Moments of characteristic polynomials of random matrices and L-functions over function fields

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I certify that all material in this thesis which is not my own work has been identified and that any material that has previously been submitted and approved for the award of a degree by this or any other University has been acknowledged.

Signed: ....

Abstract

In this thesis, we study various problems involving moments of characteristic polynomials of random matrices and also of L-functions over function fields.

In Chapter 3, we give an analytic proof of the asymptotic behaviour of the moments of moments of characteristic polynomials over $Sp(2N)$ and $SO(2N)$. We also discuss the analogous moments of moments of L-functions with symplectic or orthogonal symmetry and relate these to the shifted moment conjectures of Conrey et al. [CFK$^+$05].

In Chapter 4, we study the joint moments of the derivatives of characteristic polynomials over $Sp(2N)$, $SO(2N)$ and $O^-(2N)$. We obtain asymptotic formulae for the integer moments over all three matrix ensembles, with two alternate expressions for the leading order coefficients. Using our results, we are able to make conjectures for the corresponding joint moments of derivatives for symplectic or orthogonal families of L-functions.

Chapters 5 and 6 are on quadratic Dirichlet L-functions over function fields. Specifically, in Chapter 5, we improve the error term in a theorem of Bui, Florea and Keating [BFK23] which proves the Ratios Conjecture for these L-functions in certain ranges of the parameters. Then, in Chapter 6, we compute the first and second mollified moments of the quadratic L-functions. As an application, we obtain non-vanishing results for the L-functions and their derivatives at the central point.

Finally, in Chapter 7 we compute an asymptotic formula for the mixed second moment of derivatives of quadratic Dirichlet L-functions of prime conductor in the function field setting. We also compare our result with that predicted by our random matrix theory calculations in Chapter 4.

Acknowledgements

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Finally, I must thank my parents and family, in particular my mum, for everything they have done for me over the years. For your constant love and continued support, I am forever grateful.

Author's Declaration

I declare that the work presented in this thesis is in accordance with the regulations of the University of Exeter. The work is original except where indicated.

The work of Chapters 3 and 4 was done jointly with my supervisor Julio Andrade and has been published in Random Matrices: Theory and Applications [AB22] and Journal of Physics A: Mathematical and Theoretical [AB24], respectively. The results of Chapters 5 and 6 are in preparation for publication, again joint with Julio Andrade. I have submitted the results of Chapter 7 in a paper [Bes24] as a sole author.

No part of this thesis has been submitted for any other academic degree and has not been presented to any other university for examination.

Notation

Most of the notation used throughout will be introduced in Chapters 1 and 2 but we list the most commonly used notation here.

Random matrix theory

$X = (x_{i,j})$	An $N \times N$ matrix.
$X^T = (x_{j,i})$	The transpose of the matrix $X = (x_{i,j})$
$X^\dagger = (\overline{x_{j,i}})$	The conjugate transpose of the matrix $X = (x_{i,j})$
I_N	The $N \times N$ identity matrix.
$U(N)$	The group of $N \times N$ unitary matrices.
$Sp(2N)$	The group of $2N \times 2N$ unitary symplectic matrices.
$O(N)$	The group of $N \times N$ orthogonal matrices.
$SO(2N)$	The group of $2N \times 2N$ orthogonal matrices with determinant $+1$.
$O^-(2N)$	The set of $2N \times 2N$ orthogonal matrices with determinant -1.
$\Lambda_X(s) = \det(I - X^\dagger s)$	The characteristic polynomial of a unitary matrix X.
dX	The Haar measure on the matrix ensemble $U(N)$, $Sp(2N)$, $SO(2N)$ or $O^-(2N)$.

Analysis

O	$f(x) = O(g(x))$ if there exists a constant $c > 0$ such that $\|f(x)\| \leq cg(x)$ for all $x \geq x_0$.
$\ll$	$f(x) \ll g(x)$ if $f(x) = O(g(x))$.
o	$f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
$\sim$	$f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.
$\asymp$	$f(x) \asymp g(x)$ if both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ holds.
$\oint$	An integral over a closed contour.
$\int_{(c)}$	An integral along the line $\mathrm{Re}(s) = c$.

Number theory over function fields

$\mathbb{F}_q$	A finite field with q elements.
$\mathbb{F}_q^*$	The multiplicative group of $\mathbb{F}_q$.
$\mathbb{F}_q[t]$	The polynomial ring over $\mathbb{F}_q$.
$\mathbb{F}_q(t)$	The rational function field over $\mathbb{F}_q$.
$d(f)$	The degree of a polynomial $f \in \mathbb{F}_q[t]$.
$\|f\|$	$\|f\| = q^{d(f)}$, the norm of $f \in \mathbb{F}_q[t]$.
$\mathcal{M}$	The set of monic polynomials in $\mathbb{F}_q[t]$.
$\mathcal{H}_n$	The set of monic, square-free polynomials of degree n in $\mathbb{F}_q[t]$.
$\mathcal{P}_n$	The set of monic irreducible polynomials of degree n in $\mathbb{F}_q[t]$.
$\mu(f)$	The Möbius function on $\mathbb{F}_q[t]$.
$\varphi(f)$	The Euler-Totient function on $\mathbb{F}_q[t]$.
$\tau_k(f)$	The number of ways of writing the polynomial f as a product of k factors.

Contents

Chapter 1

Introduction

The central theme of this thesis is the link between random matrix theory and analytic number theory. The core idea underpinning the connection between the two topics is the belief that the zeros of the Riemann zeta function and other L-functions are distributed like the eigenvalues of random matrices from one of the classical compact groups. There is a growing wealth of evidence to support this belief and modelling L-functions by the characteristic polynomials of random matrices has proven an invaluable technique for furthering our understanding on the number theory side.

It is often the case that for a difficult problem involving L-functions, the analogous problem for characteristic polynomials is more tractable. Thus, a common approach for attacking these difficult problems is to first try to solve the corresponding random matrix problem. This then gives us, even if at least only conjecturally, a better understanding of the original problem and an idea of what we expect the answer to be.

In this thesis we are primarily motivated to study problems in random matrix theory due to the number theoretic applications. The results on the random matrix side are of course interesting in their own right and the deep connection to number theory only serves to increase the appeal of studying these problems. The particular problems we focus on are all connected to, or variants of, the moments of characteristic polynomials and families of L-functions.

1.1 The classical compact groups

We begin by introducing the three classical compact groups of random matrix theory, so called by Weyl [Wey66], that we will study in Chapters 3 and 4. The characteristic polynomials of matrices belonging to these groups will be used to model L-functions from various families. For an $N \times N$ matrix $X = (x_{i,j})$, we denote by $X^T = (x_{j,i})$ the transpose of X and by $X^\dagger$ the conjugate transpose of X. That is, $X^\dagger = (\overline{x_{j,i}})$. We also denote the $N \times N$ identity matrix by I_N.

1.1.1 The unitary group $U(N)$

An $N \times N$ matrix X with complex entries is said to be unitary if $XX^\dagger = I_N$. The group of $N \times N$ unitary matrices is denoted by $U(N)$. The eigenvalues of a matrix $X \in U(N)$ all lie on the unit circle so we write them as

$$e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_N} \tag{1.1.1}$$

with $\theta_j \in [0, 2\pi)$. The eigenvalues of $X^\dagger$ are then $e^{-i\theta_1}, \ldots, e^{-i\theta_N}$. The determinant of X is given by $\det(X) = \prod_{j=1}^N e^{i\theta_j}$ and so satisfies $|\det(X)| = 1$. We define the characteristic polynomial of the unitary matrix X as

$$\Lambda_X(s) := \det(I - X^\dagger s) = \prod_{j=1}^N (1 - se^{-i\theta_j}). \tag{1.1.2}$$

The characteristic polynomial satisfies the functional equation

$$\begin{aligned} \Lambda_X(s) &= (-s)^N \prod_{j=1}^N e^{-i\theta_j} \prod_{j=1}^N (1 - e^{i\theta_j} s^{-1}) \\ &= (-1)^N s^N \det(X^\dagger)\, \Lambda_{X^\dagger}(s^{-1}). \end{aligned} \tag{1.1.3}$$

We also define the $\mathcal{Z}$-function associated to the unitary matrix X by

$$\mathcal{Z}_X(s) := e^{-\pi i N/2} e^{i\sum_{j=1}^N \theta_j/2} s^{-N/2} \Lambda_X(s), \tag{1.1.4}$$

which satisfies the symmetric functional equation

$$\mathcal{Z}_X(s) = (-1)^N \mathcal{Z}_{X^\dagger}(s^{-1}), \tag{1.1.5}$$

or, equivalently,

$$\mathcal{Z}_X(s) = \overline{\mathcal{Z}_X(\overline{s}^{-1})}. \tag{1.1.6}$$

Consequently, we have that $|\mathcal{Z}_X(e^{i\theta})| = |\Lambda_X(e^{-\theta})|$ and that $\mathcal{Z}_X(e^{i\theta})$ is real for $\theta \in \mathbb{R}$.

1.1.2 The symplectic group $Sp(2N)$

A $2N \times 2N$ unitary matrix X is *symplectic* if it satisfies

$$X \Omega X^T = \Omega, \tag{1.1.7}$$

where Ω is the block matrix given by

$$\Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}. \tag{1.1.8}$$

The *symplectic group* $Sp(2N)$ is the subgroup of $U(2N)$ consisting of $2N \times 2N$ symplectic matrices. A matrix $X \in Sp(2N)$ has N pairs of complex conjugate eigenvalues $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_N}$ lying on the unit circle. Hence, $\det(X) = 1$ and the characteristic polynomial of X is of the form

$$\Lambda_X(s) = \prod_{j=1}^{N} (1 - se^{-i\theta_j})(1 - se^{i\theta_j}) = \prod_{j=1}^{N} (1 + s^2 - 2s\cos\theta_j). \tag{1.1.9}$$

In this case, $\Lambda_X(s)$ is a real polynomial and satisfies the functional equation

$$\Lambda_X(s) = s^{2N} \Lambda_X(s^{-1}). \tag{1.1.10}$$

1.1.3 The orthogonal group $O(N)$

A unitary matrix X is *orthogonal* if $XX^T = X^T X = I$. We denote the group of orthogonal $N \times N$ matrices by $O(N)$. Note that $O(N)$ consists of the unitary matrices with real entries. An orthogonal matrix has determinant equal to ± 1 so we further distinguish between these two cases.

The *special orthogonal group* $SO(2N)$ is the subgroup of $O(2N)$ consisting of those orthogonal matrices with determinant $+1$. Similarly to the symplectic case, the eigenvalues of a matrix $X \in SO(2N)$ come in N complex conjugate pairs $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_j}$ and the characteristic polynomial is of the form

$$\Lambda_X(s) = \prod_{j=1}^{N} (1 - se^{-i\theta_j})(1 - se^{i\theta_j}) = \prod_{j=1}^{N} (1 + s^2 - 2s\cos\theta_j). \tag{1.1.11}$$

The functional equation for the characteristic polynomial is again

$$\Lambda_X(s) = s^{2N} \Lambda_X(s^{-1}). \tag{1.1.12}$$

The subset of matrices in $O(2N)$ with determinant -1 is denoted by $O^-(2N)$.

A matrix $X \in O^-(2N)$ has $N-1$ pairs of complex conjugate eigenvalues and two eigenvalues at ± 1. Thus, for $X \in O^-(2N)$, the characteristic polynomial is given by

$$\Lambda_X(s) = (1-s)(1+s)\prod_{j=1}^{N-1}(1-se^{-i\theta_j})(1-se^{i\theta_j}) = (1-s)(1+s)\prod_{j=1}^{N-1}(1+s^2-2s\cos\theta_j), \tag{1.1.13}$$

and satisfies the functional equation

$$\Lambda_X(s) = -s^{2N}\Lambda_X(s^{-1}). \tag{1.1.14}$$

In particular, in this case we have $\Lambda_X(1) = 0$.

1.1.4 The Haar measure and circular ensembles

The groups of matrices $U(N)$, $Sp(2N)$ and $O(N)$ are all compact Lie groups and so may all be equipped with the normalised Haar measure to give a probability space or ensemble. When performing analysis on one of the matrix ensembles mentioned above, we will always work with the Haar measure and we will denote the Haar measure on the relevant matrix ensemble by dX.

Endowing the unitary group $U(N)$ with the Haar measure yields the *Circular Unitary Ensemble* (CUE) of random matrix theory. The other circular ensembles are the *Circular Symplectic Ensemble* (CSE) and the *Circular Orthogonal Ensemble* (COE). However, we emphasise that the CSE and COE are not simply the symplectic group $S(2N)$ or the orthogonal group $O(N)$ equipped with the Haar measure. See, for example, Table 1 in [KS99a] for a description of the realisation of the CSE and COE as spaces of matrices.

1.1.5 Probability densities and the Weyl integration formula

A matrix $X \in U(N)$ has N eigenvalues on the unit circle and any matrix conjugate to X has the same eigenvalues. Conversely, for any set of N points on the unit circle, there exists a conjugacy class of $U(N)$ whose matrices have these N point as their eigenvalues. Thus, one may identify conjugacy classes of $U(N)$ with sets of N points on the unit circle. Consequently, if f is a class function on $U(N)$, i.e. f is constant on conjugacy classes, then f depends only on the eigenvalues of X. We may therefore write $f(X) = \tilde{f}(\theta_1, \ldots, \theta_N)$ where $\tilde{f}$ is a symmetric function of the eigenangles θ_j.

Now suppose that $X \in U(N)$ is chosen randomly according to the Haar measure. Then the eigenvalues of X are N random points on the unit circle. In other words, the eigenvalues are a random N point process on the circle. The Weyl integration formula [Wey66] gives an explicit expression for the joint probability density of the eigenvalues in terms of $d\theta_1 \cdots d\theta_N$ on $[0, 2\pi)^N$. In particular, if f is a class function on $U(N)$, we have that

$$\int_{U(N)} f(X)\,dX = \frac{1}{(2\pi)^N N!} \int_{[0,2\pi)^N} \tilde{f}(\theta_1, \ldots, \theta_N) \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 \, d\theta_1 \cdots d\theta_N. \tag{1.1.15}$$

We note that the factor

$$\prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 \tag{1.1.16}$$

in the probability density of the eigenvalues causes repulsion of the eigenvalues away from each other leading to a even distribution of the eigenvalues compared to if the eigenvalues were uniformly distributed on the circle.

In the case of the symplectic or orthogonal ensembles, the non-trivial eigenvalues of the matrices come in complex conjugate pairs. Thus, one usually only considers the distribution of the eigenvalues on the upper half-circle. The probability densities for the eigenvalue distributions are also known and we list the relevant Weyl integration formulae below. For an exposition of the proof of these formulae, see, for example, Chapter 3 of [Mec19].

$$\begin{aligned} \int_{Sp(2N)} f(X)\,dX = \frac{2^N}{\pi^N N!} \int_{[0,\pi)^N} & \tilde{f}(\theta_1, \ldots, \theta_N) \\ & \times \prod_{j=1}^{N} \sin^2\theta_j \prod_{1 \leq j < k \leq N} (2\cos\theta_k - 2\cos\theta_j)^2 \; d\theta_1 \cdots d\theta_N, \end{aligned} \tag{1.1.17}$$

$$\begin{aligned} \int_{SO(2N)} f(X)\,dX = \frac{2}{(2\pi)^N N!} \int_{[0,\pi)^N} & \tilde{f}(\theta_1, \ldots, \theta_N) \\ & \times \prod_{1 \leq j < k \leq N} (2\cos\theta_k - 2\cos\theta_j)^2 \; d\theta_1 \cdots d\theta_N, \end{aligned} \tag{1.1.18}$$

and

$$\int_{O^-(2N)} f(X)\,dX = \frac{2^{N-1}}{\pi^{N-1}(N-1)!} \int_{[0,2\pi)^{N-1}} \tilde{f}(\theta_1, \ldots, \theta_{N-1})$$

$$\times \prod_{j=1}^{N-1} \sin^2\theta_j \prod_{1\le j<k\le N-1} (2\cos\theta_k - 2\cos\theta_j)^2 \; d\theta_1 \cdots d\theta_{N-1}. \tag{1.1.19}$$

Again, the probability densities above all lead to repulsion of the eigenvalues from their neighbours. In the case of Sp or O^-, we additionally see repulsion from the points ± 1 due to the $\sin^2$ factors.

1.1.5.1 The Circular β Ensemble

As mentioned earlier, one may sample a random Haar distributed matrix $X \in U(N)$ and the eigenvalues of X constitute a random point process on the unit circle. The Weyl integration formula (1.1.15) then gives us the explicit expression

$$\frac{1}{(2\pi)^N N!} \prod_{1\le j<k\le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \tag{1.1.20}$$

for the joint probability density of the eigenvalues. More generally, one may consider this random point process with the probability density

$$\frac{1}{Z_{N,\beta}} \prod_{1\le j<k\le N} |e^{i\theta_j} - e^{i\theta_k}|^\beta, \tag{1.1.21}$$

where $\beta > 0$ is a parameter to be chosen. Here, the normalisation constant is given by

$$Z_{N,\beta} = (2\pi)^N \frac{\Gamma(1+\frac{N\beta}{2})}{\Gamma(1+\frac{\beta}{2})^N}, \tag{1.1.22}$$

which follows from Selberg's integral formula. This is known as the *Circular β Ensemble* (CβE). The cases $\beta = 1, 2$ and 4 correspond the COE, the CUE and the CSE respectively. It is also possible to construct random matrix models for the general CβE, see [KN04].

1.1.6 Moments of characteristic polynomials

The statistical properties of the eigenvalues of random unitary matrices, and by association their characteristic polynomials, are the core objects of study in random matrix theory. A particular quantity of interest is the moments, or expected values, of the characteristic polynomial. Computing moments of the characteristic polynomial over one of the compact groups is a tractable problem. For instance, using the Weyl integration formula and Selberg's integral, Keating and Snaith [KS00b] proved the following.

Theorem 1.1.1 (Keating and Snaith)**.** *For* $\mathrm{Re}(k) > -1/2$ *and* $\theta \in [0, 2\pi)$,

$$\int_{U(N)} |\Lambda_X(e^{i\theta})|^{2k} \, dX = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2}. \tag{1.1.23}$$

We note that the right-hand side of (1.1.23) does not depend on θ since the Haar measure is rotationally invariant on $U(N)$. Also, by properties of the gamma function, asymptotically as $N \to \infty$ we have

$$\int_{U(N)} |\Lambda_X(e^{i\theta})|^{2k} \, dX \sim c_U(k) N^{k^2}, \tag{1.1.24}$$

where

$$c_U(k) := \lim_{N\to\infty} N^{-k^2} \int_{U(N)} |\Lambda_X(e^{i\theta})|^{2k} \, dX = \frac{\mathcal{G}(1+k)^2}{\mathcal{G}(1+2k)} \tag{1.1.25}$$

and $\mathcal{G}(s)$ is the Barnes $\mathcal{G}$-function [Bar00]. In the case that $k \in \mathbb{N}$, we have that

$$\int_{U(N)} |\Lambda_X(e^{i\theta})|^{2k} \, dX = \prod_{j=0}^{k-1} \left(\frac{j!}{(k+j)!} \prod_{i=1}^{k} (N+i+j) \right), \tag{1.1.26}$$

from which we see that the moments are given by a polynomial in N of degree k^2 and that

$$c_U(k) = \prod_{j=0}^{k-1} \frac{j!}{(k+j)!} \tag{1.1.27}$$

for integer k.

As an application of their result, Keating and Snaith [KS00b] obtained the following central limit theorem on the logarithm of the characteristic polynomial.

Theorem 1.1.2 (Keating and Snaith)**.** *Let* $B \subset \mathbb{C}$ *be a rectangle. For fixed* $\theta \in [0, 2\pi)$,

$$\lim_{N\to\infty} \mathrm{meas} \left\{ X \in U(N) : \frac{\log \Lambda_X(e^{i\theta})}{\sqrt{\frac{1}{2}\log N}} \in B \right\} = \frac{1}{2\pi} \iint_B e^{-\frac{1}{2}(x^2+y^2)} \, dx \, dy, \tag{1.1.28}$$

where the measure is the Haar measure on $U(N)$.

The result implies that both the real and imaginary parts of the logarithm of $\Lambda_X(e^{i\theta})$ tend independently to Gaussian random variables as $N \to \infty$. This example application illustrates how one can deduce a significant amount of information if one has a strong understanding of the moments of the characteristic polynomials.

In [KS00a], Keating and Snaith extended their result to the moments of $\Lambda_X(1)$ over the symplectic group $Sp(2N)$ and the special orthogonal group $SO(2N)$.

Theorem 1.1.3 (Keating and Snaith). *For* $\mathrm{Re}(k) > -1/2$

$$\int_{Sp(2N)} \Lambda_X(1)^k \, dX = 2^{2kN} \prod_{j=1}^{N} \frac{\Gamma(N+j+1)\Gamma(j+k+1/2)}{\Gamma(j+1/2)\Gamma(N+j+k+1)}, \tag{1.1.29}$$

and

$$\int_{SO(2N)} \Lambda_X(1)^k \, dX = 2^{2kN} \prod_{j=1}^{N} \frac{\Gamma(N+j-1)\Gamma(j+k-1/2)}{\Gamma(j-1/2)\Gamma(N+j+k-1)}. \tag{1.1.30}$$

In the case of integer k, the moments over $Sp(2N)$ and $SO(2N)$ are given by

$$\int_{Sp(2N)} \Lambda_X(1)^k \, dX = 2^{\frac{k(k+1)}{2}} \prod_{j=1}^{k} \frac{j!}{(2j)!} \prod_{1 \le i \le j \le k} \left(N + \tfrac{1}{2}(i+j)\right), \tag{1.1.31}$$

and

$$\int_{SO(2N)} \Lambda_X(1)^k \, dX = 2^{\frac{k(k+1)}{2}} \prod_{j=1}^{k-1} \frac{j!}{(2j)!} \prod_{1 \le i < j \le k} \left(N + \tfrac{1}{2}(i+j)\right). \tag{1.1.32}$$

Thus, the integer moments over $Sp(2N)$ and $SO(2N)$ are also polynomials in N of degree $k(k+1)/2$ and $k(k-1)/2$, respectively.

