > 0\,,$$

$$\lim_{k\to\infty} \mathrm{dis}\,\{\|\theta_k\|, [0, \lambda]\} = 0\,.$$

Here, for non-negative a and b, $\mathrm{dis}\,\{a, [0, b]\}$ is the distance from a to the interval $[0, b]$, that is

$$\mathrm{dis}\,\{a, [0, b]\} = \begin{cases} 0 & \text{if } a \leq b \\ a - b & \text{else}\,. \end{cases}$$

Remark 3. *Here, we ensure the convergence of both g_k and h_k, with the output observer error θ_k converging to a λ-strip. This adaptive framework enhances the robustness and adaptability of the observer, making it well-suited for real-world applications where continuous access to process measurements may be limited.*

3.2 Hybrid predictor-corrector numerical scheme with adaptive dead-zones

In this sub-section, we develop the hybrid predictor-corrector numerical scheme with the Lyapunov-based switching and adaptive dead zone. The key point is that if we just use the λ tracker, the gain is always increasing, so h is always decreasing, and we are back into the problem of having overly conservative small h. To overcome this, we invoke a second dead zone – a μ strip as follows:

$$h_k = \frac{1}{g_k \log\, g_k}, \quad t_{k+1} = t_k + h_k, \quad k \in \mathbb{N}_0,$$

$$g_{k+1} = \begin{cases} g_k + \|\theta_k\|^2 & \text{if} \quad \|\theta_k\| \geq \lambda \\ \rho g_k & \text{if} \quad \|\theta_k\| < \mu \\ g_k & \text{if} \quad \mu \leq \|\theta_k\| < \lambda. \end{cases} \tag{13}$$

Here $\rho \in (0, 1)$ and $\mu < \lambda$ are additional design parameters.

Remark 4. *Outside the λ-strip g_k increases, h_k decreases. Inside the μ-strip g_k decreases, h_k increases. Between the two strips, g_k and h_k are held constant. This two-dead-zone approach ensures that the gain is neither too large nor too small. The parameter ρ determines how quickly we reduce g_k in the μ-strip.*

We now combine the λ-tracking sampled data observer, also invoking the extra μ-strip, with the Euler scheme. We need to specify the switching condition via a Lyapunov function. To do this we assume that coordinates have been chosen so that:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad C = [CB\ 0], \quad \text{and} \quad B = \begin{pmatrix} I \\ 0 \end{pmatrix},$$

with A_{22} Hurwitz (minimum phase condition) and CB is totally unstable (relative degree one condition), Mareels [1984]. Then the Lyapunov function is given by

$$V(x) = x^T P x,$$

with

$$P = \begin{pmatrix} \beta I & 0 \\ 0 & P_{22} \end{pmatrix}, \qquad \text{with} \qquad P_{22}A_{22} + A_{22}^T P_{22} = -I\,, \tag{14}$$

for any $\beta > 0$. Using this Lyapunov function in the switching condition (7), we can now combine the sampled-data observer and Euler scheme through a Lyapunov-based switching condition. If:

(7) holds, $w_{k+1} = (I + hA)w_k$.

$$\text{(7) fails,} \quad \begin{cases} w_{k+1} &= w_k + h_k(A - g_k BC)w_k + h_k g_k B v_k, \\ h_k &= \frac{1}{g_k \log g_k}, \\ g_{k+1} &= \begin{cases} g_k + \|\theta_k\|^2 & \text{if} \quad \|\theta_k\| \geq \lambda, \\ \rho g_k & \text{if} \quad \|\theta_k\| < \mu, \\ g_k & \text{if} \quad \mu \leq \|\theta_k\| < \lambda. \end{cases} \end{cases} \tag{15}$$

Note that the step size is only adapted when the sampled-data observer is active. When Euler is active the step size is held constant.

In (15), the sampled process output v_k satisfies:

$$v_k = Cz_k \qquad \text{with} \quad z_{k+1} = e^{h_k A} z_k\,.$$

These formulas are not used in the scheme, but they are used in stability arguments.