In the subsequent chapters where we study the moments of moments and joint moments of derivatives of characteristic polynomials, a key ingredient in the proofs of our results are formulae for the shifted moments of these characteristic polynomials. For our purposes, we let $G(2N) \in \{Sp(2N), SO(2N), O^-(2N)\}$ and define the shifted moments by

$$I(G(2N); z_1, \ldots, z_k) := \int_{G(2N)} \Lambda_X(z_1) \cdots \Lambda_X(z_k) \, dX, \tag{1.1.33}$$

where dX is the normalised Haar measure on the relevant ensemble $G(2N)$. Using the Weyl integration formula, Conrey et al. [CFK$^+$03] have computed these shifted moments exactly. Specifically, equations (3.6), (4.8) and (4.42) in [CFK$^+$03] give us the following expressions for the shifted moments as combinatorial sums.

Theorem 1.1.4 (Conrey et al.). *For complex numbers $z_1, \ldots, z_k$, we have*

$$I(Sp(2N); z_1, \ldots, z_k) = \prod_{j=1}^{k} z_j^N \left[\sum_{\epsilon_j \in \{-1,1\}} \left(\prod_{j=1}^{k} z_j^{\epsilon_j N} \right) \prod_{1 \le i \le j \le k} \left(1 - z_i^{-\epsilon_i} z_j^{-\epsilon_j}\right)^{-1} \right], \tag{1.1.34}$$

$$I(SO(2N); z_1, \ldots, z_k) = \prod_{j=1}^{k} z_j^N \left[\sum_{\epsilon_j \in \{-1,1\}} \left(\prod_{j=1}^{k} z_j^{\epsilon_j N} \right) \prod_{1 \le i < j \le k} \left(1 - z_i^{-\epsilon_i} z_j^{-\epsilon_j}\right)^{-1} \right], \tag{1.1.35}$$

and

$$I(O^-(2N); z_1, \ldots, z_k) = \prod_{j=1}^{k} z_j^N \left[\sum_{\epsilon_j \in \{-1,1\}} \left(\prod_{j=1}^{k} \epsilon_j z_j^{\epsilon_j N} \right) \prod_{1 \le i < j \le k} \left(1 - z_i^{-\epsilon_i} z_j^{-\epsilon_j}\right)^{-1} \right]. \tag{1.1.36}$$

The shifted moments over $U(N)$ were also computed by Conrey et al., see equations (2.16) and (2.17) in [CFK+03]. We suffice to include the moment formulae in the symplectic and orthogonal cases since these are necessary for our work in future chapters.

In Chapters 3 and 4, we will make use of contour integral expressions, also due to Conrey et al. [CFK+03], for the shifted moments. These are given in the next theorem.

Theorem 1.1.5 (Conrey et al.). *For complex numbers $\alpha_1, \ldots, \alpha_k$, we have*

$$\begin{aligned} I(Sp(2N); e^{-\alpha_1}, \ldots, e^{-\alpha_k}) = \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint \cdots \oint \prod_{1 \le l \le m \le k} (1 - e^{-z_l - z_m})^{-1} \\ \times \frac{\Delta(z_1^2, \ldots, z_k^2)^2 \prod_{j=1}^{k} z_j}{\prod_{1 \le l, m \le k} (z_l^2 - \alpha_m^2)} e^{N \sum_{j=1}^{k} (z_j - \alpha_j)} \, dz_1 \cdots dz_k, \end{aligned} \tag{1.1.37}$$

$$I(SO(2N); e^{-\alpha_1}, \ldots, e^{-\alpha_k}) = \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint \cdots \oint \prod_{1 \le l < m \le k} (1 - e^{-z_l - z_m})^{-1}$$

$$\times \frac{\Delta(z_1^2,\dots,z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{1\le l,m\le k}(z_l^2-\alpha_m^2)} e^{N\sum_{j=1}^k (z_j+\alpha_j)}\, dz_1\cdots dz_k, \tag{1.1.38}$$

and

$$I(O^-(2N);e^{-\alpha_1},\dots,e^{-\alpha_k}) = \frac{(-1)^{k(k-1)/2}2^k}{(2\pi i)^k k!} \oint\cdots\oint \prod_{1\le l\le m\le k}(1-e^{-z_l-z_m})^{-1}$$
$$\times \frac{\Delta(z_1^2,\dots,z_k^2)^2 \prod_{j=1}^k \alpha_j}{\prod_{1\le l,m\le k}(z_l^2-\alpha_m^2)} e^{N\sum_{j=1}^k (z_j+\alpha_j)}\, dz_1\cdots dz_k. \tag{1.1.39}$$

In all three cases, the contours of integration encircle the poles at $\pm\alpha_m$ *for* $m = 1,\dots,k$.

The proof of Theorem 1.1.5 follows from Lemma 2.5.2 in [CFK$^+$05]. The idea is that if one evaluates the contour integrals in Theorem 1.1.5 using the residue theorem, one arrives precisely at the expressions for the moments given in Theorem 1.1.4.

1.2 The Riemann zeta function

The Riemann zeta function is defined for $\mathrm{Re}(s) > 1$ by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}. \tag{1.2.1}$$

The series is absolutely convergent for $\mathrm{Re}(s) > 1$ and uniformly convergent on the half-plane $\mathrm{Re}(s) \ge 1+\delta$ for any $\delta > 0$. Thus, $\zeta(s)$ is holomorphic for $\mathrm{Re}(s) > 1$. It was first observed by Euler [Eul44] that the zeta function may be expressed as a product over the primes:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{1.2.2}$$

The product is also absolutely convergent for $\mathrm{Re}(s) > 1$ and is non-zero in this region so $\zeta(s) \ne 0$ for $\mathrm{Re}(s) > 1$.

In his famous manuscript [Rie59], Riemann considered $\zeta(s)$ as a function of a complex variable and proved a number of important results. He showed that $\zeta(s)$

may be continued to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ with residue 1. Additionally, Riemann defined the ξ function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{1.2.3}$$

and proved that $\xi(s)$ is entire and satisfies the functional equation

$$\xi(s) = \xi(1-s). \tag{1.2.4}$$

Since the gamma function $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \dots$, the fact that $\xi(s)$ is entire implies that $\zeta(s) = 0$ for $s = -2n$ with $n \in \mathbb{N}$. These are the "trivial" zeros of the zeta function. Since $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$, it follows from the functional equation (1.2.4) that the remaining, non-trivial zeros of $\zeta(s)$ must lie in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$. Riemann conjectured that these zeros are all in fact located on the line $\mathrm{Re}(s) = 1/2$. This is the famous Riemann hypothesis (RH).

Conjecture 1.2.1 (Riemann Hypothesis)**.** *All the non-trivial zeros of $\zeta(s)$ lie on the critical line* $\mathrm{Re}(s) = 1/2$.

The Riemann hypothesis is considered by many to be the most important open problem in mathematics due to its vast number of implications. As an example of an application of the zeta function to the theory of the primes, we have the Prime Number Theorem.

Theorem 1.2.2 (Prime Number Theorem)**.** *Let $\pi(x)$ denote the number of primes that are less than or equal to x. Then, as $x \to \infty$,*

$$\pi(x) \sim \frac{x}{\log x}. \tag{1.2.5}$$

In 1896, Hadamard [Had96] and de la Valée Poussin [dlVP96] independently gave an analytic proof of the Prime Number Theorem. The key ingredient in their proof was to show that $\zeta(s)$ does not vanish on the line $\mathrm{Re}(s) = 1$. If one assumes RH, then the error term in the Prime Number Theorem may be bounded by

$$O\left(\frac{\sqrt{x}}{\log x}\right), \tag{1.2.6}$$

and it is conjectured that this is the true size of the error.

Lastly, closely related to the Riemann zeta function is Hardy's Z-function $Z(t)$, defined by

$$Z(t) = \pi^{-it/2}\frac{\Gamma(\frac{1}{4}+\frac{1}{2}it)}{|\Gamma(\frac{1}{4}+\frac{1}{2}it)|}\zeta(\tfrac{1}{2}+it). \tag{1.2.7}$$

Analogously to the $\mathcal{Z}$-function defined for a unitary matrix in the previous chapter, the Z-function satisfies the properties $Z(t) \in \mathbb{R}$ for $t \in \mathbb{R}$ and $|Z(t)| = |\zeta(\frac{1}{2} + it)|$. Note also that the non-trivial zeros of $\zeta(s)$ on the critical line correspond to real zeros of $Z(t)$. The Z-function is therefore a useful tool in the study of the zeta function and it is sometimes preferable to work with the Z-function.

1.2.1 L-functions

There is a large class of functions which satisfy properties similar to the Riemann zeta function. These are known as L-functions and in general, they are given by a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1.2.8}$$

with $a_0 = 1$ and $a_n \ll n^{\varepsilon}$ for all $\varepsilon > 0$. The series is then absolutely convergent on the half-plane $\mathrm{Re}(s) > 1$. Additionally, $L(s)$ satisfies the following properties.

1. *Analytic continuation*: The series $L(s)$ has a meromorphic continuation to the complex plane, with the only possible poles on the line $\mathrm{Re}(s) = 1$.

2. *Functional equation*: There is a function γ_L of the form

$$\gamma_L(s) = P(s)Q^s \prod_{j=1}^{w} \Gamma(w_j s + \mu_j), \tag{1.2.9}$$

   where $Q > 0$, $w_j > 0$, $\mathrm{Re}(\mu_j) \geq 0$ and $P(s)$ is a polynomial whose only zeros in the region $\mathrm{Re}(s) > 0$ are at the poles of $L(s)$, such that

$$\xi_L(s) = \gamma_L(s)L(s) \tag{1.2.10}$$

   is entire. Also, there is some number ε with $|\varepsilon| = 1$, called the sign of the functional equation, such that

$$\xi_L(s) = \varepsilon\overline{\xi_L(1 - \overline{s})}. \tag{1.2.11}$$

3. *Euler product*: For $\mathrm{Re}(s) > 1$, $L(s)$ is given by the product over primes

$$L(s) = \prod_{p} L_p(s), \tag{1.2.12}$$

where

$$L_p(s) = \sum_{k=1}^{\infty} \frac{a_{p^k}}{p^{ks}} = \exp\left(\sum_{k=0}^{\infty} \frac{b_{p^k}}{p^{ks}}\right), \tag{1.2.13}$$

with $b_n \ll n^\theta$ for some $\theta < 1/2$.

For further properties on L-functions, see Section 1.1 in [CFK$^+$05] and the references therein. It is expected that every such L-function satisfies a Riemann hypothesis, i.e. all of the non-trivial zeros lie on the line of symmetry $\mathrm{Re}(s) = 1/2$ of the functional equation. This is known as the *generalised Riemann hypothesis* (GRH). It is also widely believed that all L-functions are associated to some arithmetic object, for example characters, elliptic curves or automorphic forms. We will describe a couple of concrete examples of L-functions in Section 1.5. We refer to the L-functions and modular forms database [LMF24] for further examples of L-functions along with their properties and a list of their known zeros on the critical line.

1.3 Random matrix theory and number theory

The extremely successful application of random matrix theory to the analytic theory of the Riemann zeta and other L-functions stems from a meeting between Hugh Montgomery and Freeman Dyson in 1972. They discussed the pair correlation of both the eigenvalues of random unitary matrices and of the non-trivial zeros of the Riemann zeta function. Importantly, they found that the two expressions for the pair correlation bear a striking resemblance.

Recall that a unitary matrix $X \in U(N)$ has its N eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_N}$ lying on the unit circle. Hence, the density of eigenvalues on the circle is $N/2\pi$. Scaling the eigenvalues, we write

$$\phi_j = \frac{N\theta_j}{2\pi}, \tag{1.3.1}$$

so that these ϕ_j then have mean density 1. The pair correlation function of the eigenvalues of X is defined by

$$R_2(X; x) = \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{k=-\infty}^{\infty} \delta(x + kN - (\phi_n - \phi_m)), \tag{1.3.2}$$

where $\delta(x)$ is the Dirac delta function and Dyson [Dys62] proved the following result on the pair correlation.

Theorem 1.3.1 (Dyson). *For test functions f such that $f(x) \to 0$ as $|x| \to \infty$,*

$$\lim_{N\to\infty} \int_{U(N)} \int_{-\infty}^{\infty} f(x) R_2(X; x)\, dx\, dX = \int_{-\infty}^{\infty} f(x) \left(\delta(x) + 1 - \left(\frac{\sin(\pi x)}{\pi x} \right)^2 \right) dx, \tag{1.3.3}$$

where $\delta(x)$ is the Dirac delta function.

In particular, one may take $f(x)$ to be the indicator function on $[\alpha, \beta]$ in Theorem 1.3.1 which yields

$$\lim_{N\to\infty} \int_{U(N)} \frac{1}{N} |\{\phi_n, \phi_m : \alpha \le \phi_n - \phi_m \le \beta\}| = \int_{\alpha}^{\beta} \left(\delta(x) + 1 - \left(\frac{\sin(\pi x)}{\pi x} \right)^2 \right) dx. \tag{1.3.4}$$

Now, for the Riemann zeta function, we denote the n-th non-trivial zero by $\rho_n = \frac{1}{2} + it_n$ with $\mathrm{Re}(t_n) > 0$, where we order the zeros by height. Note that if ρ_n is a zero of $\zeta(s)$, then so is $\overline{\rho}_n$ by the functional equation so it suffices to focus on those zeros with $\mathrm{Re}(t_n) > 0$. We define the number of zeros up to height T by

$$N(T) = |\{\rho_n : 0 \le \mathrm{Re}(t_n) \le T\}|, \tag{1.3.5}$$

which is known to satisfy

$$N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi e}, \tag{1.3.6}$$

as $T \to \infty$. See, for instance, [Tit86] for a proof of this. Consequently, there are infinitely many non-trivial zeros in the critical strip and the mean density of the zeros grows like $\log T$ as T increases. We now assume the Riemann hypothesis so that $t_n \in \mathbb{R}$ for all n and we scale the zeros by defining

$$w_n = \frac{t_n}{2\pi} \log \frac{t_n}{2\pi}. \tag{1.3.7}$$

The w_n then have mean density 1 in the sense that

$$\lim_{W\to\infty} \frac{1}{W} |\{w_n \le W\}| = 1. \tag{1.3.8}$$

Montgomery [Mon72] studied the pair correlation of the zeros of the zeta function and proved that for test functions f such that the Fourier transform

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{2\pi i t}\, dx \tag{1.3.9}$$

has support in $(-1,1)$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{1\leq n,m\leq N} f(w_n-w_m)=\int_{-\infty}^{\infty} f(x)\left(\delta(x)+1-\left(\frac{\sin(\pi x)}{\pi x}\right)^2\right)dx. \quad (1.3.10)$$

The pair correlation conjecture of Montgomery is then that for fixed $\alpha\leq\beta$,

$$\lim_{T\to\infty}\frac{1}{T}|\{w_m,w_n\in[0,T]:\alpha\leq w_n-w_m\leq\beta\}|=\int_{\alpha}^{\beta}\left(\delta(x)+1-\left(\frac{\sin(\pi x)}{\pi x}\right)^2\right)dx. \quad (1.3.11)$$

Note that the conjecture follows from (1.3.10) if one takes $f(x)$ to be the indicator function on $[\alpha,\beta]$. However, this f does not satisfy the requirement that $\hat{f}$ has support in $(-1,1)$. Strong numerical evidence for Montgomery's pair correlation conjecture was provided by Odlyzko [Odl89]. Bogomolny and Keating [BK95, BK96], by assuming a certain conjectures of Hardy and Littlewood on the correlations between prime numbers, have shown that the n-point correlations of the zeros of the zeta function also agree with the prediction based on random matrix theory.

The clear similarity between (1.3.4) and (1.3.11) suggests that the distribution of the non-trivial zeros of the zeta function may be modelled by the eigenvalues of random unitary matrices. As these eigenvalues are the zeros of the characteristic polynomials of the matrices, modelling the zeta function itself by the characteristic polynomials of unitary matrices is a natural extension of these ideas. We will see throughout the rest of this thesis that this has been an extremely successful tactic.

Another example that suggests a similarity between the value distributions of the Riemann zeta function on the critical line and the characteristic polynomials of random unitary matrices is the following central limit theorem of Selberg [Sel46].

Theorem 1.3.2 (Selberg)**.** *Let $B\subset\mathbb{C}$ be a rectangle. Then*

$$\lim_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{T\leq t\leq 2T:\frac{\log\zeta(\frac{1}{2}+it)}{\sqrt{\frac{1}{2}\log\log\frac{t}{2\pi}}}\in B\right\}=\frac{1}{2\pi}\iint_B e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy. \quad (1.3.12)$$

Comparing Theorems 1.3.2 and 1.1.2, the resemblance is clear between the two results. Consequently, similarly to the case of the characteristic polynomial, both the real and imaginary parts of the logarithm of the zeta function tend independently to Gaussian random variables.

1.3.1 The Katz-Sarnak philosophy

Given an individual primitive L-function $L(s)$, i.e. an L-function that cannot be written as the product of two or more L-functions, Rudnick and Sarnak [RS96] showed that one also expects the distribution of the zeros high on the critical line to also be governed in distribution like the eigenvalues of Haar distributed unitary matrices. However, Katz and Sarnak [KS99a, KS99b] argued that the statistics of zeros of L-functions in certain families can also be modelled by the corresponding eigenvalue statistics of one of the classical compact groups $U(N)$, $O(N)$ or $Sp(2N)$. The matrix ensemble that one should use for comparison is determined by the symmetry type of the family: unitary, orthogonal or symplectic. Specifically, let $\mathcal{F}$ be a family of L-functions with each L-function associated to an $f \in \mathcal{F}$. Also, for each L-function there is a number c_f called the conductor of the L-function which allows us to partially order the family. Assuming the Riemann Hypothesis for the L-functions in the family, we may write the j-th zero of each L-function on the critical line as $1/2 + \gamma_f^j$. The belief is that the distribution of the zeros close to $1/2$ in the family depends on the symmetry type of the family.

As an example of a zero statistic that depends on the symmetry type of the family, we consider the one-level density of low-lying zeros. Let $\mathcal{F}_X$ denote the members of the family $\mathcal{F}$ with conductor less than or equal to X. For a suitable test function ϕ, the one-level density of the zeros in the family is defined as

$$D(\mathcal{F}, X, \phi) = \sum_{f \in \mathcal{F}_X} \sum_{\gamma_f^j} \phi\left(\frac{\log c_f}{2\pi} \gamma_f^j\right), \tag{1.3.13}$$

where the factor of $\log(c_f)/2\pi$ stems from the density of the zeros close to $1/2$. The density conjecture of Katz and Sarnak is that

$$\lim_{X \to \infty} \frac{D(\mathcal{F}, X, \phi)}{|\mathcal{F}_X|} = \int_{-\infty}^{\infty} \phi(x) W_{\mathcal{F}}(x)\, dx, \tag{1.3.14}$$

where $W_{\mathcal{F}}(x) = W_G(x)$ is the one-level density function for the matrix ensemble G from which $\mathcal{F}$ takes its symmetry type. These density functions are given by

$$\begin{aligned} W_U(x) &= 1, \\ W_{Sp}(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x}, \\ W_O(x) &= 1 + \frac{1}{2}\delta(x), \end{aligned}$$

$$
\begin{aligned}
W_{SO}(x) &= 1 + \frac{\sin(2\pi x)}{2\pi x}, \\
W_{O^-}(x) &= \delta(x) + 1 + \frac{\sin(2\pi x)}{2\pi x},
\end{aligned} \tag{1.3.15}
$$

where $\delta(x)$ is the Dirac delta function again.

Determining the symmetry type of a family of L-functions is not an easy task in general. To provide evidence for their arguments and conjectures, Katz and Sarnak [KS99b] considered L-functions associated to zeta functions of curves over finite fields. In this case the families are natural to define and the symmetry type is given by the monodromy of the family. In [CF00], Conrey and Farmer provided further evidence that the behaviour of the moments of a family of L-functions is also governed by the symmetry type. See also the survey paper [KS03] for further details on the applications of random matrix theory to the theory of the Riemann zeta function and other families of L-functions.

1.4 Moments of the Riemann zeta function

A core object of study in analytic number theory are the mean values or moments of L-functions. In the case of the Riemann zeta function, we are interested in the mean values of $\zeta(s)$ on the critical line $\mathrm{Re}(s) = 1/2$. The $2k$-th moment of the zeta function on the critical line is defined as

$$
M_k(T) := \frac{1}{T} \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} \, dt. \tag{1.4.1}
$$

One of the reasons for studying the moments of $\zeta(\frac{1}{2} + it)$ is their connection to a conjecture of Lindelöf.

Conjecture 1.4.1 (Lindelöf Hypothesis)**.** *For all $\varepsilon > 0$, we have*

$$
\zeta(\tfrac{1}{2} + it) \ll t^{\varepsilon}. \tag{1.4.2}
$$

It is shown, for example in [Tit86], that the Lindelöf hypothesis is equivalent to the bound

$$
M_k(T) \ll T^{\varepsilon}, \tag{1.4.3}
$$

for every positive integer k and all $\varepsilon > 0$. Asymptotic formulae for the $2k$-th moments of the zeta function are only know for $k = 1$ and $k = 2$. In 1916, Hardy and Littlewood [HL16] proved an asymptotic formula for the second moment.

Theorem 1.4.2 (Hardy and Littlewood)**.** *As $T \to \infty$, we have*

$$M_1(T) \sim \log T. \tag{1.4.4}$$

Subsequently, Ingham [Ing27] improved the error and obtained a lower order term in the asymptotic of Hardy and Littlewood.

Theorem 1.4.3 (Ingham)**.** *As $T \to \infty$, we have*

$$M_1(T) \sim \log \frac{T}{2\pi} + (2\gamma - 1) + O(T^{-\frac{1}{2}} \log T), \tag{1.4.5}$$

where γ is Euler's constant.

Ingham also gave the following asymptotic formula for the fourth moment.

Theorem 1.4.4 (Ingham)**.** *As $T \to \infty$, we have*

$$M_2(T) \sim \frac{1}{2\pi^2} \log^4 T + O(\log^3 T). \tag{1.4.6}$$

Heath-Brown [HB79] later obtained all the main terms in the fourth moment with a power saving error.

Theorem 1.4.5 (Heath-Brown)**.** *There exists constants a_0, a_1, a_2, a_3, a_4 such that, for $T \geq 2$ and $\varepsilon > 0$,*

$$M_2(T) = \sum_{n=0}^{4} a_n \log^n T + O(T^{-\frac{1}{8}+\varepsilon}). \tag{1.4.7}$$

The constants a_4 and a_3 are given by

$$a_4 = \frac{1}{2\pi^2} \text{ and } a_3 = 2\big(4\gamma - 1 \log(2\pi) - 12\zeta'(2)\pi^{-2}\big)\pi^{-2}, \tag{1.4.8}$$

where γ is Euler's constant.

There are currently no known asymptotic formulae for the higher moments. Using number theoretic heuristics, Conrey and Ghosh [CG92] put forward the conjecture that

$$M_3(T) \sim \frac{42}{9!} \prod_p \left(\left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \right) (\log T)^9. \tag{1.4.9}$$

Conrey and Gonek [CG01] then considered the eighth moment and conjectured that

$$M_4(T) \sim \frac{24024}{16!} \prod_p \left(\left(1 - \frac{1}{p}\right)^9 \left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right) \right) (\log T)^{16}. \tag{1.4.10}$$

The general and long-standing conjecture for the $2k$-th moment is the following.

Conjecture 1.4.6. *For* $\mathrm{Re}(k) > -1/2$*, as* $T \to \infty$*,*

$$M_k(T) \sim a_\zeta(k) c_\zeta(k) (\log T)^{k^2}, \tag{1.4.11}$$

where $a_\zeta(k)$ *is an arithmetic factor given by*

$$a_\zeta(k) = \prod_p \left(\left(1 - \frac{1}{p}\right)^{k^2} \left(\sum_{m=0}^{\infty} \left(\frac{\Gamma(k+m)}{m!\Gamma(k)} \right)^2 p^{-m} \right) \right), \tag{1.4.12}$$

and $c_\zeta(k)$ *is some coefficient depending on* k*.*

The arithmetic factor $a_\zeta(k)$ and its properties are well understood, see the works [CG84, CG92, Gon89], and so the challenge for a long time was to give an explicit description of the factor $c_\zeta(k)$. The known results and previous conjectures mentioned above imply that $c_\zeta(1) = 1$, $c_\zeta(2) = 1/12$ and conjecturally, $c_\zeta(3) = 42/9!$ and $c_\zeta(4) = 24024/16!$. Note that for these values, $c_\zeta(k) \cdot (k^2)!$ is an integer. Using the philosophy that the zeta function may be modelled by the characteristic polynomials of random unitary matrices, Keating and Snaith [KS00b] put forward the conjecture that $c_\zeta(k) = c_U(k)$, where recall that

$$\begin{aligned} c_U(k) &= \lim_{N\to\infty} \frac{1}{N^{k^2}} \int_{U(N)} |\Lambda_X(1)|^{2k} \, dX \\ &= \frac{\mathcal{G}(1+k)^2}{\mathcal{G}(1+2k)}, \end{aligned} \tag{1.4.13}$$

with $\mathcal{G}(s)$ the Barnes $\mathcal{G}$-function. The identification of N with $\log T$ in the asymptotics for the moments comes naturally from equating the densities of the eigenvalues of the characteristic polynomials on the unit circle and the zeros of the zeta function up to height T on the critical line. We've seen earlier that for $k \in \mathbb{N}$, we have

$$c_U(k) = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}. \tag{1.4.14}$$

Thus, $c_U(k) \cdot (k^2)!$ is an integer for $k \in \mathbb{N}$ and more importantly, this expression matches the previous known and conjectural values for $c_\zeta(k)$.

An explanation for the leading order coefficient of the $2k$-th moment of the zeta function splitting as an arithmetic term and a term arising from random matrix theory was given by Gonek, Hughes and Keating in [GHK07]. They show that one may approximate the zeta function using a hybrid Euler-Hadamard product. In particular, one has $\zeta(s) = P_X(s)Z_X(s)(1 + o(1))$ where $P_X(s)$ is a truncated Euler product and $Z_X(s)$ is a truncated form of the Hadamard product over the non-trivial zeros of the zeta function. It is conjectured in [GHK07] that the moments of $\zeta(\frac{1}{2}+it)$ factor as the moments of $P_X(\frac{1}{2} + it)$ multiplied by the moments of $Z_X(\frac{1}{2} + it)$ and this is known as the Splitting Conjecture. Evaluating the $2k$-th moment of $P_X(\frac{1}{2}+it)$ leads to the arithmetic factor $a(k)$ and the moments of $Z_X(\frac{1}{2} + it)$ yield the random matrix coefficient $c_U(k)$. It is also shown in [GHK07] that the Splitting Conjecture holds in the cases $k = 1$ and $k = 2$.

While computing asymptotic formulae for higher moments of the zeta function are beyond the reach of current techniques, there are now sharp lower and upper bounds for the moments. First, Ramachandra [Ram80] proved that for $k \in \mathbb{N}$,

$$M_k(T) \gg (\log T)^{k^2}. \tag{1.4.15}$$

This lower bound was extended to all rational $k > 0$ by Heath-Brown [HB81]. Radziwiłł and Soundararajan [RS13] then obtained lower bounds of the conjectured order for all real $k \geq 1$. Heap and Soundararajan [HS22], with a new method, showed that the above lower bound holds for all real $k > 0$.

Assuming the Riemann Hypothesis, Soundararajan [Sou09] obtained the upper bound

$$M_k(T) \ll (\log T)^{k^2+\varepsilon} \tag{1.4.16}$$

for all real $k > 0$ and $\varepsilon > 0$. By refining Soundararajan's argument, Harper [Har13] was able to remove the ε in the exponent of the $\log T$. Unconditional upper bounds have been obtained by Heap, Radziwiłł, and Soundararajan [HRS19] who proved that for real $0 \leq k \leq 2$, we have

$$M_k(T) \ll (\log T)^{k^2}. \tag{1.4.17}$$

Precise conjectures for the $2k$-th moment of $\zeta(\frac{1}{2} + it)$ including lower order terms were formulated by Conrey et al. in [CFK$^+$05]. They gave a method, now known as the "recipe", that allows one to make conjectures for the integral moments of various families of L-functions, including shifts. For instance, the conjecture for the $2k$-th moment of the zeta function is that

$$M_k(T) = Q_k\big(\log \tfrac{T}{2\pi}\big) + o(1), \tag{1.4.18}$$

where $Q_k(x)$ is an polynomial of degree k^2 which may be expressed explicitly in the form of a sum over permutations or as a contour integral, see Conjecture 1.5.1. in [CFK+05]. Interestingly, the conjectural expression has almost identical structure to that for the moments of the characteristic polynomial over $U(N)$ computed in [CFK+03], despite not using random matrix theory to develop the conjectures.

In a series of papers, Conrey and Keating [CK15b, CK15c, CK15d, CK16, CK18] gave an alternate viewpoint for making conjectures on the moments of the zeta function. For sets $\mathbf{A} = \{\alpha_1, \ldots, \alpha_k\}$ and $\mathbf{B} = \{\beta_1, \ldots, \beta_k\}$ of small shifts, they consider

$$\int_0^\infty \Big(\prod_{\alpha\in\mathbf{A}} \zeta(\tfrac{1}{2} + it + \alpha)\Big)\Big(\prod_{\beta\in\mathbf{B}} \zeta(\tfrac{1}{2} - it + \beta)\Big)\psi\Big(\frac{t}{T}\Big)\, dt \tag{1.4.19}$$

where ψ is some smooth function with compact support. Conrey and Keating show that one may heuristically evaluate the quantity in (1.4.19) by careful examination of the Dirichlet series

$$\prod_{\alpha\in\mathbf{A}} \zeta(\tfrac{1}{2} + it + \alpha) = \sum_{n=1}^\infty \frac{\tau_{\mathbf{A}}(n)}{n^s}, \tag{1.4.20}$$

where $\tau_{\mathbf{A}}(n)$ is the generalised divisor function defined by the above series. By making suitable assumptions on correlations of the divisor function, Conrey and Keating arrive at a conjecture that agrees with the conjecture of [CFK+05].

1.4.1 The Ratios Conjecture

As well as moments of powers of the Riemann zeta function on the critical line, another quantity of interest are the mean values of ratios of zeta functions. For instance, Farmer [Far93] originally conjectured that as $T \to \infty$,

$$\frac{1}{T}\int_0^T \frac{\zeta(\frac{1}{2} + it + \alpha)\zeta(\frac{1}{2} - it + \beta)}{\zeta(\frac{1}{2} + it + \gamma)\zeta(\frac{1}{2} - it + \delta)} dt \sim \frac{(\alpha + \delta)(\beta + \gamma)}{(\alpha + \beta)(\gamma + \delta)} - T^{-(\alpha+\beta)}\frac{(\alpha - \gamma)(\beta - \delta)}{(\alpha + \beta)(\gamma + \delta)}, \tag{1.4.21}$$

where $\alpha, \beta, \gamma, \delta$ are complex shifts with real parts of size $\asymp 1/\log T$. This a deep conjecture with significant implications for the distribution of the zeros of $\zeta(s)$. In particular, Farmer's conjecture implies the pair correlation conjecture of Montgomery discussed in the previous section.

More general conjectures for the mean values of arbitrary numbers of zeta functions in the numerator or denominator were developed by Conrey, Farmer and Zirnbauer [CFZ08] by extending the recipe of Conrey et al. [CFK$^+$05]. The conjectures of Conrey, Farmer and Zirnbauer apply to different families of L-functions as well and are now known collectively as the Ratios Conjecture. The Ratios Conjecture has a large number of applications to number theoretic problems as demonstrated by Conrey and Snaith in [CS07]. For example, if one assumes the Ratios Conjecture, then one may compute the one-level density of zeros of a family of L-functions, including lower order terms, with no restriction on the Fourier transform of the test function. At leading order, the result agrees with the density conjecture of Katz and Sarnak.

Since we expect that the zeta function and other L-functions can be modelled well by the characteristic polynomials of random matrices, it is natural to expect that the ratios of L-functions will behave similarly to those of the characteristic polynomials. In the random matrix case, the mean values of ratios of characteristic polynomials over the classical compact groups may be computed exactly, see for example the works [CFS05, BS06, HPZ16]. Similarly to the shifted moments, the expressions for the ratios in both the L-function and characteristic polynomial case appear almost identical, the significant difference being the presence of arithmetic factors on the L-function side.

1.5 Families of L-functions

Here we give an example of a family of L-functions with symplectic symmetry and another with orthogonal symmetry.

1.5.1 Quadratic Dirichlet L-functions

A *fundamental discriminant* is an integer $d \neq 1$ such $d \equiv 1 \pmod 4$ and d square-free or $d = 4k$ with $k \equiv 2$ or $3 \pmod 4$ and k square-free. In other words, fundamental discriminants are the discriminants of quadratic number fields $\mathbb{Q}(\sqrt{d})$. Given a fundamental discriminant d, we define the quadratic Dirichlet character

$$\chi_d(n) = \left(\frac{d}{n}\right), \tag{1.5.1}$$

where $(\frac{d}{n})$ is the Kronecker symbol. This χ_d is a primitive, real Dirichlet character of modulus $|d|$. The quadratic Dirichlet L-function attached to the character χ_d is given for $\mathrm{Re}(s) > 1$ by the Dirichlet series and Euler product

$$L(s,\chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s} = \prod_p \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1}. \tag{1.5.2}$$

The L-function has an analytic continuation to the complex plane and satisfies the functional equation

$$L(s,\chi_d) = X_d(s)L(1-s,\chi_d), \tag{1.5.3}$$

where $X_d(s) = |d|^{1/2-s}X(s,a)$ with $a = 0$ if $d > 0$ and $a = 1$ if $d < 0$, and

$$X(s,a) = \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1+a-s}{2}\right)\Gamma\left(\frac{s+a}{2}\right)^{-1}. \tag{1.5.4}$$

The family of quadratic Dirichlet L-functions is an example of a symplectic family ordered by the conductor $|d|$. The k-th moment of this family is defined as

$$\frac{1}{D^*}\sum_{|d|\le D}^{*} L(\tfrac{1}{2},\chi_d)^k, \tag{1.5.5}$$

where the sum is only over fundamental discriminants d and D^* is the number of terms in the sum. Given the symplectic symmetry of this family, we use the characteristic polynomials of random symplectic matrices to model the L-functions. In particular, Keating and Snaith [KS00a] made the following conjecture for the moments of $L(\frac{1}{2},\chi_d)$.

Conjecture 1.5.1 (Keating and Snaith)**.** *As $D \to \infty$, we have*

$$\frac{1}{D}\sum_{|d|\le D}^{*} L(\tfrac{1}{2},\chi_d)^k \sim a_{L_D}(k)c_{L_D}(k)\cdot(\log D)^{\frac{k(k+1)}{2}}, \tag{1.5.6}$$

where $a_{L_D}(k)$ is an arithmetic factor given by

$$a_{L_D}(k) = \prod_p \left(1-\frac{1}{p}\right)^{\frac{k(k+1)}{2}}\left(\frac{1}{2}\left(\left(1-\frac{1}{\sqrt{p}}\right)^{-k}+\left(1+\frac{1}{\sqrt{p}}\right)^{-k}\right)+\frac{1}{p}\right)\left(1+\frac{1}{p}\right)^{-1}, \tag{1.5.7}$$

and $c_{L_D}(k)$ is the random matrix coefficient given by

$$\begin{aligned} c_{L_D}(k) &= \lim_{N\to\infty} N^{-\frac{k(k+1)}{2}}\int_{Sp(2N)} \Lambda_X(1)^k\,dX \\ &= 2^{k^2/2}\frac{\mathcal{G}(k+1)\sqrt{\Gamma(k+1)}}{\sqrt{\mathcal{G}(2k+1)\Gamma(2k+1)}}. \end{aligned} \tag{1.5.8}$$

In particular, for integer k, we have that

$$c_{L_D}(k) = 2^{\frac{k(k+1)}{2}} \prod_{j=1}^{k} \frac{j!}{(2j)!}. \tag{1.5.9}$$

Asymptotic formula for the first and second moment of this family were obtained by Jutila [Jut81] with the following results.

Theorem 1.5.2 (Jutila). *As $D \to \infty$, we have*

$$\sum_{0<d\leq D}^{*} L(\tfrac{1}{2}, \chi_d) = \frac{P(1)}{4\zeta(2)} D \left(\log\left(\frac{D}{\pi}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right) + 4\gamma - 1 + 4\frac{P'}{P}(1) \right) + O(D^{\frac{3}{4}+\varepsilon}), \tag{1.5.10}$$

where γ is Euler's constant and

$$P(s) = \prod_{p} \left(1 - \frac{1}{p^s(p+1)}\right). \tag{1.5.11}$$

Theorem 1.5.3 (Jutila). *As $X \to \infty$, we have*

$$\sum_{0<d\leq D}^{*} L(\tfrac{1}{2}, \chi_d)^2 = \frac{c}{\zeta(2)} D(\log D)^3 + O(D(\log D)^{\frac{5}{2}+\varepsilon}), \tag{1.5.12}$$

where

$$c = \frac{1}{48} \prod_{p} \left(1 - \frac{4p^2 - 3p + 1}{p^4 + p^3}\right). \tag{1.5.13}$$

Goldfeld and Hoffstein [GH85], using the method of multiple Dirichlet series, improved the bound on the error term in the first moment to $O(D^{\frac{19}{32}+\varepsilon})$. Young [You09] considered the smoothed first moment and using a recursive method, obtained an error of size $O(D^{\frac{1}{2}+\varepsilon})$. Based on numerical computations of the moments, Alderson and Rubinstein [AR12] conjecture that the error in the first moment is $O(D^{\frac{1}{4}+\varepsilon})$.

In [Sou00], using Poisson summation and generalised Gauss sums, Soundararajan was able to obtain the second and third moments with a power saving error. Restricting to discriminants of the form $8d$ where d is an odd, square-free integer for the sake of simplicity to ensure that χ_{8d} is a primitive character of conductor $8d$ with $\chi_{8d}(-1) = 1$, Soundararajan proved the following.

Theorem 1.5.4 (Soundararajan). *There is a polynomial Q of degree 3 and a polynomial R of degree 6 such that*

$$\sum_{0<d\leq D}^{*} L(\tfrac{1}{2}, \chi_{8d})^2 = D\, Q(\log D) + O(D^{\frac{5}{6}+\varepsilon}), \tag{1.5.14}$$

and

$$\sum_{0<d\le D}^{*} L(\tfrac{1}{2},\chi_{8d})^3 = D\,R(\log D) + O(D^{\frac{11}{12}+\varepsilon}), \tag{1.5.15}$$

where the sums are over fundamental discriminants $8d$.

Sono [Son20] improved the error term in the smoothed second moment to $O(D^{\frac{1}{2}+\varepsilon})$. The method of multiple Dirichlet series was applied to the third moment by Diaconu, Goldfeld and Hoffstein in [DGH03] who were able to improve the error term by proving that

$$\sum_{|d|\le D}^{*} L(\tfrac{1}{2},\chi_d)^3 = D\sum_{j=0}^{6} c_j(\log D)^j + O(D^{\theta+\varepsilon}), \tag{1.5.16}$$

where the c_j are computable constants and $\theta \sim 0.853$. Young [You12] further improved the error to $O(D^{\frac{3}{4}+\varepsilon})$ by considering a smoothed third moment similarly to [You09]. Diaconu and Whitehead [DW21] also considered the smoothed third moment and proved the existence of a secondary main term of size $D^{\frac{3}{4}}$ with the error bounded by $O(D^{\frac{2}{3}+\varepsilon})$ for all $\varepsilon > 0$.

An asymptotic formula for the fourth moment of $L(\frac{1}{2},\chi_d)$ was obtained, conditional on GRH, by Shen [She21].

Theorem 1.5.5 (Shen)**.** *Assume GRH for $L(s,\chi_d)$ for all fundamental discriminants d. For any $\varepsilon > 0$, we have*

$$\sum_{\substack{0<d\le D\\(d,2)=1}}^{*} L(\tfrac{1}{2},\chi_{8d})^4 = cD(\log D)^{10} + O\big(D(\log D)^{9.75+\varepsilon}\big), \tag{1.5.17}$$

for some explicit constant c.

The constant c in Shen's theorem is shown to match that predicted by Conjecture 1.5.1. Unconditionally, Shen also proved the lower bound

$$\sum_{\substack{0<d\le D\\(d,2)=1}}^{*} L(\tfrac{1}{2},\chi_{8d})^4 \ge (c+o(1))D(\log D)^{10}. \tag{1.5.18}$$

More recently, Shen and Stucky [SS24] have obtained an unconditional asymptotic formula for the smoothed fourth moment with the first four main terms. They proved that for $\Phi : (0,\infty) \to \mathbb{R}$ a smooth, compactly supported function, we have

$$\sum_{(d,2)=1}^{*} L(\tfrac{1}{2},\chi_d)^4\Phi(\tfrac{d}{D}) = D\,Q_{10}(\log\tfrac{8D}{\pi}) + O\big(D(\log D)^{6+\varepsilon}\big), \tag{1.5.19}$$

where $Q_{10}(x)$ is a polynomial of degree 10.

For this family of L-functions, sharp lower bounds of the form

$$\sum_{0<d\le D}^{*} L(\tfrac{1}{2},\chi_d)^k \gg_k D(\log D)^{\frac{k(k+1)}{2}} \tag{1.5.20}$$

were first obtained for even integers k by Rudnick and Soundararajan [RS06]. The works of Radziwiłł and Soundararajan [RS13] and Heap and Soundararajan [HS22] extend this lower bound to all real $k>0$. The sharp upper bound

$$\sum_{0<d\le D}^{*} L(\tfrac{1}{2},\chi_d)^k \ll_k D(\log D)^{\frac{k(k+1)}{2}} \tag{1.5.21}$$

follows, conditional on GRH, using the method of Soundararajan [Sou09] and its refinement by Harper [Har13]. Gao [Gao21], applying the argument of Radziwiłł and Soundararajan [RS15], showed that unconditionally for $0\le k\le 2$, we have

$$\sum_{\substack{0<d\le D\\(d,2)=1}}^{*} |L(\tfrac{1}{2},\chi_{8d})|^k \ll_k D(\log D)^{\frac{k(k+1)}{2}}. \tag{1.5.22}$$

We emphasise that all of the results mentioned above are consistent with Conjecture 1.5.1. We also note that conjectures for the integral shifted moments of this family produced using the recipe of CFKRS have been formulated, see Conjecture 1.5.3 in [CFK+05]. The conjecture for the shifted moments are very similar in structure to the shifted moment formulae for the characteristic polynomials over $Sp(2N)$ in Theorem 1.1.5. At the central point, the conjecture is that

$$\sum_{|d|\le D}^{*} L(\tfrac{1}{2},\chi_d) = Q_k(\log D) + o(1), \tag{1.5.23}$$

where Q_k is an explicit polynomial of degree $k(k+1)/2$ and where one can show that the leading order term of Q_k agrees with that previously put forward in Conjecture 1.5.1, see, for example Section 5 in [KO08].