4. SIMULATIONS AND RESULTS

In this section, we illustrate the effectiveness of the hybrid numerical scheme (7) and (15) based on dead-zone adaptive λ tracking. We apply the scheme to 3-dimensional process with a stiff but stable A matrix and output C and

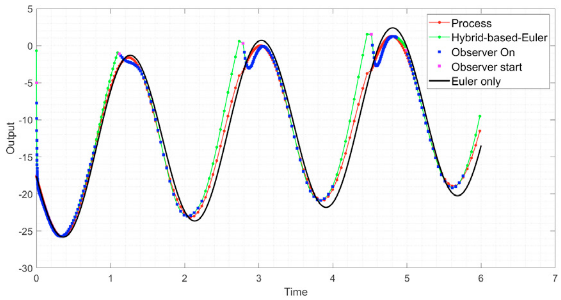

Figure 1. Hybrid numerical scheme (7) and (15) with adaptive dead-zones. Process output $y(t_k)$ and numerical output Cw_k: Euler active (green); Observer active (blue); Observer start (magenta).

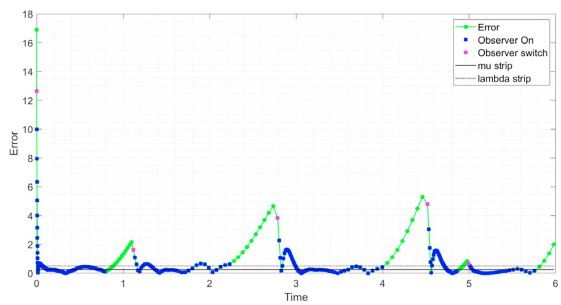

Figure 2. Hybrid numerical scheme (7) and (15) with adaptive dead-zones. Output error θ_k: Euler active (green); Observer active (blue); Observer start (magenta).

choose B so that the system (A, B, C) is minimum phase and relative degree one.

Example 1. We consider the specific process as follows

$$\dot{x}(t) = \begin{pmatrix} 10.86 & 104.08 & 305.37 \\ -1 & -11 & -28 \\ 0 & 1 & 0 \end{pmatrix} x(t). \qquad (16)$$

Here, A has one negative eigenvalue and a pair of complex conjugate eigenvalues:

$$-0.1022, \quad -0.0189 \pm 3.5519i\,.$$

For the stability of Euler we need a step size:

$$h < h_c = \frac{2(0.0189)}{0.0189^2 + 3.5519^2} = 0.003.$$

The step size threshold h_c is depicted in Figure 4.

We assume a process output:

$$y(t) = (1 \quad 0 \quad 0)\, x(t)\,. \qquad (17)$$

We choose,

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \qquad (18)$$

In this case, the triple (A, B, C) is a relative degree one and minimum-phase system with stable zeros -4 and -7. We consider (7), (14) and (15) with parameters:

$$\gamma = 0.9, \quad \lambda = 0.5, \quad \mu = 0.25, \quad \rho = 0.95, \quad \beta = 0.01\,.$$

The simulation is given in Figures 1 to 4. We run Euler alone with the maximum step size $h = h_c$. We observe the following:

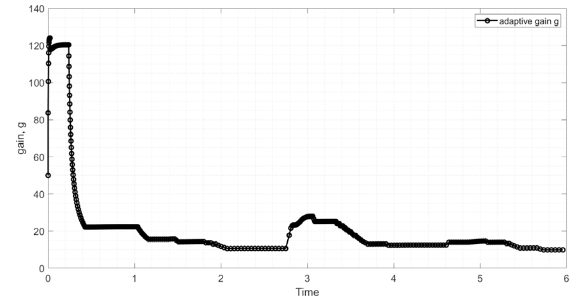

Figure 3. Increasing/decreasing observer gain g_k in hybrid scheme (15).

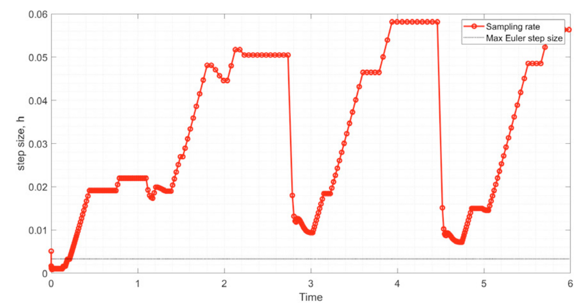

Figure 4. Increasing/decreasing step size h_k in hybrid scheme (15). The dotted line depicts the maximum allowed step size h_c for stable Euler.