1.5.2 Quadratic twists of elliptic curve L-functions

For an example of a family of L-functions with orthogonal symmetry, we consider the quadratic twists of an elliptic curve L-function. To construct the family, let E be an elliptic curve defined over $\mathbb{Q}$ with conductor Q_E. The L-function attached to the elliptic curve E is defined by

$$L_E(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^{s+1/2}} = \prod_{p|Q_E} \left(1 - \frac{a_p}{p^{s+1/2}}\right)^{-1} \prod_{p\nmid Q_E} \left(1 - \frac{a_p}{p^{s+1/2}} + \frac{1}{p^{2s}}\right)^{-1}. \quad (1.5.24)$$

Here, for primes p, the coefficients a_p are given by $a_p = p+1-|E(\mathbb{F}_p)|$, where $|E(\mathbb{F}_p)|$ is the number of points on E, including the point at infinity, over $\mathbb{F}_p$. Hasse's bound gives that $|a(n)| \le \tau(n)\sqrt{n}$ where $\tau(n)$ is the divisor function and so the Dirichlet series and Euler product converges absolutely for $\mathrm{Re}(s) > 1$. It follows from the works [Wil95, TW95, BCDT01] on modular elliptic curves over $\mathbb{Q}$ and Fermat's last theorem that $L_E(s)$ is a modular L-function. In particular, $L_E(s)$ has an be analytic continuation to the complex plane and satisfies the functional equation

$$L_E(s) = w_E Y(s) L_E(1-s), \quad (1.5.25)$$

where the root number $w_E = \pm 1$ and

$$Y(s) = \left(\frac{\sqrt{Q_E}}{2\pi}\right)^{1-2s} \Gamma\left(\frac{3}{2} - s\right) \Gamma\left(\frac{1}{2} + s\right)^{-1}. \quad (1.5.26)$$

Now, for a fundamental discriminant d with $(d, Q_E) = 1$, the quadratic twist of the L-function $L_E(s)$ by the quadratic character $\chi_d(n) = (\frac{d}{n})$ is defined for $\mathrm{Re}(s) > 1$ by

$$L_E(s, \chi_d) = \sum_{n=1}^{\infty} \frac{a_n \chi_d(n)}{n^{s+\frac{1}{2}}}. \quad (1.5.27)$$

This is the L-function of the elliptic curve E_d, the quadratic twist of E by d and thus it also admits an analytic continuation to $\mathbb{C}$. For $(d, Q_E) = 1$, the conductor of E_d is $d^2 Q_E$ and the twisted L-function satisfies the functional equation

$$L_E(s, \chi_d) = w_E \chi_d(-Q_E) Y_d(s) L_E(1-s, \chi_d), \quad (1.5.28)$$

where $Y_d(s) = |d|^{1-2s} Y(s)$.

The family $\{L_E(s, \chi_d) : d \text{ a fundamental discriminant with } (d, Q_E) = 1\}$ is an example of an orthogonal family. Just as matrices in the orthogonal group $O(2N)$ have determinant ± 1, we see that the sign of the functional equation of the L-functions $L_E(s, \chi_d)$ is also ± 1. So, as we do with the orthogonal matrices, it is useful to split the family of L-functions according to whether their sign is $+1$ or -1. The family

$$\{L_E(s,\chi_d) : d \text{ a fundamental discriminant with } (d,Q_E)=1,\ w_E\chi_d(-Q_E)=+1\} \tag{1.5.29}$$

then has *even* orthogonal symmetry and we use matrices from the special orthogonal group $SO(2N)$ to model the family. Conversely, the family

$$\{L_E(s,\chi_d) : d \text{ a fundamental discriminant with } (d,Q_E)=1,\ w_E\chi_d(-Q_E)=-1\} \tag{1.5.30}$$

has *odd* orthogonal symmetry and we use the negative coset $O^-(2N)$ of orthogonal matrices with determinant -1 for comparison.

Turning to the moments of this family of quadratic twists, by the functional equation, those L-functions with sign -1 necessarily have $L_E(\frac{1}{2},\chi_d)=0$. Therefore we restrict our attention to the family of L-functions with sign $+1$. Recall that the moments of the characteristic polynomial $\Lambda_X(s)$ at $s=1$ over $SO(2N)$ were computed by Keating and Snaith [KS00a] and are given in Theorem 1.1.3. Using their result, Keating and Snaith made the following conjecture for the moments of $L_E(\frac{1}{2},\chi_d)$.

Conjecture 1.5.6 (Keating and Snaith)**.** *As $D\to\infty$, we have*

$$\frac{1}{D^*}\sideset{}{^*}\sum_{\substack{|d|\le D\\ w_E\chi_d(-Q_E)=+1}} L_E(\tfrac{1}{2},\chi_d)^k \sim a_{L_E}(k)c_{L_E}(k)(\log D)^{\frac{k(k-1)}{2}}, \tag{1.5.31}$$

where the sum is over fundamental discriminants d, D^* is the number of terms in the sum and $a_{L_E}(k)$ is an arithemtic factor depending on the curve E. Also, $c_{L_E}(k)$ is the random matrix coefficient given by

$$\begin{aligned} c_{L_E}(k) &= \lim_{N\to\infty} N^{-\frac{k(k-1)}{2}}\int_{SO(2N)} \Lambda_X(1)^k\,dX \\ &= 2^{\frac{k^2}{2}}\frac{\mathcal{G}(k+1)\sqrt{\Gamma(2k+1)}}{\sqrt{\mathcal{G}(2k+1)\Gamma(k+1)}} \end{aligned} \tag{1.5.32}$$

In particular, for integer k, we have that

$$c_{L_E}(k) = 2^{\frac{k(k+1)}{2}}\prod_{j=0}^{k-1}\frac{j!}{(2j)!}. \tag{1.5.33}$$

The arithmetic factor $a_{L_E}(k)$ in the conjecture depends both on the elliptic curve E and the set of discriminants being summed over and is not simple to write down in general. In practice, when computing these moments, one usually makes simplifying assumptions such as the conductor Q_E being square-free (which restricts the discriminants to certain residue classes). In these cases the arithmetic factor can be written down explicitly, see for instance Section 4.4 of [CFK$^+$05] or Section 3 of [CKRS06].

Compared to the family of quadratic Dirichlet L-functions, computing moments in this family of quadratic twists is a more difficult problem. The first moment was studied in [BFH90, Iwa90, MM91], primarily with a view to obtaining non-vanishing results for the L-functions $L_E(s, \chi_d)$ and thus inferring information about the analytic ranks of the elliptic curves E_d.

The second moment of this family, which is the limit of current techniques, was considered by Soundararajan and Young in [SY10]. Their result, conditional on GRH, may be stated as follows.

Theorem 1.5.7 (Soundararajan and Young)**.** *Assuming GRH, we have*

$$\sideset{}{^*}\sum_{\substack{0<8d\leq D \\ (d,2)=1 \\ w_E\chi_d(-Q_E)=+1}} L_E(\tfrac{1}{2}, \chi_{8d})^2 = \big(c + o(1)\big) D \log D, \tag{1.5.34}$$

for some explicit constant c.

The constant c in the theorem agrees precisely with that conjectured by Keating and Snaith in Conjecture 1.5.6. Soundararajan and Young also argue that if they consider the smoothed second moment, then they obtain an error of size $O(D(\log D)^{\frac{3}{4}+\varepsilon})$ and unconditionally, they prove the lower bound

$$\sideset{}{^*}\sum_{\substack{0<8d\leq D \\ (d,2)=1 \\ w_E\chi_d(-Q_E)=+1}} L_E(\tfrac{1}{2}, \chi_{8d})^2 \geq \big(c + o(1)\big) D \log D. \tag{1.5.35}$$

We note the similarity between the results of Soundararajan and Young and those of Shen in [She21]. This supports the idea that computing a fourth moment for the family of quadratic Dirichlet L-function is comparable in difficulty to computing the second moment in this family of quadratic twists. Recently, Soundararajan and Young's theorem was made unconditional by Li in [Li24].

For the family of quadratic twists where the sign of the functional equation is -1, as the L-function itself vanishes at the central point, one may instead consider the moments of the derivative of $L_E(s, \chi_d)$ at $s = 1/2$. Various results of this nature were obtained by Petrow in [Pet12].

Finally, we remark that the works of [Sou09, Har13] on upper bounds and those of [RS06, RS15, HS22] on lower bounds allow one to obtain the sharp bounds

$$D(\log D)^{\frac{k(k-1)}{2}} \ll_k \sideset{}{^*}\sum_{\substack{|d|\leq D \\ w_E\chi_d(-Q_E)=+1}} L_E(\tfrac{1}{2},\chi_d)^k \ll_k D(\log D)^{\frac{k(k-1)}{2}}, \tag{1.5.36}$$

for all $k > 0$. As before, the lower bounds here are unconditional and the upper bounds are conditional on GRH.

1.6 Problems considered in this thesis

Here we give a brief introduction to and summary of the problems we consider in this thesis. A more detailed background and literature review will be reserved for the relevant chapters.

1.6.1 Moments of moments

The first problem we consider are the moments of moments of characteristic polynomials of random matrices. The moments of moments consist of an average of the characteristic polynomial over the unit circle first and then an average over the matrix group, hence the name. Specifically, they are defined as

$$\mathrm{MoM}_{G(N)}(k,\beta) := \int_{G(N)} \left(\frac{1}{2\pi}\int_0^{2\pi} |\Lambda_X(e^{-i\theta})|^{2\beta} d\theta\right)^k dX, \tag{1.6.1}$$

where dX is the Haar measure on $G(N)$. The moments of moments are of interest due to their link to the maximum value attained by $\log|\Lambda(s)|$ on the unit circle. In particular, the moments of moments and the maximum value of the characteristic polynomial are the subject of conjectures of Fyodorov, Hiary and Keating [FHK12] and Fyodorov and Keating [FK14]. In Chapter 3, we consider the moments of moments over the symplectic group $Sp(2N)$ and the special orthogonal group $SO(2N)$. Using analytic techniques, we prove asymptotic formulae for $\mathrm{MoM}_{G(N)}(k,\beta)$ when $k,\beta \in \mathbb{N}$. We also discuss the analogous moments of moments of families of L-functions with symplectic or orthogonal symmetry.

1.6.2 Moments of derivatives

The moments of derivatives of characteristic polynomials are also of interest, with an application being to gaining an insight into the moments of derivatives of the Riemann zeta and other L-functions. For example, in [CRS06] Conrey, Rubinstein and Snaith considered the $2k$-th moment

$$\int_{U(N)} |\Lambda_X'(1)|^{2k}\, dX \tag{1.6.2}$$

of the derivative of the characteristic polynomial over $U(N)$. They obtained an asymptotic formula for integer k and used their result to make a conjecture for the analogous moments

$$\frac{1}{T}\int_0^T |\zeta'(\tfrac{1}{2}+it)|^{2k}\, dt \tag{1.6.3}$$

of the derivative of the Riemann zeta function. A particular motivation for the study of the derivative of the zeta function is a theorem of Speiser [Spe35] which states that the Riemann hypothesis is equivalent to $\zeta'(s)$ having no zeros in the region $0 < \mathrm{Re}(s) < 1/2$.

More recently, moments and joint moments of higher order derivatives have seen a considerable amount of interest. In Chapter 4, we study the joint moments of arbitrary order derivatives of the characteristic polynomials over $Sp(2N)$, $SO(2N)$ and $O^-(2N)$. We obtain asymptotic formulae for the joint moments for all three matrix ensembles and use our results to make conjectures for the joint moments of derivatives of L-functions with symmetry type Sp, SO or O^-.

In Chapter 7, we consider a mixed second moment of derivatives of quadratic Dirichlet L-functions with prime conductor over function fields. We prove an asymptotic formula for arbitrary order derivatives and compare the result with our conjecture made in Chapter 4.

1.6.3 Mollified moments

The technique of mollifying the Riemann zeta function was first successfully applied by Selberg [Sel42] in 1942 to show that a positive proportion of the non-trivial zeros lie on the critical line $\mathrm{Re}(s) = 1/2$. In 1974, Levinson [Lev74] gave a new method for obtaining a lower bound on the proportion of non-trivial zeros on the critical line by evaluating the mollified second moment.

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)M_N(\tfrac{1}{2}+it)|^2\,dt. \tag{1.6.4}$$

Here, the mollifier $M_N(s)$ is a Dirichlet polynomial of length N whose purpose is to approximate $\zeta(s)^{-1}$. Levinson was able to evaluate the mollified moment above for $N = T^\theta$ with $\theta < 1/2$ and thus deduced that at least $1/3$ of the non-trivial zeros are on the line. One can also mollify the moments of other families of L-functions with a particular application being to obtain non-vanishing results. For instance, Soundararajan [Sou00] computed the mollified first and second moment of quadratic Dirichlet L-functions $L(s,\chi_d)$ and consequently showed that for at least 87.5% of fundamental discriminants d, we have $L(\frac{1}{2},\chi_d) \neq 0$. Computing mollified moments is in general rather complicated but it was shown by Conrey and Snaith [CS07] that one may relatively easily obtain the mollified moments by assuming the relevant ratios conjecture for the L-functions involved.

In Chapter 5, we look at the result of Bui, Florea and Keating [BFK23] which proves the ratios conjecture for quadratic Dirichlet L-functions over function fields in certain ranges. In particular, we improve the bound on the error term in the case of two L-functions in the numerator and the denominator. Then, in Chapter 6, we prove asymptotic formulae for the mollified first and second moments of these quadratic L-functions in the function field setting by making use of the result of [BFK23] and our improved error bound in Chapter 5. As an application, we obtain non-vanishing results on the derivatives of the completed L-function at $s = 1/2$.

Chapter 2

Background on L-functions over function fields

In this chapter, we introduce the necessary background on L-functions over function fields. We use Rosen's book [Ros02] as a general reference.

2.1 Polynomials over finite fields

Let $\mathbb{F}_q$ be a finite field with q elements. We will denote by $\mathbb{A} = \mathbb{F}_q[t]$ the ring of polynomials with coefficients in $\mathbb{F}_q$. The ring of integers $\mathbb{Z}$ and $\mathbb{A}$ have many properties in common. For example, they are both unique factorisation domains, both have infinitely many prime members and they both have a finite number of units. Thus, many number theoretic questions and results have analogues in the polynomial case.

An element $f \in \mathbb{A}$ is of the form

$$f(t) = \alpha_n t^n + \alpha_{n-1} t^{n-1} + \cdots + \alpha_0, \tag{2.1.1}$$

where $\alpha_j \in \mathbb{F}_q$. If $\alpha_n \neq 0$, the *degree* of f is given by n and we write $\deg(f) = n$. Moreover, if $\alpha_n \neq 0$, we define the *sign* of f to be $\alpha_n \in \mathbb{F}_q^*$ and write $\mathrm{sgn}(f) = \alpha_n$, where $\mathbb{F}_q^*$ is the multiplicative group of non-zero elements in $\mathbb{F}_q$. We also set $\mathrm{sgn}(0) = 0$ and $\deg(0) = -\infty$. Some useful properties of the degree and sign are given in the following proposition.

Proposition 2.1.1. *Let $f, g \in \mathbb{A}$ be non-zero polynomials. Then we have*

1. $\deg(fg) = \deg(f) + \deg(g)$,
2. $\mathrm{sgn}(fg) = \mathrm{sgn}(f)\mathrm{sgn}(g)$,

3. $\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$, *with equality if* $\deg(f) \neq \deg(g)$.

The polynomial f is *monic* if $\mathrm{sgn}(f) = 1$. We will denote the set of monic polynomials in $\mathbb{A}$ by $\mathcal{M}$. Also, we denote by $\mathcal{M}_n$ and $\mathcal{M}_{\leq n}$ the sets of monic polynomials of degree n and of degree at most n, respectively. The monic polynomials are the analogue of the positive integers.

A polynomial $f \in \mathbb{A}$ is *irreducible* if it cannot be written as a product $f(t) = g(t)h(t)$ with $\deg(g) > 0$ and $\deg(h) > 0$. The polynomial f is *reducible* otherwise. We denote by $\mathcal{P}$ the set of monic irreducible polynomials in $\mathbb{A}$ and by $\mathcal{P}_n$ the set of monic irreducibles of degree n. The monic irreducible polynomials play the role of the prime numbers so we refer to them as the "prime" polynomials. Importantly, every non-zero $f \in \mathbb{A}$ has a unique factorisation

$$f = \alpha P_1^{e_1} P_2^{e_2} \cdots P_r^{e_r}, \tag{2.1.2}$$

where $\alpha \in \mathbb{F}_q^*$, each P_i is a prime polynomial, $P_i \neq P_j$ for $i \neq j$ and e_i is a non-negative integer. We will generally reserve the letter P to represent a prime polynomial.

The *norm* of a polynomial $f \in \mathbb{A}$ is defined to be $|f| = q^{\deg(f)}$ if $f \neq 0$ and 0 if $f = 0$. This norm satisfies $|fg| = |f||g|$ for all $f, g, \in \mathbb{A}$.

The zeta function of $\mathbb{A}$ is defined for $\mathrm{Re}(s) > 1$ by the Dirichlet series and Euler product

$$\zeta_q(s) = \sum_{f \in \mathcal{M}} \frac{1}{|f|^s} = \prod_{P \in \mathcal{P}} \left(1 - \frac{1}{|P|^s}\right)^{-1}. \tag{2.1.3}$$

Since there are q^n monic polynomials of degree n, we have that

$$\zeta_q(s) = \sum_{n=0}^{\infty} \sum_{f \in \mathcal{M}_n} \frac{1}{|f|^s} = \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} = \frac{1}{1 - q^{1-s}}. \tag{2.1.4}$$

This expression for the zeta function immediately yields a meromorphic continuation to the complex plane with a simple pole at $s = 1$ with residue $1/\log q$. Furthermore, we see that $\zeta_q(s) \neq 0$ for all $s \in \mathbb{C}$ so this zeta function satisfies a Riemann hypothesis. We define the gamma function of $\mathbb{A}$ by

$$\Gamma_q(s) = \frac{1}{1 - q^{-s}}. \tag{2.1.5}$$

Then, by combing the above observations we have the following.

Theorem 2.1.2. *The zeta function* $\zeta_q(s)$ *has a meromorphic continuation to the complex plane with a simple pole at* $s = 1$ *with residue* $1/\log q$*. Moreover, the* *function $\xi_q(s) = q^{-s}\Gamma_q(s)\zeta_q(s)$ satisfies the functional equation*

$$\xi_q(s) = \xi_q(1-s). \tag{2.1.6}$$

The above result highlights the analogy between the zeta function $\zeta_q(s)$ and the Riemann zeta function. The function $\zeta_q(s)$ is a much simpler object though due to the simple expression in (2.1.4) and the lack of zeros. In various scenarios it will be useful to make the change of variables $u = q^{-s}$. For the zeta function, we therefore define $\mathcal{Z}(u) = \zeta_q(s)$, i.e.

$$\mathcal{Z}(u) = \frac{1}{1-qu}. \tag{2.1.7}$$

Using the Euler product for $\zeta_q(s)$, one can count the number of prime polynomials of degree n. This leads to the following result which is the analogue of the Prime Number Theorem.

Theorem 2.1.3 (Prime Polynomial Theorem)**.** *The number of prime polynomials of degree n is*

$$|\mathcal{P}_n| = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right). \tag{2.1.8}$$

Remark 2.1.4.

1. *If we set $x = q^n$, then the result of the Prime Polynomial Theorem reads*

$$|\mathcal{P}_n| = \frac{x}{\log_q x} + O\left(\frac{\sqrt{x}}{\log_q x}\right), \tag{2.1.9}$$

which resembles the conjectured exact form of the Prime Number Theorem.

2. *The number of prime polynomials of degree n may in fact be counted exactly as*

$$|\mathcal{P}_n| = \frac{1}{n}\sum_{d|n} \mu(d) q^{\frac{n}{d}}, \tag{2.1.10}$$

where $\mu(n)$ is the usual Möbius function. See Proposition 2.1 and the Corollary in [Ros02].

2.2 Arithmetic functions on $\mathbb{F}_q[t]$

The usual arithmetic functions defined on the integers have natural analogues is the case of polynomials over a finite field. The Möbius function on $\mathbb{F}_q[t]$ is defined by

$$\mu(f) = \begin{cases} (-1)^r & \text{if } f = \alpha P_1 \cdots P_r, \\ 0 & \text{otherwise,} \end{cases} \tag{2.2.1}$$

where the P_j's are distinct prime polynomials. The Euler totient function $\varphi(f)$ on $\mathbb{F}_q[t]$ is defined as

$$\varphi(f) = \sum_{\substack{g \in \mathcal{M} \\ \deg(g) < \deg(f) \\ (f,g)=1}} 1. \tag{2.2.2}$$

We denote the divisor function on $\mathbb{F}_q[t]$ by $\tau(f)$. That is, $\tau(f)$ is the number of monic divisors of f. By [Ros02, Proposition 2.5], the partial sums of the divisor function satisfy

$$\sum_{f \in \mathcal{M}_n} \tau(f) = q^n(n+1). \tag{2.2.3}$$

We will also make use of the following generalised divisor function $\tau_{\mathbf{A}}(f)$. For a set $\mathbf{A} = \{\alpha_1, \ldots, \alpha_k\}$ of complex numbers, the divisor function $\tau_{\mathbf{A}}(f)$ is defined as the coefficient in the Dirichlet series

$$\prod_{j=1}^{k} \zeta_q(s + \alpha_j) = \sum_{f \in \mathcal{M}} \frac{\tau_{\mathbf{A}}(f)}{|f|^s}. \tag{2.2.4}$$

Specifically, the divisor function $\tau_{\mathbf{A}}(f)$ is given by

$$\tau_{\mathbf{A}}(f) = \sum_{f = f_1 \cdots f_k} \frac{1}{|f_1|^{\alpha_1} \cdots |f_k|^{\alpha_k}}, \tag{2.2.5}$$

where the sum is over monic divisors of f.

2.3 Dirichlet characters and L-functions

Dirichlet L-functions over function fields are defined analogously to those in the number field setting. We begin with Dirichlet characters defined on $\mathbb{A}$.

Definition 2.3.1. *Let $Q \in \mathcal{M}$. A Dirichlet character modulo Q is a function $\chi : \mathbb{A} \to \mathbb{C}$ such that*

1. *$\chi(f + gQ) = \chi(f)$ for all $f, g \in \mathbb{A}$,*

2. $\chi(f)\chi(g) = \chi(fg)$ *for all* $f, g \in \mathbb{A}$,

3. $\chi(f) \neq 0$ *if and only if* $(f, Q) = 1$.