- The error converges to the λ-strip, with fluctuations in and out of the μ-strip.
- With these parameters, $\mu = 0.25$, at half of $\lambda = 0.5$, is relatively large, whilst the parameter $\rho = 0.95$ is relatively large. So the size of the μ strip is the main driver for increasing the step size h. Notice that with the increasing step size mechanism provided by the μ strip, the adaptive step sizes from the λ-tracking observer are almost all the time much larger than the maximum step size for stability of Euler alone. This can be seen in Figure 4. In fact, the adaptive step size reaches values 20-fold the value h_c.
- Notice also that the adaptive step size is held constant either when the error is between the μ and λ strips, or when the observer is inactive – see green plots in Figure 2.

Example 2. We consider another system as follows:

$$\dot{x}(t) = \begin{pmatrix} 8.96 & 82.74 & 179.41 \\ -1 & -9 & -20 \\ 0 & 1 & 0 \end{pmatrix} x(t). \qquad (19)$$

Here, A has one negative eigenvalue and a pair of complex conjugate imaginary eigenvalues:

$$-0.0095, \quad -0.0152 \pm 4.7010i.$$

For stability of Euler we need a step size:

$$h < h_c = \frac{2(0.0152)}{0.0152^2 + 4.7010^2} = 0.0014.$$

We assume a process output:

$$y(t) = (1 \quad 0 \quad 0)\, x(t)\,. \qquad (20)$$

We choose:

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \qquad (21)$$

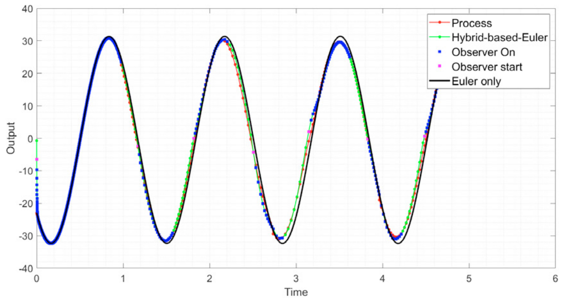

Figure 5. Hybrid numerical scheme (7) and (15) with adaptive dead-zones. Process output $y(t_k)$ and numerical output Cw_k: Euler active (green); Observer active (blue); Observer start (magenta).

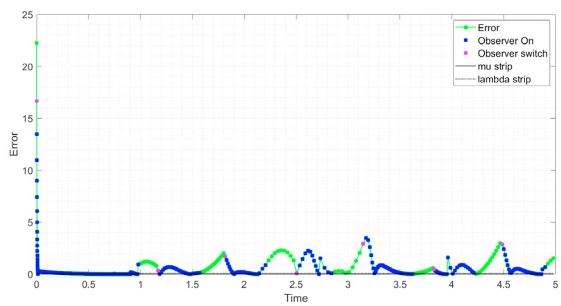

Figure 6. Hybrid numerical scheme (7) and (15) with adaptive dead-zones. Output error θ_k: Euler active (green); Observer active (blue); Observer start (magenta).

In this case, the triple (A, B, C) is a relative degree one and minimum-phase system with stable zeros -5 and -4.

Here we consider (7), (14) and (15) with parameters:

$$\gamma = 0.9 \quad \lambda = 0.1, \quad \mu = 0.01, \quad \rho = 0.5, \quad \beta = 0.01\,.$$

The simulation is given in Figures 5 to 8. We run Euler alone with the maximum step size $h = h_c$. We observe the following:

- The error converges to the λ-strip, with fluctuations in and out of the μ-strip.
- With these parameters, $\mu = 0.01$, at one tenth of $\lambda = 0.1$, is relatively small, whilst the parameter $\rho = 0.5$ is relatively small. So the size of ρ is the main driver for increasing the step size h. Notice that with the increasing step size mechanism provided by ρ, the adaptive step sizes from the λ-tracking observer are almost all the time much larger than the maximum step size for stability of Euler alone. This can be seen in Figures 8 and 10. In fact, the adaptive step size reaches values almost 30 times the value h_c.
- Looking at the zoomed Figures 9 and 10 we see that the adaptive step size is held constant either when the error is between the μ and λ strips, or when the observer is inactive (green portions in Figure 9). The step size decreases if the observer is active and the error is outside the λ strip (blue portions in Figure 9), and increases if the error is inside the μ strip.