There are $\varphi(Q)$ Dirichlet characters modulo Q. The trivial character modulo Q is denoted by χ_0 and is given by

$$\chi_0(f) = \begin{cases} 1 & \text{if } (f, Q) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.1}$$

The character χ is *even* if $\chi(\alpha) = 1$ for all $\alpha \in \mathbb{F}_q^*$ and is *odd* otherwise. Furthermore, we have the following orthogonality relations for these Dirichlet characters [Ros02, Propostion 4.2].

Proposition 2.3.2. *Let* χ *and* ψ *be two Dirichlet characters modulo* Q *and* f *and* g *two elements of* $\mathbb{A}$ *relatively prime to* Q. *Then*

1.

$$\sum_f \chi(f)\overline{\psi(f)} = \begin{cases} \varphi(Q) & \textit{if } \chi = \psi, \\ 0 & \textit{otherwise.} \end{cases} \tag{2.3.2}$$

2.

$$\sum_\chi \chi(f)\overline{\chi(g)} = \begin{cases} \varphi(Q) & \textit{if } f \equiv g \pmod{Q}, \\ 0 & \textit{otherwise.} \end{cases} \tag{2.3.3}$$

where the first sum is over any set of representatives of $\mathbb{A}/Q\mathbb{A}$ *and the second sum is over all Dirichlet characters modulo* Q.

Given a Dirichlet character χ modulo Q, the Dirichlet L-function associated to χ is defined for $\mathrm{Re}(s) > 1$ by the Dirichlet series and Euler product

$$L(s, \chi) = \sum_{f \in \mathcal{M}} \frac{\chi(f)}{|f|^s} = \prod_{P \in \mathcal{P}} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1}. \tag{2.3.4}$$

If $\chi = \chi_0$ is the trivial character then

$$L(s, \chi_0) = \prod_{P | Q} \left(1 - \frac{1}{|P|^s}\right) \zeta_q(s), \tag{2.3.5}$$

so that $L(s, \chi_0)$ has a meromorphic continuation to the complex plane with a simple pole at $s = 1$ arsinign from the zeta function. On the other hand, if χ is a non-trivial character, then we may write

$$L(s, \chi) = \sum_{n=0}^{\infty} \Big(\sum_{f \in \mathcal{M}_n} \chi(f) \Big) q^{-ns}. \tag{2.3.6}$$

From the orthogonality relations, we then have the following result.

Proposition 2.3.3. *Let χ be a non-trivial character modulo Q. Then, for $n \geq$* $\deg(Q)$,

$$\sum_{f \in \mathcal{M}_n} \chi(f) = 0. \tag{2.3.7}$$

Consequently, the L-function $L(s, \chi)$ is given by a polynomial in q^{-s} of degree at most $\deg(Q) - 1$.

As a result of the proposition, we have that the Dirichlet L-function $L(s, \chi)$ has an analytic continuation to the entire complex plane.

2.4 Quadratic Dirichlet L-functions

2.4.1 The Reciprocity Law

Let $P \in \mathcal{P}$ be a prime polynomial and d a divisor of $q - 1$. Then if $f \in \mathbb{A}$ and $P \nmid f$, by Proposition 1.10 in [Ros02], we have that the congruence $x^d \equiv f \pmod{P}$ is solvable if and only if

$$f^{\frac{|P|-1}{d}} \equiv 1 \pmod{P}. \tag{2.4.1}$$

Since $f^{\frac{|P|-1}{d}}$ is an element of order dividing d in $(\mathbb{A}/P\mathbb{A})^*$, there is a unique element $\alpha \in \mathbb{F}_q^*$ such that

$$f^{\frac{|P|-1}{d}} \equiv \alpha \pmod{P}. \tag{2.4.2}$$

Definition 2.4.1. *If $P \nmid f$, let $(f/P)_d$ be the unique element of $\mathbb{F}_q^*$ such that*

$$f^{\frac{|P|-1}{d}} \equiv \left(\frac{f}{P}\right)_d \pmod{P}. \tag{2.4.3}$$

If $P|f$ define $(f/P)_d = 0$. The symbol $(f/P)_d$ is called the d-th power residue symbol.

Specialising to the case $d = 2$, we define the quadratic residue symbol $(f/P) \in \{\pm 1\}$ by

$$\left(\frac{f}{P}\right) \equiv f^{\frac{|P|-1}{2}} \pmod{P} \tag{2.4.4}$$

if $P \nmid f$, and define $(f/P) = 0$ if $P|f$. This is the analogue of the usual Legendre symbol. For a monic polynomial Q, we may then define the Jacobi symbol (f/Q) as follows. Suppose that Q has the prime factorisation $Q = P_1^{e_1} \cdots P_r^{e_r}$, then we define the Jacobi symbol by

$$\left(\frac{f}{Q}\right) = \prod_{j=1}^{r} \left(\frac{f}{P_j}\right)^{e_j}. \tag{2.4.5}$$

Similarly to the case over the integers, these quadratic residue symbols satisfy a reciprocity law.

Theorem 2.4.2 (Quadratic Reciprocity Law)**.** *Let $f, g \in \mathcal{M}$ be relatively prime, non-zero polynomials. Then,*

$$\left(\frac{f}{g}\right) = \left(\frac{g}{f}\right)(-1)^{\frac{q-1}{2}\deg(f)\deg(g)}. \tag{2.4.6}$$

Note that when $q \equiv 1 \pmod 4$, the reciprocity law reads

$$\left(\frac{f}{g}\right) = \left(\frac{g}{f}\right). \tag{2.4.7}$$

For the rest of the chapter and in Chapters 5, 6 and 7, we will assume for simplicity that $q \equiv 1 \pmod 4$.

2.4.2 Quadratic characters

We denote by $\mathcal{H}$ the set of monic, square-free polynomials in $\mathbb{F}_q[t]$ and by $\mathcal{H}_n$ the set of monic, square-free polynomials of degree n. The cardinality of $\mathcal{H}_n$ is

$$|\mathcal{H}_n| = \begin{cases} 1 & n = 0, \\ q^n(1 - q^{-1}) & n \geq 1, \end{cases} \tag{2.4.8}$$

as can be seen by considering the coefficient of q^{-ns} in the series

$$\sum_{n=0}^{\infty} \frac{|\mathcal{H}_n|}{q^{ns}} = \sum_{f \in \mathcal{H}} \frac{1}{|f|^s} = \frac{\zeta_q(s)}{\zeta_q(2s)}. \tag{2.4.9}$$

Given a $D \in \mathcal{H}$ with $\deg(D) > 0$, we define the quadratic character χ_D using the quadratic residue symbol by

$$\chi_D(f) = \left(\frac{D}{f}\right). \tag{2.4.10}$$

In other words, for a prime polynomial P, we have

$$\chi_D(P) = \begin{cases} 1 & \text{if } P \nmid D \text{ and } D \text{ is a square modulo } P, \\ -1 & \text{if } P \nmid D \text{ and } D \text{ is not a square modulo } P, \\ 0 & \text{if } P|D, \end{cases} \tag{2.4.11}$$

and then χ_D is extended to all $f \in \mathcal{M}$ completely multiplicatively. The character χ_D is a real Dirichlet character modulo D.

The quadratic Dirichlet L-function attached to the character χ_D is defined for $\mathrm{Re}(s) > 1$ by the Dirichlet series and Euler product

$$L(s, \chi_D) = \sum_{f \in \mathcal{M}} \frac{\chi_D(f)}{|f|^s} = \prod_{P \in \mathcal{P}} \left(1 - \frac{\chi_D(P)}{|P|^s}\right)^{-1}. \tag{2.4.12}$$

With the change of variables $u = q^{-s}$, we define $\mathcal{L}(u, \chi_D) = L(s, \chi_D)$. Then, for $|u| < q^{-1}$, we have that $\mathcal{L}(u, \chi_D)$ is given by the power series

$$\mathcal{L}(u, \chi_D) = \sum_{f \in \mathcal{M}} \chi_D(f) u^{\deg(f)} = \sum_{n=0}^{\infty} \Big(\sum_{f \in \mathcal{M}_n} \chi_D(f) \Big) u^n. \tag{2.4.13}$$

By Proposition 2.3.3, the coefficient of u^n in this series vanishes if $n \geq \deg(D)$ and so $\mathcal{L}(u, \chi_D)$ is a polynomial in u of degree at most $\deg(D) - 1$. Following the argument in [Rud10], we have that $\mathcal{L}(u, \chi_D)$ has a "trivial" zero at $u = 1$ if and only if $\deg(D)$ if even. Thus, we define the completed L-function $\mathcal{L}^*(u, \chi_D)$ by

$$\mathcal{L}(u, \chi_D) = (1-u)^{\lambda} \mathcal{L}^*(u, \chi_D), \qquad \lambda = \begin{cases} 1, & \deg(D) \text{ even}, \\ 0, & \deg(D) \text{ odd}. \end{cases} \tag{2.4.14}$$

The completed L-function $\mathcal{L}^*(u, \chi_D)$ is then a polynomial of even degree

$$2\delta = \deg(D) - 1 - \lambda \tag{2.4.15}$$

and satisfies the functional equation

$$\mathcal{L}^*(u, \chi_D) = (qu^2)^\delta \mathcal{L}^*\left(\frac{1}{qu}, \chi_D\right). \tag{2.4.16}$$

Similarly, for a prime polynomial $P \in \mathcal{P}$, we define the quadratic character χ_P by the quadratic residue symbol

$$\chi_P(f) = \left(\frac{f}{P}\right). \tag{2.4.17}$$

This is again a real Dirichlet character modulo P and the associated L-function is given by

$$L(s, \chi_P) = \sum_{f \in \mathcal{M}} \frac{\chi_P(f)}{|f|^s}. \tag{2.4.18}$$

Note that since we are assuming that $q \equiv 1 \pmod 4$, we have

$$\chi_P(f) = \left(\frac{f}{P}\right) = \left(\frac{P}{f}\right). \tag{2.4.19}$$

Thus, this definition of the character χ_P coincides with the definition of the characters χ_D defined above. Since the set of prime polynomials is a subset of the set of square-free polynomials, all of the above discussion holds for these quadratic L-functions to a prime modulus as well.

2.4.3 The hyperelliptic ensemble $\mathcal{H}_{2g+1}$

Let $D \in \mathcal{H}$. Then the quadratic character χ_D is an even character if $\deg(D)$ is even and is an odd character otherwise. As we've seen previously, the L-function $\mathcal{L}(u, \chi_D)$ has a trivial zero at $u = 1$ if and only if $\deg(D)$ is even. For simplicity, in this thesis we will restrict our focus to odd characters and therefore we choose $D \in \mathcal{H}_{2g+1}$ for some integer $g \geq 0$.

Given a $D \in \mathcal{H}_{2g+1}$, by summarising the previous few sections, we have that the L-function

$$\mathcal{L}(u, \chi_D) = \sum_{n=0}^{2g} \left(\sum_{f \in \mathcal{M}_n} \chi_D(f) \right) u^n \tag{2.4.20}$$

is a polynomial of degree $2g$ which satisfies the functional equation

$$\mathcal{L}(u, \chi_D) = (qu^2)^g \mathcal{L}\left(\frac{1}{qu}, \chi_D\right). \tag{2.4.21}$$

Equivalently, we have that $L(s, \chi_D)$ is a polynomial of degree $2g$ in q^{-s} and satisfies

$$L(s,\chi_D) = q^{g(1-2s)} L(1-s,\chi_D). \tag{2.4.22}$$

We also define the ξ-function $\xi(s,\chi_D) := q^{\frac{g}{2}(2s-1)} L(s,\chi_D)$ which satisfies the symmetric functional equation

$$\xi(s,\chi_D) = \xi(1-s,\chi_D). \tag{2.4.23}$$

We consider $\mathcal{H}_{2g+1}$ as a probability space (or ensemble) with the uniform probability measure. The expected value of any function F on $\mathcal{H}_{2g+1}$ is then

$$\langle F \rangle = \frac{1}{\mathcal{H}_{2g+1}} \sum_{D \in \mathcal{H}_{2g+1}} F(D). \tag{2.4.24}$$

We will similarly consider the space $\mathcal{P}_{2g+1}$ of prime polynomials of odd degree when working with quadratic L-functions to a prime modulus.

2.4.4 Zeta functions of curves

For $D \in \mathcal{H}_{2g+1}$, one may consider the hyperelliptic curve C_D given by the affine equation

$$C_D : y^2 = D(x). \tag{2.4.25}$$

The curve C_D defines a smooth, projective, geometrically connected curve of genus g, say. For $|u| < q^{-1}$, the zeta function of the curve introduced by Artin [Art24] is defined by

$$Z_{C_D}(u) = \exp\left(\sum_{n=1}^{\infty} N_n(C_D) \frac{u^n}{n} \right), \tag{2.4.26}$$

where $N_n(C_D)$ is the number of points on C_D over $\mathbb{F}_{q^n}$, including the point at infinity. Weil [Wei48] proved that $Z_{C_D}(u)$ is a rational function of the form

$$Z_{C_D}(u) = \frac{P_{C_D}(u)}{(1-u)(1-qu)}, \tag{2.4.27}$$

where $P_{C_D}(u) \in \mathbb{Z}[u]$ is a polynomial of degree $2g$ with integer coefficients and $P_{C_D}(0) = 1$. It was also proven by Weil that $P_{C_D}(u)$ satisfies the functional equation

$$P_{C_D}(u) = (qu^2)^g P_{C_D}\left(\frac{1}{qu}\right), \tag{2.4.28}$$

and that all of the zeros of $P_{C_D}(u)$ lie on the circle $|u| = q^{-\frac{1}{2}}$. It was proven in Artin's thesis that the polynomial $P_{C_D}(u)$ coincides with the L-function $\mathcal{L}(u, \chi_D)$. In particular, all of the zeros of $\mathcal{L}(u, \chi_D)$ are on the circle $|u| = q^{-\frac{1}{2}}$ or equivalently, all of the zeros of $L(s, \chi_D)$ are on the line $\mathrm{Re}(s) = 1/2$. Therefore the L-functions $L(s, \chi_D)$ satisfy a Riemann hypothesis.

2.4.5 A spectral interpretation

For $D \in \mathcal{H}_{2g+1}$, the zeros of $\mathcal{L}(u, \chi_D)$ all lie on the circle $|u| = q^{-\frac{1}{2}}$ so we may write the zeros of $\mathcal{L}(u, \chi_D)$ as $q^{-\frac{1}{2}} e^{\pm i\theta_j}$ with $\theta_j \in [0, 2\pi)$ for $j = 1, \dots, 2g$. In this case, there exists a unitary symplectic matrix $\Theta_D \in Sp(2g)$, defined up to conjugacy and called the Frobenius of $\mathcal{L}(u, \chi_D)$, such that $e^{i\theta_j}$ are its eigenvalues and consequently

$$\mathcal{L}(u, \chi_D) = \det(I - \sqrt{q} u \Theta_D). \tag{2.4.29}$$

This provides a clear spectral interpretation for the zero of these L-functions.

Katz and Sarnak [KS99b] showed that if one fixes the genus g, then as $q \to \infty$ the conjugacy classes (or Frobenius classes) $\{\Theta_D : D \in \mathcal{H}_{2g+1}\}$ become equidistributed with respect to Haar measure on the unitary symplectic group $Sp(2g)$. This implies that for any continuous function F on $Sp(2g)$, we have that

$$\lim_{q \to \infty} \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} F(\Theta_D) = \int_{Sp(2g)} F(X)\, dX. \tag{2.4.30}$$

where as usual, dX denotes the Haar measure. Thus, one may obtain quantities such as the moments of $\mathcal{L}(u, \chi_D)$ in the large q limit using a random matrix theory computation. For example, the moments at the central point $s = 1/2$ are given by

$$\lim_{q \to \infty} \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \mathcal{L}(q^{-1/2}, \chi_D)^k = \int_{Sp(2g)} \det(I - X)^k\, dX, \tag{2.4.31}$$

and the moments of the characteristic polynomial on the right are given by Theorem 1.1.3 of Keating and Snaith:

$$\int_{Sp(2N)} \det(I - X)^k\, dX = 2^{2kN} \prod_{j=1}^{N} \frac{\Gamma(N + j + 1)\Gamma(j + k + 1/2)}{\Gamma(j + 1/2)\Gamma(N + j + k + 1)}. \tag{2.4.32}$$

In the other regime of q fixed and $g \to \infty$, there is no such equidistribution result. The large g regime is also more similar the the number field case and is the regime we consider in this thesis.

2.5 Moments of L-functions over function fields

We conclude this background chapter by discussing the literature on the moments of quadratic Dirichlet L-functions over function fields. The first moment of $L(\frac{1}{2}, \chi_D)$ over the hyperelliptic ensemble $\mathcal{H}_{2g+1}$ was considered by Andrade and Keating [AK12] who proved the following asymptotic formula.

Theorem 2.5.1 (Andrade and Keating). *Let q be the fixed cardinality of the ground field $\mathbb{F}_q$ and assume for simplicity that $q \equiv 1 \pmod 4$. Then*

$$\sum_{D \in \mathcal{H}_{2g+1}} L(\tfrac{1}{2}, \chi_D) = \frac{P(1)}{2\zeta_q(2)}|D| \left(\log_q |D| + 1 + \frac{4}{\log q} \frac{P'}{P}(1) \right) + O(|D|^{\frac{3}{4} + \frac{\log_q 2}{2}}), \quad (2.5.1)$$

where $|D| = q^{2g+1}$ and

$$P(s) = \prod_{P \in \mathcal{P}} \left(1 - \frac{1}{|P|^s(|P|+1)} \right). \quad (2.5.2)$$

Comparing the above result to the result of Jutila in Theorem 1.5.2, we see the similarity between the number field and function field cases. Florea [Flo17c] showed that there is a secondary main term in the above asymptotic for the first moment of the form $q^{\frac{2g+1}{3}} R(2g+1)$, where R is an explicit polynomial of degree 1, and was able to bound the error term by $O(q^{\frac{g}{2}(1+\varepsilon)})$ for any $\varepsilon > 0$. Florea's approach also allowed her to compute the second and third moments with a power saving error in [Flo17b].

Theorem 2.5.2 (Florea). *Let q be a prime with $q \equiv 1 \pmod 4$. Then*

$$\sum_{D \in \mathcal{H}_{2g+1}} L(\tfrac{1}{2}, \chi_D)^2 = \frac{q^{2g+1}}{\zeta_q(2)} P(2g+1) + O(q^{g(1+\varepsilon)}) \quad (2.5.3)$$

and

$$\sum_{D \in \mathcal{H}_{2g+1}} L(\tfrac{1}{2}, \chi_D)^3 = \frac{q^{2g+1}}{\zeta_q(2)} Q(2g+1) + O(q^{\frac{3g}{2}(1+\varepsilon)}), \quad (2.5.4)$$

where $P(x)$ is a polynomial of degree 3 and $Q(x)$ is a polynomial of degree 6 whose coefficients can be computed explicitly.

In this case we see a clear resemblance between Florea's result and that of Soundararajan in Theorem 1.5.4. Using the method of multiple Dirichlet series, Diaconu [Dia19] proved the existence of a secondary main term in the third moment of size $q^{\frac{3g}{2}}$.

The fourth moment was also computed by Florea in [Flo17a] although without a power saving error.

Theorem 2.5.3 (Florea)**.** *Let q be a prime with $q \equiv 1 \pmod 4$. Then*

$$\sum_{D\in\mathcal{H}_{2g+1}} L(\tfrac{1}{2},\chi_D)^4 = q^{2g+1}(a_{10}g^{10} + a_9 g^9 + a_8 g^8) + O(q^{2g+1}g^{7+\frac{1}{2}+\varepsilon}), \tag{2.5.5}$$

where the coefficients a_{10}, a_9, a_8 are arithmetic factors which can be written down explicitly.

By adapting the recipe of Conrey et al. [CFK+05] to the function field setting, Andrade and Keating [AK14] put forward the conjecture that

$$\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}} L(\tfrac{1}{2},\chi_D)^k = P_k(2g+1) + o(1), \tag{2.5.6}$$

where $P_k(2g+1)$ is a polynomial of degree $k(k+1)/2$ given explicitly in the form of a contour integral. All of the results mentioned above agree precisely with the conjecture. Furthermore, numerical evidence in support of the conjecture was provided by Rubinstein and Wu in [RW15].

Sharp lower bounds for the k-th moment of $L(\frac{1}{2},\chi_D)$ for all odd q and even k were obtained by Andrade [And16] using the method of Rudnick and Soundararajan [RS05, RS06]. Almost sharp upper bounds for prime $q \equiv 1 \pmod 4$ and real $k > 0$ were proven by Florea in [Flo17a]. By applying the principles of Soundararajan [Sou09] and Harper [Har13] for upper bounds and those of Heap and Soundararajan [HS22] for lower bounds, Gao and Zhao [GZ23] obtained sharp upper and lower bounds for the moments of the quadratic Dirichlet L-functions over function fields.

Theorem 2.5.4 (Gao and Zhao)**.** *Suppose that $q \equiv 1 \pmod 4$ is a prime number. For every real number $k \geq 0$ we have for large g,*

$$\sum_{D\in\mathcal{H}_{2g+1}} L(\tfrac{1}{2},\chi_D)^k \asymp_k q^{2g+1}(2g+1)^{\frac{k(k+1)}{2}}, \tag{2.5.7}$$

and

$$\sum_{P\in\mathcal{P}_{2g+1}} L(\tfrac{1}{2},\chi_P)^k \asymp_k q^{2g+1}(2g+1)^{\frac{k(k+1)}{2}-1}. \tag{2.5.8}$$

Recently, by applying methods from algebraic topology, Bergström, Diaconu, Petersen and Westerland [BDPW24] and Miller, Patzt, Petersen and Randal-Williams

[MPPRW24] have proven that for fixed integer k and q a sufficiently large odd prime power, the conjecture of Andrade and Keating on the moments of $L(\frac{1}{2}, \chi_D)$ does indeed hold. Specifically, Theorem 1.5 in [MPPRW24] states that

$$\frac{1}{q^{2g+1}} \sum_{D \in \mathcal{H}_{2g+1}} L(\tfrac{1}{2}, \chi_D)^k = Q_k(2g+1) + O(4^{g(k+1)} q^{-\frac{g+6}{12}}), \tag{2.5.9}$$

where $Q_k(x)$ is an explicit polynomial of degree $k(k+1)/2$.

For the moments of $L(\frac{1}{2}, \chi_P)$ over prime polynomials $P \in \mathcal{P}_{2g+1}$, less is known in this case as averaging over primes is a more difficult problem than averaging over square-frees. The first moment was evaluated by Andrade and Keating in [AK13] who proved the following.

Theorem 2.5.5 (Andrade and Keating). *Let $\mathbb{F}_q$ be a fixed finite field of odd cardinality with $q \equiv 1 \pmod 4$. Then for every $\varepsilon > 0$, we have*

$$\sum_{P \in \mathcal{P}_{2g+1}} (\log_q |P|) L(\tfrac{1}{2}, \chi_P) = \frac{|P|}{2} (\log_q |P| + 1) + O(|P|^{\frac{3}{4}+\varepsilon}) \tag{2.5.10}$$

In the same paper, Andrade and Keating also proved an asymptotic formula for the second moment.

Theorem 2.5.6 (Andrade and Keating). *Let $\mathbb{F}_q$ be a fixed finite field of odd cardinality with $q \equiv 1 \pmod 4$. Then we have*

$$\sum_{P \in \mathcal{P}_{2g+1}} L(\tfrac{1}{2}, \chi_P)^2 = \frac{1}{24\zeta_q(2)} |P| (\log_q |P|)^2 + O(|P| \log_q |P|). \tag{2.5.11}$$

Note that $|P|$ is constant over $\mathcal{P}_{2g+1}$ so one has $|P| = q^{2g+1}$ and $\log_q |P| = 2g+1$ in the above two theorems. In [BF20], Bui and Florea were able to improve the error in the second moment and identify a lower order term by showing that

$$\frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P \in \mathcal{P}_{2g+1}} L(\tfrac{1}{2}, \chi_P)^2 = \frac{g^3}{3\zeta_q(2)} + g^2 \left(\frac{3}{2} + \frac{1}{2q} \right) + O(q^{\frac{3}{2}+\varepsilon}). \tag{2.5.12}$$

Florea's method involving Poisson summation does not apply to summations over prime polynomials so one cannot currently obtain a power saving error in the second moment of $L(\frac{1}{2}, \chi_P)$ or compute moments higher than the second. Conjectures for the integral moments over $\mathcal{P}_{2g+1}$ were made by Andrade, Jung and Shamesaldeen [AJS21] by applying the recipe of [CFK+05]. Specifically, they conjecture that for every integer $k \geq 1$,

$$\frac{1}{|\mathcal{P}_{2g+1}|}\sum_{P\in\mathcal{P}_{2g+1}} L(\tfrac{1}{2},\chi_P)^k = R_k(2g+1)+o(1), \tag{2.5.13}$$

where $R_k(x)$ is an explicit polynomial of degree $k(k+1)/2$.

Chapter 3

Moments of moments of symplectic and orthogonal characteristic polynomials

3.1 Moments of moments of characteristic polynomials

Let $G(N) \in \{U(N), Sp(2N), SO(2N)\}$, where as usual, $U(N)$ is the group of $N \times N$ unitary matrices, $Sp(2N)$ is the the group of $2N \times 2N$ unitary symplectic matrices and $SO(2N)$ is the group of $2N \times 2N$ orthogonal matrices with determinant $+1$. In this chapter we consider the *moments of moments* of the characteristic polynomial

$$\Lambda_X(e^{-i\theta}) = \det(I - Ae^{-i\theta}) \tag{3.1.1}$$

on the unit circle. The moments of moments consist of an average over the unit circle first and then an average over the matrix ensemble, hence the name. Specifically, they are defined as

$$\mathrm{MoM}_{G(N)}(k,\beta) := \int_{G(N)} \left(\frac{1}{2\pi} \int_0^{2\pi} |\Lambda_X(e^{-i\theta})|^{2\beta} d\theta \right)^k dX, \tag{3.1.2}$$

where dX is the Haar measure on $G(N)$. The principal motivation for studying the moments of moments is their link to the maximum value of the characteristic polynomials on the unit circle. For example, in the case of the unitary group $U(N)$, Fyodorov, Hiary and Keating [FHK12] and subsequently Fyodorov and Keating [FK14], made conjectures on the moments of moments and for the maximum value of $\log |\Lambda_X(e^{-i\theta})|$ for $0 \leq \theta < 2\pi$. The idea behind the link between the moments of moments and the maximum value is that one can think of $\mathrm{MoM}_{G(N)}(k,\beta)$ as the k-th moment of the random variable

$$Z_N(X;\beta) := \frac{1}{2\pi}\int_0^{2\pi} |\Lambda_X(e^{-i\theta})|^{2\beta} d\theta. \tag{3.1.3}$$

Now, define $V_N(X;\theta) := -2\log|\Lambda_X(e^{-i\theta})|$. Then, borrowing the terminology of statistical mechanics,

$$Z_N(X;\beta) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left(-\beta V_N(X;\theta)\right) d\theta \tag{3.1.4}$$

is a *partition function* of the field $\theta \mapsto \log|\Lambda_X(e^{-i\theta})|$ and $\beta > 0$ is the *inverse temperature*, see for instance [Isi71, Chapter 3]. The quantity

$$F(\beta) = -\frac{1}{\beta}\log Z_N(X;\beta) \tag{3.1.5}$$

is known as the *free energy* of the system and the maximum of $\log|\Lambda_X(e^{-i\theta})|$ can be recovered as

$$\lim_{\beta\to\infty} F(\beta) = -2 \max_{\theta\in[0,2\pi)} \log|\Lambda_X(e^{-i\theta})|. \tag{3.1.6}$$

The conjecture of Fyodorov and Keating [FK14] on the maximum of the characteristic polynomial is the following.

Conjecture 3.1.1 (Fyodorov–Keating (2014))**.** *For $X \in U(N)$ sampled uniformly with respect to the Haar measure,*

$$\max_{\theta\in[0,2\pi)} \log|\Lambda_X(e^{-i\theta})| = \log N - \frac{3}{4}\log\log N + y_{X,N}, \tag{3.1.7}$$

where $(y_{X,N})_{N\in\mathbb{N}}$ is a sequence of random variables which converge in distribution.

Remark 3.1.2. *By "converge in distribution", we mean that the cumulative distribution function of $y_{X,N}$ converges to some limiting distribution function as $N \to \infty$. It is further conjectured by Fyodorov and Keating that $y_{X,N}$ should converge in distribution to the sum of two independent Gumbel random variables.*

Briefly, the justification for Conjecture 3.1.1 is that $V_N(X;\theta)$ behaves similarly to a log-correlated field with respect to θ. The conjecture was verified up to leading order by Arguin, Belius and Bourgade [ABB17] and Paquette and Zeitouni [PZ17] have proven that the subleading term is also correct. Chhaibi, Madaule and Najnudel [CMN18] have proven the strongest result which includes both the main terms in

Conjecture 3.1.1 and shows that the random variables which constitute the error are tight. Their results also apply more general to the case of the Circular β Ensemble (CβE) [1].

Naturally, the conjecture for the characteristic polynomials of matrices from $U(N)$ can be used to formulate an analogous conjecture for the Riemann zeta function with the usual correspondence of N with $\log T$. The conjecture for the maximum of the zeta function on the critical line, now known as the Fyodorov-Hiary-Keating conjecture, is the following.

Conjecture 3.1.3. *Let $t \sim [T, 2T]$, that is, t is chosen uniformly from the interval $[T, 2T]$. Then*

$$\max_{h \in [0, 2\pi)} \log |\zeta(\tfrac{1}{2} + i(t+h))| = \log\log T - \frac{3}{4} \log\log\log T + x_t, \tag{3.1.8}$$

where the random variable x_t is expected to have a limiting value distribution as $t \to \infty$.

Najnudel [Naj18], assuming RH, proved Conjecture 3.1.3 up to leading order. Arguin, Belius, Bourgade, Radziwiłł and Soundararajan [ABB+19] then showed that one may remove the assumption of RH from Najnudel's result. Concerning the lower order terms, Harper [Har19] proved an almost sharp upper bound for the maximum including the subleading term. Finally, in [ABR20], Arguin, Bourgade and Radziwiłł were able to prove the expected upper bound for the maximum of the zeta function and further bound the tail of the random variable x_t. Then, in [ABR23], they prove the corresponding lower bound, thus confirming the prediction of the Fyodorov-Hiary-Keating conjecture. For a more in depth discussion of the conjectures of [FHK12, FK14] and work in their direction, we refer to the excellent survey article [BK22].

Returning to the moments of moments, one of the conjectures of [FK14] is that as $N \to \infty$,

$$\mathrm{MoM}_{U(N)}(k, \beta) \sim \begin{cases} \left(\frac{\mathcal{G}(1+\beta)^2}{\mathcal{G}(1+2\beta)\Gamma(1-\beta^2)}\right)^k \Gamma(1-k\beta^2) N^{k\beta^2} & \text{if } k < 1/\beta^2, \\ c(k,\beta) N^{k^2\beta^2 - k + 1} & \text{if } k > 1/\beta^2, \end{cases} \tag{3.1.9}$$

where $\mathcal{G}(s)$ is the Barnes $\mathcal{G}$-function and $c(k, \beta)$ is some unspecified function of k and β. At the transition point $k = 1/\beta^2$, the moments of moments are conjectured to grow like $N \log N$. The justification for the conjecture in (3.1.9) is that for $k \in \mathbb{N}$, one can write $\mathrm{MoM}_{U(N)}(k, \beta)$ as a k-fold integral of a Toeplitz determinant. Then,

[1] Not to be confused with the parameter β appearing in the moments of moments

one uses the Fisher-Hartwig asymptotic formula and, provided that $k < 1/\beta^2$, the Selberg integral can be applied to yield to conjecture in this regime. In the case that $k > 1/\beta^2$, the Fisher-Hartwig singularities will coalesce, leading to a much more difficult analysis and hence the lack of an expression for $c(k, \beta)$ in this case.

The above asymptotics were confirmed in the case that $k = 2$ and $\beta > -1/4$ is real by Claeys and Krasovsky [CK15a] by proving asymptotics for Toeplitz determinants using Riemann-Hilbert problem techniques. Their approach also established a link between the leading order coefficient $c(2, \beta)$ and the Painlevé V equation. Fahs [Fah21] then extended these results to general $k \in \mathbb{N}$ and non-negative, real β but without an explicit expression for $c(k, \beta)$. The case of $k = 2$ and $\beta \in \mathbb{N}$ was established in [KRRGR18] via two alternate methods. The first is complex analytic and the second is combinatorial, leading to two different expressions for the leading order coefficient $c(2, \beta)$. It was then shown in [BGR18] how the combinatorial expression for $c(2, \beta)$ can also be linked to Painlevé V.

In recent work, Keating and Wong [KW22], through the perspective of Gaussian multiplicative chaos, obtain an asymptotic formula for $\mathrm{MoM}_{U(N)}(k, \beta)$ at the critical point $k\beta^2 = 1$ for $k \geq 2$ an integer. Their result confirms that the moments of moments are of the order $N \log N$ as $N \to \infty$. They also conjecture that this asymptotic result holds for all $k > 1$ and provide a heuristic argument in support of this.

Bailey and Keating [BK19] obtained an asymptotic formula for $\mathrm{MoM}_{U(N)}(k, \beta)$ when $k, \beta \in \mathbb{N}$ by generalising the analytic argument deployed in [KRRGR18] with the following result.

Theorem 3.1.4 (Bailey-Keating (2019))**.** *For $k, \beta \in \mathbb{N}$, as $N \to \infty$,*

$$\mathrm{MoM}_{U(N)}(k, \beta) = c(k, \beta) N^{k^2\beta^2 - k + 1}\big(1 + O\big(N^{-1}\big)\big), \tag{3.1.10}$$

where $c(k, \beta)$ can be written explicitly in the form of an integral. Furthermore, $\mathrm{MoM}_{U(N)}(k, \beta)$ is a polynomial in N of degree $k^2\beta^2 - k + 1$.

The proof of Theorem 3.1.4 uses the fact that for $k \in \mathbb{N}$, one can change the order of integration to obtain

$$\mathrm{MoM}_{U(N)}(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} I_{k,\beta}(U(N), \theta_1, \ldots, \theta_k)\, d\theta_1 \ldots d\theta_k, \tag{3.1.11}$$

where for $\underline{\theta} = (\theta_1, \ldots, \theta_k)$,

$$I_{k,\beta}(U(N),\underline{\theta}) = \int_{U(N)} \prod_{j=1}^{k} |\Lambda_X(e^{-i\theta_j})|^{2\beta}\, dX. \tag{3.1.12}$$

The function $I_{k,\beta}(U(N),\underline{\theta})$ is an autocorrelation function of the characteristic polynomials and was computed by Conrey et al. [CFK$^+$03]. Two equivalent expressions for $I_{k,\beta}(U(N),\underline{\theta})$ are given in [CFK$^+$03]; one takes the form of a combinatorial sum and the other is as a multiple contour integral. The first of these was used in [BK19] to prove that $\mathrm{MoM}_{U(N)}(k,\beta)$ is a polynomial in N and then an intricate analysis of the contour integral representation was used to determine the asymptotic behaviour.

Assiotis and Keating [AK21] gave an alternate proof of the asymptotic formula in Theorem 3.1.4 using a combinatorial approach involving constrained Gelfand-Tsetlin patterns. They therefore obtained a different expression for the leading order coefficient $c(k,\beta)$ as the volume of a certain region. It is remarked in [AK21] that their expression for $c(k,\beta)$ appears to be very difficult to obtain from the expression obtained in [BK19].

The combinatorial approach used in [AK21] was then applied by Assiotis, Bailey and Keating [ABK22] to the symplectic and special orthogonal groups to determine the asymptotic behaviour of the moments of moments for $k,\beta \in \mathbb{N}$. Their results are stated below.

Theorem 3.1.5 (Assiotis, Bailey and Keating)**.** *Let $k,\beta \in \mathbb{N}$. Then,* $\mathrm{MoM}_{Sp(2N)}(k,\beta)$ *is a polynomial function in N. Moreover, as $N \to \infty$,*

$$\mathrm{MoM}_{Sp(2N)}(k,\beta) = \mathfrak{c}_{Sp}(k,\beta) N^{k\beta(2k\beta+1)-k}\left(1 + O\left(N^{-1}\right)\right), \tag{3.1.13}$$

where the leading order term coefficient $\mathfrak{c}_{Sp}(k,\beta)$ is the volume of a certain convex region and is strictly positive.

Theorem 3.1.6 (Assiotis-Bailey-Keating)**.** *Let $k,\beta \in \mathbb{N}$. Then,* $\mathrm{MoM}_{SO(2N)}(k,\beta)$ *is a polynomial function in N. Moreover, as $N \to \infty$,*

$$\mathrm{MoM}_{SO(2N)}(1,1) = 2(N+1), \tag{3.1.14}$$

otherwise,

$$\mathrm{MoM}_{SO(2N)}(k,\beta) = \mathfrak{c}_{SO}(k,\beta) N^{k\beta(2k\beta-1)-k}\left(1 + O\left(N^{-1}\right)\right), \tag{3.1.15}$$

where the leading order term coefficient $\mathfrak{c}_{SO}(k,\beta)$ is given as a sum of volumes of certain convex regions and is strictly positive.

Our main goal in this chapter is to apply the analytic approach used by Bailey and Keating [BK19] on the unitary group to the symplectic and orthogonal cases. We therefore provide an analytic proof of the asymptotic formulae given in Theorems 3.1.5 and 3.1.6 and in particular, we obtain alternate expressions for the leading order coefficients explicitly in the form of a multiple integral. Our results are as follows.

Theorem 3.1.7. *For $k, \beta \in \mathbb{N}$, as $N \to \infty$,*

$$\mathrm{MoM}_{Sp(2N)}(k,\beta) = \gamma_{Sp}(k,\beta) N^{k\beta(2k\beta+1)-k} \left(1 + O\left(N^{-1}\right)\right), \tag{3.1.16}$$

where $\gamma_{Sp}(k,\beta)$ is given explicitly in the form of an integral, see (3.2.58).

Theorem 3.1.8. *For $k, \beta \in \mathbb{N}$ with $(k,\beta) \neq (1,1)$, as $N \to \infty$,*

$$\mathrm{MoM}_{SO(2N)}(k,\beta) = \gamma_{SO}(k,\beta) N^{k\beta(2k\beta-1)-k} \left(1 + O\left(N^{-1}\right)\right), \tag{3.1.17}$$

where $\gamma_{SO}(k,\beta)$ is given explicitly in the form of an integral, see (3.3.14).

We will prove Theorems 3.1.7 and 3.1.8 in Sections 3.2.1 and 3.3.1, respectively.

In general, when computing asymptotics for moments of moments, the combinatorial approach still seems to be the simpler method, especially when $k > 2$. For example, the alternate proof of Theorem 3.1.4 given in [AK21] is much shorter than that given in [BK19]. The main difficulty in proving these asymptotic formulae is showing that the leading order coefficient obtained is non-zero, and this appears to be much easier when it is expressed as a volume using the combinatorial approach. In our proofs of Theorems 3.1.7 and 3.1.8, we are able to avoid this problem and infer that the leading order coefficients obtained are non-zero by comparison of our formulae with those given in [ABK22]. In particular, we do not need to explicitly evaluate the integral expression for our coefficients to check that they are non-zero as was necessary in [BK19]. We also note that the combinatorial approach has been successfully applied to the moments of moments in the more general case of the CβE by Assiotis in [Ass22].

3.1.1 Moments of moments of L-functions

Also considered in [FHK12, FK14] were the moments of moments of the Riemann zeta function. Analogously to the moments of moments of the characteristic polynomials, these consist of an average first over a short piece of the critical line and then an average over these intervals. Specifically, the moments of moments of $\zeta(s)$ are defined for $T > 0$ and $\mathrm{Re}(\beta) > -1/2$ by [2]

$$\mathrm{MoM}_{\zeta_T}(k, \beta) := \frac{1}{T} \int_0^T \left(\int_t^{t+1} |\zeta(\tfrac{1}{2} + ih)|^{2\beta} dh \right)^k dt. \tag{3.1.18}$$

Bailey and Keating [BK21], using the philosophy that the Riemann zeta function on the critical line can be modelled by the characteristic polynomials of random unitary matrices and Theorem 3.1.4, made the following conjecture.

Conjecture 3.1.9 (Bailey-Keating (2021))**.** *For $k, \beta \in \mathbb{N}$, as $T \to \infty$,*

$$\mathrm{MoM}_{\zeta_T}(k, \beta) = \alpha(k, \beta) c(k, \beta) \left(\log \tfrac{T}{2\pi} \right)^{k^2\beta^2 - k + 1} \left(1 + O_{k,\beta} \left(\log^{-1} T \right) \right), \tag{3.1.19}$$

where $c(k, \beta)$ is the same coefficient appearing in (3.1.10)*, and $\alpha(k, \beta)$ contains the arithmetic information in the form of an Euler product.*

It was then proven in [BK21] that Conjecture 3.1.9 follows from the conjecture of Conrey et al. [CFK$^+$05] for the shifted moments of the zeta function. Explicitly, they prove that a function which, according to the conjecture of [CFK$^+$05] approximates $\mathrm{MoM}_{\zeta_T}(k, \beta)$ up to a power saving in T, does indeed behave asymptotically as Conjecture 3.1.9 predicts $\mathrm{MoM}_{\zeta_T}(k, \beta)$ does. The proof is similar to that of Theorem 3.1.4 due to the similarity of the integral expressions for the shifted moments of the unitary characteristic polynomials and the Riemann zeta function. In the case $k = 2$, Curran [Cur23] has obtained sharp upper bounds for $\mathrm{MoM}_{\zeta_T}(2, \beta)$ for $0 \leq \beta \leq 1$ and lower bounds of the conjectured order for $\beta \geq 0$.

Finally, in [BK21], Bailey and Keating also considered the moments of moments of families of L-functions with symplectic or orthogonal symmetry. For each of the symmetry types, the moments of moments consist of an average over a short interval near the symmetry point and then an average through the family. Using Theorems 3.1.5 and 3.1.6, Bailey and Keating made conjectures for the asymptotic growth of the moments of moments of these families in the same spirit as that of Conjecture 3.1.9. Following the proof of both Theorems 3.1.7 and 3.1.8 in Sections 3.2.1 and 3.3.1, we will discuss the examples of a symplectic and orthogonal family of L-functions considered in [BK21] and show that the corresponding conjectures of Bailey and Keating also follow from the relevant shifted moments conjectures of [CFK$^+$05].

[2] In [FHK12, FK14], the intervals being averaged over were of length 2π rather than 1. However, in [BK21], the intervals were taken to be of length 1 for convenience. The analysis of [BK21] holds for any interval that is $O(1)$ as $T \to \infty$.