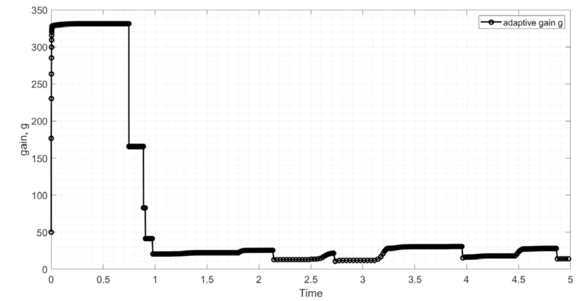

Figure 7. Increasing/decreasing observer gain g_k in hybrid scheme (15).

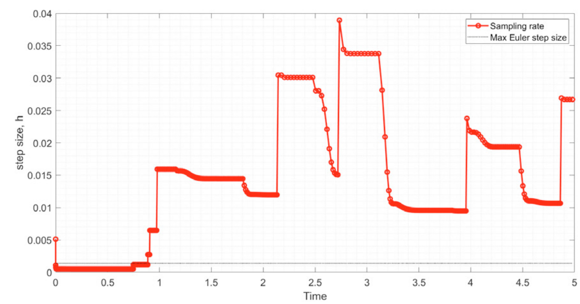

Figure 8. Increasing/decreasing step size h_k in hybrid scheme (15). The dotted line depicts the maximum allowed step size h_c for stable Euler.

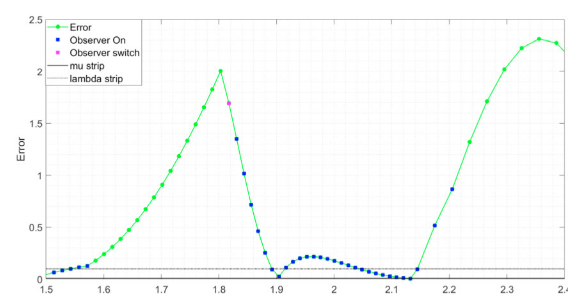

Figure 9. The region is zoomed in for the error in and out strips, highlighted from 1.5 to 2.4 from Figure 6.

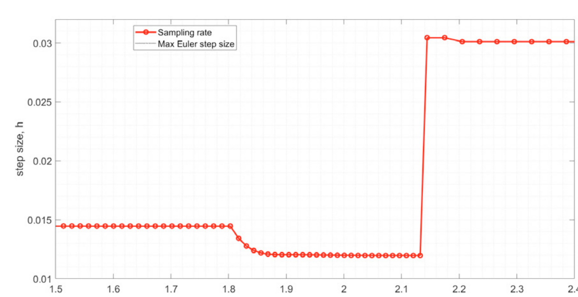

Figure 10. The region is zoomed in for step size h, highlighted from 1.5 to 2.4 from Figure 8

5. CONCLUSIONS AND FUTURE WORK

In this work, we have further developed the hybrid observer-based numerical scheme over existing literature, by drawing on ideas from adaptive high-gain stabilization Sontag et al. [1989]. Specifically, we combine a sampled-data high-gain λ-tracking observer with the Euler scheme to create a predictor-corrector scheme with adaptive step sizes. The adaptive step size is derived from an adaptive observer gain, which acts to counter the potential stiffness of the observed process to be computed.

The observer based ODE solver may be helpful in numerical solution of stiff: Byrne and Hindmarsh [1987], unstable, highly oscillatory: Gurfil and Klein [2007], Wu and Wang [2021], nonlinear chaotic systems. Development of such solvers for implicit, differential algebraic equations Hosea and Shampine [1996], Shampine and Reichelt [1997], using variable step, variable order: Shampine [2002] and higher order methods have been of continued research over last few decades. The current approach shows a potential solution for flipping between different ODE solvers for initial value problems (IVPs) and boundary problems (BVPs), distributed ODEs and coupled partial differential equations: Wang et al. [2021], although a detailed comparison with all of these existing solvers are beyond the current which could be investigated in future research, beyond the well known Runge Kutta family of solvers of low, medium, high and variable orders Bogacki and Shampine [1989], Dormand and Prince [1980], Verner [2010].

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