3.2 Moments of moments over $Sp(2N)$

3.2.1 Proof of Theorem 3.1.7

In this section we will prove Theorem 3.1.7. The argument is based on that used in [BK19] and makes use of the complex analytic techniques deployed in [KO08] and [KRRGR18]. The eigenvalues of matrices in $Sp(2N)$ lie on the unit circle and come in complex conjugate pairs $e^{i\phi_1}, e^{-i\phi_1}, \dots, e^{i\phi_N}, e^{-i\phi_N}$. Hence, we have that

$$\overline{\Lambda_X(e^{-i\theta})} = \Lambda_X(e^{i\theta}), \tag{3.2.1}$$

or equivalently,

$$\overline{\tilde{\Lambda}_A(\theta)} = \tilde{\Lambda}_A(-\theta), \tag{3.2.2}$$

where $\tilde{\Lambda}_A(\theta) := \Lambda_X(e^{-i\theta})$. Thus, for $k, \beta \in \mathbb{N}$, we can change the order of integration by Fubini's theorem and use (3.2.1) to see that

$$\mathrm{MoM}_{Sp(2N)}(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} I_{k,\beta}(Sp(2N), \theta_1, \dots, \theta_k)\, d\theta_1 \cdots d\theta_k, \tag{3.2.3}$$

where

$$I_{k,\beta}(Sp(2N), \underline{\theta}) := \int_{Sp(2N)} \prod_{j=1}^{k} \Lambda_X(e^{-i\theta_j})^\beta \Lambda_X(e^{i\theta_j})^\beta \, dX. \tag{3.2.4}$$

Explicit expressions for the shifted moments $I_{k,\beta}(Sp(2N), \underline{\theta})$ were computed by Conrey et al. [CFK+03]. In particular, using Theorem 1.1.5, we may write it in the form of a multiple contour integral as

$$\begin{aligned} I_{k,\beta}(Sp(2N), \underline{\theta}) = \frac{(-1)^{k\beta} 2^{2k\beta}}{(2\pi i)^{2k\beta}(2k\beta)!} \oint \cdots \oint \prod_{1 \le m \le n \le 2k\beta} \left(1 - e^{-z_m - z_n}\right)^{-1} \\ \times \frac{\Delta(z_1^2, \dots, z_{2k\beta}^2)^2 \prod_{n=1}^{2k\beta} z_n}{\prod_{n=1}^{2k\beta} \prod_{m=1}^{k} (z_n - i\theta_m)^{2\beta} (z_n + i\theta_m)^{2\beta}} e^{N \sum_{n=1}^{2k\beta} z_n} \, dz_1 \cdots dz_{2k\beta}, \end{aligned} \tag{3.2.5}$$

where $\Delta(z_1, \dots, z_n)$ is the Vandermonde determinant and the contours encircle the poles at $\pm i\theta_m$ for $m = 1, \dots, k$.

Now, each of the $2k\beta$ contours in (3.2.5) can be deformed into $2k$ small circles, one around each of the poles $\pm i\theta_m$, with connecting straight lines whose contributions to the integral will cancel out. The multiple integral $I_{k,\beta}(Sp(2N),\underline{\theta})$ can therefore be written as a sum of $(2k)^{2k\beta}$ integrals as follows. For $\varepsilon_j \in \{\pm 1, \ldots, \pm k\}$, let C_{ε_j} denote a small circular contour around $i\theta_{\varepsilon_j}$ if $\varepsilon_j > 0$ and a small circular contour around $-i\theta_{-\varepsilon_j}$ if $\varepsilon_j < 0$. Then we have that

$$I_{k,\beta}(Sp(2N),\underline{\theta}) = \frac{(-1)^{k\beta}2^{2k\beta}}{(2\pi i)^{2k\beta}(2k\beta)!} \sum_{\varepsilon_j \in \{\pm 1,\ldots,\pm k\}} J_{k,\beta}(\underline{\theta}; \varepsilon_1, \ldots, \varepsilon_{2k\beta}), \tag{3.2.6}$$

where the vector $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{2k\beta})$ determines the pole around which each contour is centred and

$$\begin{aligned} J_{k,\beta}(\underline{\theta};\underline{\varepsilon}) = \int_{C_{\varepsilon_{2k\beta}}} \cdots \int_{C_{\varepsilon_1}} \prod_{1\le m\le n\le 2k\beta} \left(1 - e^{-z_m - z_n}\right)^{-1} \\ \times \frac{\Delta(z_1^2, \ldots, z_{2k\beta}^2)^2 \prod_{n=1}^{2k\beta} z_n}{\prod_{n=1}^{2k\beta}\prod_{m=1}^{k}(z_n - i\theta_m)^{2\beta}(z_n + i\theta_m)^{2\beta}} e^{N\sum_{n=1}^{2k\beta} z_n}\, dz_1 \cdots dz_{2k\beta}. \end{aligned} \tag{3.2.7}$$

Remark 3.2.1. *Equation (3.2.6) can also be seen as an application of the residue theorem to the contour integral expression for $I_{k,\beta}(Sp(2N),\underline{\theta})$ but where we leave the residues in the form of an integral.*

Many of the summands in (3.2.6) are in fact zero as the following lemma demonstrates.

Lemma 3.2.2. *For a choice of contours $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{2k\beta})$ in (3.2.6) and for $j \in \{1, \ldots, k\}$, let m_j and n_j be the number of occurrences of j and $-j$ in $\underline{\varepsilon}$, respectively. Then, if $m_j + n_j > 2\beta$ for some j, we have that $J_{k,\beta}(\underline{\theta};\underline{\varepsilon})$ is identically zero.*

Proof. The proof is similar to that of Lemma 3.2 in [BK19] and Lemma 4.11 in [KRRGR18]. Without loss of generality, we may assume that $m_1 + n_1 > 2\beta$ so that $n_1 \ge 2\beta + 1 - m_1$. Note that since the integrand of $J_{k,\beta}(\underline{\theta};\underline{\varepsilon})$ is a symmetric function of the variables $z_1, \ldots, z_{2k\beta}$, we may suitably relabel the z_j's after permuting the entries of $\underline{\varepsilon}$ to leave $J_{k,\beta}(\underline{\theta};\underline{\varepsilon})$ unchanged. Hence, by permuting the entries of $\underline{\varepsilon}$ if necessary, we may assume that $\underline{\varepsilon}$ is of the form

$$\underline{\varepsilon} = (\underbrace{1, \ldots, 1}_{m_1}, \underbrace{-1, \ldots, -1}_{2\beta+1-m_1}, \ldots). \tag{3.2.8}$$

Next, for simplicity, we assume that $m_1 = 0$. If this is not the case, then to $J_{k,\beta}(\underline{\theta}; \underline{\varepsilon})$ we would make the change of variables $z_j \mapsto -z_j$ for $1 \leq j \leq m_1$ and the same argument applies. Thus, this is the only case we need consider.

By making the change of variables $z_j \mapsto z_j - i\theta_1$, the contours of integration for $z_1, \ldots, z_{2\beta+1}$ are now small circles around 0 and the integrand of $J_{k,\beta}(\underline{\theta}; \underline{\varepsilon})$ becomes

$$\frac{G(z_1, \ldots, z_{2\beta+1})\Delta\left((z_1 - i\theta_1)^2, \ldots, (z_{2k\beta} - i\theta_1)^2\right) dz_1 \cdots dz_{2k\beta}}{\prod_{n=1}^{2\beta+1} z_n^{2\beta}}, \tag{3.2.9}$$

where

$$\begin{aligned} G(z_1, \ldots, z_{2\beta+1}) = & \frac{\prod_{1\leq m\leq n\leq 2k\beta}\left(1 - e^{-z_m - z_n + 2i\theta_1}\right)^{-1}}{\prod_{n=1}^{2k\beta}\prod_{m=2}^{k}\left(z_n - i(\theta_1 + \theta_m)\right)^{2\beta}\left(z_n + i(\theta_m - \theta_1)\right)^{2\beta}} \\ & \times \frac{\Delta\left((z_1 - i\theta_1)^2, \ldots, (z_{2k\beta} - i\theta_1)^2\right)\prod_{n=1}^{2k\beta}(z_n - i\theta_1)\, e^{N\sum_{n=1}^{2k\beta}(z_n - i\theta_1)}}{\prod_{n=1}^{2k\beta}(z_n - 2i\theta_1)^{2\beta}\prod_{n=2\beta+2}^{2k\beta} z_n^{2\beta}} \end{aligned} \tag{3.2.10}$$

is analytic in a neighbourhood of the origin. The idea now is to show that the coefficient of $\prod_{n=1}^{2\beta+1} z_n^{-1}$ in the integrand of $J_{k,\beta}(\underline{\theta}; \underline{\varepsilon})$ is zero and hence by the residue theorem, so is the integral. We have that $G(z_1, \ldots, z_{2\beta+1})$ is analytic around zero and we can write the Vandermonde factor as

$$\begin{aligned} \Delta\left((z_1 - i\theta_1)^2, \ldots, (z_{2k\beta} - i\theta_1)^2\right) &= \Delta\left((z_1^2 - 2i\theta_1 z_1), \ldots, (z_{2k\beta}^2 - 2i\theta_1 z_{2k\beta})\right) \\ &= \sum_{\sigma\in S_{2k\beta}} \operatorname{sign}(\sigma)\prod_{n=1}^{2k\beta}(z_n^2 - 2i\theta_1 z_n)^{\sigma(n)-1}. \end{aligned} \tag{3.2.11}$$

For each permutation $\sigma \in S_{2k\beta}$, we must have $\sigma(n) - 1 \geq 2\beta$ for at least one $n \in \{1, 2, \ldots, 2\beta + 1\}$. It follows that there are no terms in the expansion of the Vandermonde of the form $\prod_{n=1}^{2\beta+1} z_n^{a(n)}$ with $a(n) \leq 2\beta - 1$ for all $n \leq 2\beta + 1$. Thus, as $G(z_1, \ldots, z_{2\beta+1})$ is analytic around zero, the coefficient of $\prod_{n=1}^{2\beta+1} z_n^{-1}$ in the integrand of $J_{k,\beta}(\underline{\theta}; \underline{\varepsilon})$ is zero which completes the proof. □

Lemma 3.2.2 implies that the non-zero summands in (3.2.6) are given by those $\underline{\varepsilon}$ for which $m_j + n_j = 2\beta$ for all j. This, and the fact that the integrand of $J_{k,\beta}(\underline{\theta}; \underline{\varepsilon})$ is a symmetric function of $z_1, \ldots, z_{2k\beta}$, means that we can rewrite (3.2.6) as

$$I_{k,\beta}(Sp(2N), \underline{\theta}) = \frac{(-1)^{k\beta}2^{2k\beta}}{(2\pi i)^{2k\beta}(2k\beta)!}\sum_{l_1=0}^{2\beta}\cdots\sum_{l_k=0}^{2\beta} c_{\underline{l}}(k,\beta)J_{k,\beta}(\underline{\theta}; l_1, \ldots, l_k), \tag{3.2.12}$$

where $J_{k,\beta}(\underline{\theta};\underline{l})$ is defined to be $J_{k,\beta}(\underline{\theta};\underline{\varepsilon})$ with $\underline{\varepsilon}$ given by

$$\underline{\varepsilon}=(\underbrace{1,\dots,1}_{l_1},\underbrace{-1,\dots,-1}_{2\beta-l_1},\underbrace{2,\dots,2}_{l_2},\underbrace{-2,\dots,-2}_{2\beta-l_2},\dots,\underbrace{k,\dots,k}_{l_k},\underbrace{-k,\dots,-k}_{2\beta-l_k}),\quad (3.2.13)$$

and

$$c_{\underline{l}}(k,\beta)=\binom{2k\beta}{l_1}\binom{2k\beta-l_1}{2\beta-l_1}\binom{(2k-2)\beta}{l_2}\binom{(2k-2)\beta-l_2}{2\beta-l_2}\cdots\binom{2\beta}{l_k}\quad (3.2.14)$$

counts the number of ways in which the entries of $\underline{\varepsilon}$ can be permuted. In the next lemma, we obtain an expression for the asymptotic behaviour of $I_{k,\beta}(Sp(2N),\underline{\theta})$.

Lemma 3.2.3. *As $N\to\infty$, we have*

$$I_{k,\beta}(Sp(2N),\underline{\theta})\sim\sum_{l_1,\dots,l_k=0}^{2\beta}\frac{(-1)^{k\beta+\sum_{m=1}^k l_m}c_{\underline{l}}(k,\beta)}{(2\pi i)^{2k\beta}(2k\beta)!}N^{|\mathcal{A}_{k,\beta;\underline{l}}|}e^{iN\sum_{n=1}^{2k\beta}\mu_n}$$
$$\times\int_{C_0}\cdots\int_{C_0}\prod_{\substack{1\le m\le n\le 2k\beta\\ \mu_m+\mu_n\neq 0}}\left(1-e^{-\frac{(v_m+v_n)}{N}-i(\mu_m+\mu_n)}\right)^{-1}f(\underline{v};\underline{l})\prod_{n=1}^{2k\beta}dv_n,$$
$$(3.2.15)$$

where C_0 denotes a small circular contour around the origin. The set $\mathcal{A}_{k,\beta;\underline{l}}$ and the function $f(\underline{v};\underline{l})$ are defined in the proof, see (3.2.34) *and* (3.2.41) *respectively. Also, the μ_n are defined in terms of the θ_m in* (3.2.17).

Proof. In view of (3.2.12), we focus on $J_{k,\beta}(\underline{\theta};\underline{l})$. For a given $\underline{l}=(l_1,\dots,l_k)$, we make the change of variables

$$z_n=\frac{v_n}{N}+i\mu_n,\quad (3.2.16)$$

where [3]

$$\mu_n = \begin{cases} \theta_1, & \text{if } 1 \le n \le l_1 \\ -\theta_1, & \text{if } \ l_1 + 1 \le n \le 2\beta \\ \theta_2, & \text{if } 2\beta + 1 \le n \le 2\beta + l_2 \\ -\theta_2, & \text{if } 2\beta + l_2 + 1 \le n \le 4\beta \\ \vdots & \vdots \\ \theta_k, & \text{if } (2k-2)\beta + 1 \le n \le (2k-2)\beta + l_k \\ -\theta_k, & \text{if } (2k-2)\beta + l_k + 1 \le n \le 2k\beta. \end{cases} \tag{3.2.17}$$

The contours of integration are then all small circles around the origin and the integrand of $J_{k,\beta}(\underline{\theta};\underline{l})$ becomes

$$\prod_{1 \le m \le n \le 2k\beta} \left(1 - e^{-\frac{(v_m+v_n)}{N} - i(\mu_m+\mu_n)}\right)^{-1} \Delta\left(\left(\tfrac{v_1}{N} + i\mu_1\right)^2, \ldots, \left(\tfrac{v_{2k\beta}}{N} + i\mu_{2k\beta}\right)^2\right)^2$$
$$\times \frac{\prod_{n=1}^{2k\beta}\left(\frac{v_n}{N} + i\mu_n\right)}{\prod_{n=1}^{2k\beta}\prod_{m=1}^{k}\left(\frac{v_n}{N} + i(\mu_n - \theta_m)\right)^{2\beta}\left(\frac{v_n}{N} + i(\mu_n + \theta_m)\right)^{2\beta}} e^{\sum_{n=1}^{2k\beta} v_n} e^{iN\sum_{n=1}^{2k\beta}\mu_n} \prod_{n=1}^{2k\beta} \frac{dv_n}{N}. \tag{3.2.18}$$

We record here the facts that for each $1 \le j \le k$, there are 2β values of $1 \le n \le 2k\beta$ with $\mu_n^2 = \theta_j^2$. Thus, we have

$$|\{(m,n) : 1 \le m < n \le 2k\beta : \mu_m^2 = \mu_n^2 = \theta_j^2\}| = \sum_{n=1}^{2\beta}(n-1) = \beta(2\beta - 1), \tag{3.2.19}$$

and

$$|\{(m,n) : 1 \le m < n \le 2k\beta : \mu_m^2 = \mu_n^2\}| = \sum_{j=1}^{k}\beta(2\beta-1) = k\beta(2\beta-1). \tag{3.2.20}$$

Also, we have

$$|\{(m,n) : 1 \le m < n \le 2k\beta : \mu_m^2 \ne \mu_n^2\}|$$

[3] The μ_n naturally depend on the choice $\underline{l}$ but we do not make this explicit in the notation.

$$= |\{(m,n) : 1 \le m < n \le 2k\beta\}| - |\{(m,n) : 1 \le m < n \le 2k\beta : \mu_m^2 = \mu_n^2\}|$$
$$= k\beta(2k\beta - 1) - k\beta(2\beta - 1) = 2k\beta^2(k-1). \tag{3.2.21}$$

We approximate the integrand of $J_{k,\beta}(\underline{\theta}; \underline{l})$ as $N \to \infty$ using the following estimates. For the Vandermonde factor, we have

$$\begin{aligned}
&\Delta\left(\left(\tfrac{v_1}{N} + i\mu_1\right)^2, \ldots, \left(\tfrac{v_{2k\beta}}{N} + i\mu_{2k\beta}\right)^2\right) \\
&= \prod_{m<n}\left(\left(\tfrac{v_n}{N} + i\mu_n\right)^2 - \left(\tfrac{v_m}{N} + i\mu_m\right)^2\right) \\
&= \prod_{m<n}\left(\frac{v_n}{N} + i\mu_n + \frac{v_m}{N} + i\mu_m\right)\left(\frac{v_n}{N} + i\mu_n - \frac{v_m}{N} - i\mu_m\right) \\
&= \left(1 + O(N^{-1})\right) \prod_{\substack{m<n \\ \mu_n+\mu_m=0}} \left(\frac{v_n + v_m}{N}\right) \prod_{\substack{m<n \\ \mu_n-\mu_m=0}} \left(\frac{v_n - v_m}{N}\right) \\
&\quad \times \prod_{\substack{m<n \\ \mu_n+\mu_m\neq 0}} (i\mu_n + i\mu_m) \prod_{\substack{m<n \\ \mu_n-\mu_m\neq 0}} (i\mu_n - i\mu_m).
\end{aligned} \tag{3.2.22}$$

Next, we have

$$\prod_{n=1}^{2k\beta}\left(\frac{v_n}{N} + i\mu_n\right) = \left(1 + O(N^{-1})\right)\prod_{n=1}^{2k\beta}(i\mu_n), \tag{3.2.23}$$

and

$$\begin{aligned}
&\prod_{n=1}^{2k\beta}\prod_{m=1}^{k}\left(\tfrac{v_n}{N} + i(\mu_n - \theta_m)\right)^{2\beta}\left(\tfrac{v_n}{N} + i(\mu_n + \theta_m)\right)^{2\beta} \\
&= \prod_{\substack{n=1 \\ \mu_n-\theta_m=0}}^{2k\beta}\prod_{m=1}^{k}\left(\frac{v_n}{N}\right)^{2\beta}\prod_{\substack{n=1 \\ \mu_n-\theta_m\neq 0}}^{2k\beta}\prod_{m=1}^{k}\left(i(\mu_n - \theta_m) + O(N^{-1})\right)^{2\beta} \\
&\quad \times \prod_{\substack{n=1 \\ \mu_n+\theta_m=0}}^{2k\beta}\prod_{m=1}^{k}\left(\frac{v_n}{N}\right)^{2\beta}\prod_{\substack{n=1 \\ \mu_n+\theta_m\neq 0}}^{2k\beta}\prod_{m=1}^{k}\left(i(\mu_n + \theta_m) + O(N^{-1})\right)^{2\beta} \\
&= \left(1 + O(N^{-1})\right)\prod_{n=1}^{2k\beta}\left(\frac{v_n}{N}\right)^{2\beta}\prod_{\substack{n=1 \\ \mu_n-\theta_m\neq 0}}^{2k\beta}\prod_{m=1}^{k}(i\mu_n - i\theta_m)^{2\beta}\prod_{\substack{n=1 \\ \mu_n+\theta_m\neq 0}}^{2k\beta}\prod_{m=1}^{k}(i\mu_n + i\theta_m)^{2\beta}.
\end{aligned} \tag{3.2.24}$$

Lastly, we use the Laurent expansion

$$(1-e^{-s})^{-1} = \frac{1}{s} + O(1) \tag{3.2.25}$$

of $(1-e^{-s})^{-1}$ about its pole at $s=0$ for the $(1-e^{-\frac{(v_m+v_n)}{N}-i(\mu_m+\mu_n)})^{-1}$ terms with $\mu_m+\mu_n=0$. Putting these estimates together, we have that as $N\to\infty$, the integrand of $J_{k,\beta}(\underline{\theta};\underline{l})$ is

$$\begin{aligned}
&= \big(1+O(N^{-1})\big)N^{-2k\beta}\prod_{\substack{1\le m\le n\le 2k\beta\\ \mu_m+\mu_n\neq 0}}\left(1-e^{-\frac{(v_m+v_n)}{N}-i(\mu_m+\mu_n)}\right)^{-1} e^{\sum_{n=1}^{2k\beta}v_n}e^{iN\sum_{n=1}^{2k\beta}\mu_n}\\
&\times\prod_{n=1}^{2k\beta}(i\mu_n)\frac{\prod_{\substack{1\le m<n\le 2k\beta\\ \mu_m+\mu_n\neq 0}}(i\mu_m+i\mu_n)^2\prod_{\substack{1\le m<n\le 2k\beta\\ \mu_m-\mu_n\neq 0}}(i\mu_m-i\mu_n)^2}{\prod_{\substack{1\le m<n\le 2k\beta\\ \mu_m+\mu_n=0}}\left(\frac{N}{v_n+v_m}\right)\prod_{\substack{1\le m<n\le 2k\beta\\ \mu_m-\mu_n=0}}\left(\frac{N}{v_n-v_m}\right)^2}\\
&\times\frac{\prod_{n=1}^{2k\beta}\left(\frac{v_n}{N}\right)^{-2\beta}}{\prod_{\substack{n=1\\ \mu_n-\theta_m\neq 0}}^{2k\beta}\prod_{m=1}^{k}(i\mu_n-i\theta_m)^{2\beta}\prod_{\substack{n=1\\ \mu_n+\theta_m\neq 0}}^{2k\beta}\prod_{m=1}^{k}(i\mu_n+i\theta_m)^{2\beta}}\prod_{n=1}^{2k\beta}dv_n.
\end{aligned}\tag{3.2.26}$$

Simplifying this expression gives us

$$\begin{aligned}
&\big(1+O(N^{-1})\big)(-1)^{k\beta}N^{4k\beta^2-2k\beta}\prod_{\substack{1\le m\le n\le 2k\beta\\ \mu_m+\mu_n\neq 0}}\left(1-e^{-\frac{(v_m+v_n)}{N}-i(\mu_m+\mu_n)}\right)^{-1}\\
&\times e^{\sum_{n=1}^{2k\beta}v_n}e^{iN\sum_{n=1}^{2k\beta}\mu_n}\prod_{n=1}^{2k\beta}\mu_n\frac{\prod_{\substack{m<n\\ \mu_m^2=\mu_n^2}}(2i\mu_n)^2\prod_{\substack{m<n\\ \mu_m^2\neq\mu_n^2}}(\mu_m^2-\mu_n^2)^2}{\prod_{\substack{1\le m<n\le 2k\beta\\ \mu_m+\mu_n=0}}\left(\frac{N}{v_n+v_m}\right)\prod_{\substack{1\le m<n\le 2k\beta\\ \mu_m-\mu_n=0}}\left(\frac{N}{v_n-v_m}\right)^2}\\
&\times\frac{1}{\prod_{\substack{n=1\\ \mu_n^2=\theta_m^2}}^{2k\beta}\prod_{m=1}^{k}(2i\theta_m)^{2\beta}\prod_{\substack{n=1\\ \mu_n^2\neq\theta_m^2}}^{2k\beta}\prod_{m=1}^{k}(\mu_n^2-\theta_m^2)^{2\beta}}\prod_{n=1}^{2k\beta}\frac{dv_n}{v_n^{2\beta}}.
\end{aligned}\tag{3.2.27}$$

Now, from the definition of μ_n in (3.2.17), we have that

$$\prod_{n=1}^{2k\beta}\mu_n = \prod_{m=1}^{k}\left((-1)^{2\beta-l_m}\theta_m^{2\beta}\right) = (-1)^{\sum_{m=1}^{k}l_m}\prod_{m=1}^{k}\theta_m^{2\beta}. \tag{3.2.28}$$

Also, as there are 2β values of $1\le n\le 2k\beta$ with $\mu_n^2=\theta_m^2$ for a given m, we have

$$\prod_{\substack{n=1\\ \mu_n^2=\theta_m^2}}^{2k\beta}\prod_{m=1}^{k}(2i\theta_m)^{2\beta} = \prod_{m=1}^{k}(2i\theta_m)^{4\beta^2} = 2^{4k\beta^2}\prod_{m=1}^{k}\theta_m^{4\beta^2}, \tag{3.2.29}$$

and, using (3.2.19),

$$\prod_{\substack{m<n\\ \mu_m^2=\mu_n^2}} (2i\mu_n)^2 = \prod_{j=1}^{k} \prod_{\substack{m<n\\ \mu_m^2=\mu_n^2=\theta_j^2}} (2i\theta_j)^2 = \prod_{j=1}^{k}(2i\theta_j)^{4\beta^2-2\beta} = (-1)^{k\beta}2^{4k\beta^2-2k\beta}\prod_{j=1}^{k}\theta_j^{4\beta^2-2\beta}. \tag{3.2.30}$$

Similarly, we write

$$\prod_{\substack{m<n\\ \mu_m^2\neq\mu_n^2}} (\mu_m^2-\mu_n^2)^2 = (-1)^{|\{m<n\,:\,\mu_m^2\neq\mu_n^2\}|} \prod_{\substack{m\neq n\\ \mu_m^2\neq\mu_n^2}} (\mu_m^2-\mu_n^2), \tag{3.2.31}$$

where by (3.2.21), the (-1) factor is equal to $(-1)^{2k\beta^2(k-1)} = 1$. Then, we have

$$\prod_{\substack{m\neq n\\ \mu_m^2\neq\mu_n^2}} (\mu_m^2-\mu_n^2) = \prod_{j=1}^{k} \prod_{\substack{m\neq n\\ \mu_m^2\neq\mu_n^2\\ \mu_n^2=\theta_j}} (\mu_m^2-\theta_j^2) = \prod_{j=1}^{k}\prod_{\substack{m=1\\ \mu_n^2\neq\theta_m^2}}^{2k\beta}(\mu_m^2-\theta_j^2)^{2\beta}, \tag{3.2.32}$$

again as there are 2β values of $1\leq n\leq 2k\beta$ with $\mu_n^2=\theta_j^2$. Taking into account the above calculations, we see that many of the products in (3.2.27) cancel and our final expression for the integrand is then

$$\begin{aligned} &= \big(1+O(N^{-1})\big)\frac{(-1)^{\sum_{m=1}^{k} l_m}}{2^{2k\beta}}N^{4k\beta^2-2k\beta}\prod_{\substack{1\leq m\leq n\leq 2k\beta\\ \mu_m+\mu_n\neq 0}}\left(1-e^{-\frac{(v_m+v_n)}{N}-i(\mu_m+\mu_n)}\right)^{-1}\\ &\quad\times e^{\sum_{n=1}^{2k\beta} v_n}e^{iN\sum_{n=1}^{2k\beta}\mu_n}\prod_{\substack{1\leq m<n\leq 2k\beta\\ \mu_m+\mu_n=0}}\left(\frac{v_n+v_m}{N}\right)\prod_{\substack{1\leq m<n\leq 2k\beta\\ \mu_m-\mu_n=0}}\left(\frac{v_n-v_m}{N}\right)^2\prod_{n=1}^{2k\beta}\frac{dv_n}{v_n^{2\beta}}. \end{aligned} \tag{3.2.33}$$

The power of N coming from the products in the second line of (3.2.33) is determined by the size of the following sets:

$$\mathcal{A}_{k,\beta;\underline{l}} := \{(m,n) : 1\leq m<n\leq 2k\beta,\ \mu_m+\mu_n=0\}, \tag{3.2.34}$$

and

$$\mathcal{B}_{k,\beta;\underline{l}} := \{(m,n) : 1\leq m<n\leq 2k\beta,\ \mu_m-\mu_n=0\}. \tag{3.2.35}$$

Using the definition of μ_n in (3.2.17), we have that

$$|\mathcal{A}_{k,\beta;\underline{l}}| = \sum_{m=1}^{k} l_m(2\beta - l_m) \tag{3.2.36}$$

since for each $m \in \{1, \dots, k\}$, there are l_m of the μ_n's equal to θ_m and $2\beta - l_m$ of the μ_n's equal to $-\theta_m$. Also, we have

$$|\mathcal{A}_{k,\beta;\underline{l}}| + |\mathcal{B}_{k,\beta;\underline{l}}| = |\{(m,n) : 1 \leq m < n \leq 2k\beta,\ \mu_m^2 = \mu_n^2\}| = k\beta(2\beta - 1), \tag{3.2.37}$$

and so

$$|\mathcal{B}_{k,\beta;\underline{l}}| = k\beta(2\beta - 1) + \sum_{m=1}^{k} l_m(l_m - 2\beta). \tag{3.2.38}$$

In particular, the power of N in the second line of (3.2.33) is

$$-|\mathcal{A}_{k,\beta;\underline{l}}| - 2|\mathcal{B}_{k,\beta;\underline{l}}| = |\mathcal{A}_{k,\beta;\underline{l}}| - 2k\beta(2\beta - 1). \tag{3.2.39}$$

Therefore, the integrand of $J_{k,\beta}(\underline{\theta};\underline{l})$ as $N \to \infty$ is

$$\begin{aligned} &\left(1 + O\left(\tfrac{1}{N}\right)\right) \frac{(-1)^{\sum_{m=1}^{k} l_m}}{2^{2k\beta}} N^{|\mathcal{A}_{k,\beta;\underline{l}}|} \prod_{\substack{1 \leq m \leq n \leq 2k\beta \\ \mu_m + \mu_n \neq 0}} \left(1 - e^{-\frac{(v_m + v_n)}{N} - i(\mu_m + \mu_n)}\right)^{-1} \\ &\quad \times e^{iN \sum_{n=1}^{2k\beta} \mu_n} f(\underline{v};\underline{l}) \prod_{n=1}^{2k\beta} dv_n, \end{aligned} \tag{3.2.40}$$

where

$$f(\underline{v};\underline{l}) := \frac{\prod_{\substack{1 \leq m < n \leq 2k\beta \\ \mu_m + \mu_n = 0}} (v_n + v_m) \prod_{\substack{1 \leq m < n \leq 2k\beta \\ \mu_m - \mu_n = 0}} (v_n - v_m)^2}{\prod_{n=1}^{2k\beta} v_n^{2\beta}} e^{\sum_{n=1}^{2k\beta} v_n} \tag{3.2.41}$$

denotes the terms which do not depend on $\theta_1, \dots, \theta_k$. Hence, by using the expression for the integrand of $J_{k,\beta}(\underline{\theta}, \underline{l})$ in (3.2.40) and returning to (3.2.12), we have that as $N \to \infty$,

$$I_{k,\beta}(Sp(2N), \underline{\theta}) \sim \sum_{l_1, \dots, l_k = 0}^{2\beta} \frac{(-1)^{k\beta + \sum_{m=1}^{k} l_m} c_{\underline{l}}(k,\beta)}{(2\pi i)^{2k\beta}(2k\beta)!} N^{|\mathcal{A}_{k,\beta;\underline{l}}|} e^{iN \sum_{n=1}^{2k\beta} \mu_n}$$

$$\times \int_{C_0} \cdots \int_{C_0} \prod_{\substack{1 \le m \le n \le 2k\beta \\ \mu_m + \mu_n \neq 0}} \left(1 - e^{-\frac{(v_m+v_n)}{N} - i(\mu_m+\mu_n)}\right)^{-1} f(\underline{v};\underline{l}) \prod_{n=1}^{2k\beta} dv_n, \tag{3.2.42}$$

as claimed. □

We can now obtain an asymptotic formula for $\mathrm{MoM}_{Sp(2N)}(k,\beta)$.

Proposition 3.2.4. *As $N \to \infty$, we have*

$$\mathrm{MoM}_{Sp(2N)}(k,\beta) \sim \gamma_{Sp}(k,\beta) N^{k\beta(2k\beta+1)-k}, \tag{3.2.43}$$

where $\gamma_{Sp}(k,\beta)$ is given in the form of an integral and is defined on the proof, see (3.2.58)*.*

Proof. By recalling (3.2.3) and using Lemma 3.2.3, we have

$$\begin{aligned}
&\mathrm{MoM}_{Sp(2N)}(k,\beta) \\
&= \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} I_{k,\beta}(Sp(2N),\theta_1,\ldots,\theta_k)\, d\theta_1 \cdots d\theta_k \\
&\sim \frac{1}{(2\pi)^k} \sum_{l_1,\ldots,l_k=0}^{2\beta} \frac{(-1)^{k\beta+\sum_{m=1}^k l_m} c_{\underline{l}}(k,\beta)}{(2\pi i)^{2k\beta}(2k\beta)!} N^{|\mathcal{A}_{k,\beta;\underline{l}}|} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{iN\sum_{n=1}^{2k\beta}\mu_n} \\
&\times \oint_{C_0} \cdots \oint_{C_0} \prod_{\substack{1 \le m \le n \le 2k\beta \\ \mu_m + \mu_n \neq 0}} \left(1 - e^{-\frac{(v_m+v_n)}{N} - i(\mu_m+\mu_n)}\right)^{-1} f(\underline{v};\underline{l}) \prod_{n=1}^{2k\beta} dv_n \prod_{m=1}^{k} d\theta_m.
\end{aligned} \tag{3.2.44}$$

Changing the order of integration, we have that

$$\begin{aligned}
\mathrm{MoM}_{Sp(2N)}(k,\beta) \sim \sum_{l_1,\ldots,l_k=0}^{2\beta} &\frac{(-1)^{k\beta+\sum_{m=1}^k l_m} c_{\underline{l}}(k,\beta)}{(2\pi)^k(2\pi i)^{2k\beta}(2k\beta)!} N^{|\mathcal{A}_{k,\beta;\underline{l}}|} \oint_{C_0} \cdots \oint_{C_0} f(\underline{v};\underline{l}) \\
&\times \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{\substack{1 \le m \le n \le 2k\beta \\ \mu_m + \mu_n \neq 0}} \left(1 - e^{-\frac{(v_m+v_n)}{N} - i(\mu_m+\mu_n)}\right)^{-1} \\
&\times e^{iN\sum_{n=1}^{2k\beta}\mu_n} \prod_{m=1}^{k} d\theta_m \prod_{n=1}^{2k\beta} dv_n,
\end{aligned} \tag{3.2.45}$$

and we now seek to determine the N dependence of the inner integrals over $\theta_1,\ldots,\theta_k$. The first step is to write the integrand explicitly in terms of $\theta_1,\ldots,\theta_k$ using the definition of $\mu_1, \ldots, \mu_{2k\beta}$. For instance, the exponential term is

$$\exp\left(iN\sum_{n=1}^{2k\beta}\mu_n\right) = \exp\left(2iN\sum_{m=1}^{k}(l_m-\beta)\theta_m\right). \tag{3.2.46}$$

For the product of $(1-e^{-z_m-z_n})^{-1}$ terms, we define the set

$$\begin{aligned}\mathcal{T}_{k,\beta;\underline{l}} &:= \{(m,n) : 1 \le m \le n \le 2k\beta,\ \mu_m+\mu_n \neq 0\}\\ &= \{(m,n) : 1 \le m \le n \le 2k\beta\} \setminus \mathcal{A}_{k,\beta;\underline{l}},\end{aligned} \tag{3.2.47}$$

and the following disjoint subsets of $\mathcal{T}_{k,\beta;\underline{l}}$ for $1 \le \sigma \le \tau \le k$:

$$\mathcal{U}^{+}_{\sigma,\tau;\underline{l}} := \{(m,n) \in \mathcal{T}_{k,\beta;\underline{l}} : \mu_m+\mu_n = \theta_\sigma+\theta_\tau\}, \tag{3.2.48}$$

$$\mathcal{U}^{-}_{\sigma,\tau;\underline{l}} := \{(m,n) \in \mathcal{T}_{k,\beta;\underline{l}} : \mu_m+\mu_n = -(\theta_\sigma+\theta_\tau)\}, \tag{3.2.49}$$

and

$$\mathcal{V}^{+}_{\sigma,\tau;\underline{l}} := \{(m,n) \in \mathcal{T}_{k,\beta;\underline{l}} : \mu_m+\mu_n = \theta_\sigma-\theta_\tau\}, \tag{3.2.50}$$

$$\mathcal{V}^{-}_{\sigma,\tau;\underline{l}} := \{(m,n) \in \mathcal{T}_{k,\beta;\underline{l}} : \mu_m+\mu_n = -(\theta_\sigma-\theta_\tau)\}. \tag{3.2.51}$$

Note that $\mathcal{V}^{+}_{\sigma,\tau;\underline{l}} = \mathcal{V}^{-}_{\sigma,\tau;\underline{l}} = \varnothing$ for $\sigma=\tau$. The product of $(1-e^{-z_m-z_n})^{-1}$ terms can then be written as

$$\begin{aligned}&\prod_{\substack{1\le m\le n\le 2k\beta\\ \mu_m+\mu_n\neq 0}}\left(1-e^{-\frac{(v_m+v_n)}{N}-i(\mu_m+\mu_n)}\right)^{-1}\\ &= \prod_{1\le\sigma\le\tau\le k}\ \prod_{(m,n)\in\mathcal{U}^{+}_{\sigma,\tau;\underline{l}}}\left(1-e^{-\frac{(v_m+v_n)}{N}-i(\theta_\sigma+\theta_\tau)}\right)^{-1}\prod_{(m,n)\in\mathcal{U}^{-}_{\sigma,\tau;\underline{l}}}\left(1-e^{-\frac{(v_m+v_n)}{N}+i(\theta_\sigma+\theta_\tau)}\right)^{-1}\\ &\quad\times\prod_{(m,n)\in\mathcal{V}^{+}_{\sigma,\tau;\underline{l}}}\left(1-e^{-\frac{(v_m+v_n)}{N}-i(\theta_\sigma-\theta_\tau)}\right)^{-1}\prod_{(m,n)\in\mathcal{V}^{-}_{\sigma,\tau;\underline{l}}}\left(1-e^{-\frac{(v_m+v_n)}{N}+i(\theta_\sigma-\theta_\tau)}\right)^{-1}.\end{aligned} \tag{3.2.52}$$

Now, we make the change of variables $t_m = N\theta_m$. As $N \to \infty$, by the Laurent expansion of $(1-e^{-s})^{-1}$ about $s=0$, the above product is then

$$\prod_{\substack{1\leq m\leq n\leq 2k\beta \\ \mu_m+\mu_n\neq 0}} \left(1-e^{-\frac{(v_m+v_n)}{N}-i(\mu_m+\mu_n)}\right)^{-1}$$
$$\sim \prod_{1\leq\sigma\leq\tau\leq k} \prod_{(m,n)\in\mathcal{U}^+_{\sigma,\tau;\underline{l}}} \frac{N}{v_m+v_n+i(t_\sigma+t_\tau)} \prod_{(m,n)\in\mathcal{U}^-_{\sigma,\tau;\underline{l}}} \frac{N}{v_m+v_n-i(t_\sigma+t_\tau)}$$
$$\times \prod_{(m,n)\in\mathcal{V}^+_{\sigma,\tau;\underline{l}}} \frac{N}{v_m+v_n+i(t_\sigma-t_\tau)} \prod_{(m,n)\in\mathcal{V}^-_{\sigma,\tau;\underline{l}}} \frac{N}{v_m+v_n-i(t_\sigma-t_\tau)}. \quad (3.2.53)$$

The power of N coming from this product is

$$|\mathcal{T}_{k,\beta;\underline{l}}| = k\beta(2k\beta+1) - |\mathcal{A}_{k,\beta;\underline{l}}|. \quad (3.2.54)$$

We therefore have that as $N\to\infty$, the integrals over $\theta_1,\ldots,\theta_k$ are

$$\int_0^{2\pi}\cdots\int_0^{2\pi} \prod_{\substack{1\leq m\leq n\leq 2k\beta \\ \mu_m+\mu_n\neq 0}} \left(1-e^{-\frac{(v_m+v_n)}{N}-i(\mu_m+\mu_n)}\right)^{-1} e^{iN\sum_{n=1}^{2k\beta}\mu_n} \prod_{m=1}^{k} d\theta_m$$
$$\sim \int_0^{2N\pi}\cdots\int_0^{2N\pi} N^{k\beta(2k\beta+1)-k-|\mathcal{A}_{k,\beta;\underline{l}}|}\, e^{2i\sum_{m=1}^{k}(l_m-\beta)t_m}$$
$$\times \prod_{1\leq\sigma\leq\tau\leq k} \frac{\prod_{(m,n)\in\mathcal{U}^+_{\sigma,\tau;\underline{l}}}(v_m+v_n+i(t_\sigma+t_\tau))^{-1}\prod_{(m,n)\in\mathcal{U}^-_{\sigma,\tau;\underline{l}}}(v_m+v_n-i(t_\sigma+t_\tau))^{-1}}{\prod_{(m,n)\in\mathcal{V}^+_{\sigma,\tau;\underline{l}}}(v_m+v_n+i(t_\sigma-t_\tau))\prod_{(m,n)\in\mathcal{V}^-_{\sigma,\tau;\underline{l}}}(v_m+v_n-i(t_\sigma-t_\tau))}$$
$$\times\, dt_1\cdots dt_k$$
$$\sim\; N^{k\beta(2k\beta+1)-k-|\mathcal{A}_{k,\beta;\underline{l}}|}\Psi_{k,\beta}(\underline{v};\underline{l}), \quad (3.2.55)$$

where

$$\Psi_{k,\beta}(\underline{v};\underline{l}) := \int_0^{\infty}\cdots\int_0^{\infty} e^{2i\sum_{m=1}^{k}(l_m-\beta)t_m}$$
$$\times \prod_{1\leq\sigma\leq\tau\leq k} \frac{\prod_{(m,n)\in\mathcal{U}^+_{\sigma,\tau;\underline{l}}}(v_m+v_n+i(t_\sigma+t_\tau))^{-1}\prod_{(m,n)\in\mathcal{U}^-_{\sigma,\tau;\underline{l}}}(v_m+v_n-i(t_\sigma+t_\tau))^{-1}}{\prod_{(m,n)\in\mathcal{V}^+_{\sigma,\tau;\underline{l}}}(v_m+v_n+i(t_\sigma-t_\tau))\prod_{(m,n)\in\mathcal{V}^-_{\sigma,\tau;\underline{l}}}(v_m+v_n-i(t_\sigma-t_\tau))}$$
$$\times\, dt_1\cdots dt_k. \quad (3.2.56)$$

Returning now to (3.2.45) and using (3.2.55), we have that

$$\mathrm{MoM}_{Sp(2N)}(k,\beta) \sim \gamma_{Sp}(k,\beta) N^{k\beta(2k\beta+1)-k}, \quad (3.2.57)$$

where

$$\gamma_{Sp}(k,\beta) := \sum_{l_1,\ldots,l_k=0}^{2\beta} \frac{(-1)^{k\beta+\sum_{m=1}^{k} l_m} c_{\underline{l}}(k,\beta)}{(2\pi)^k(2\pi i)^{2k\beta}(2k\beta)!} \oint_{C_0} \cdots \oint_{C_0} f(\underline{v};\underline{l})\Psi_{k,\beta}(\underline{v};\underline{l}) \prod_{n=1}^{2k\beta} dv_n, \tag{3.2.58}$$

and this completes the proof. □

Proof of theorem 3.1.7. To complete the proof of Theorem 3.1.7, we compare the asymptotic formula in Proposition 3.2.4 to that of Theorem 3.1.5 to show that $\gamma_{Sp}(k,\beta) \neq 0$. We see that as $\mathrm{MoM}_{Sp(2N)}(k,\beta)$ is a polynomial in N, we must have that $\gamma_{Sp}(k,\beta) = \mathfrak{c}_{Sp}(k,\beta) > 0$ and that the error is certainly $O(N^{k\beta(2k\beta+1)-k-1})$ which concludes the proof.

□

3.2.2 Moments of moments of symplectic L-functions

An example of a symplectic family of L-functions considered by Bailey and Keating in [BK21] is the family of quadratic Dirichlet L-functions $L(s,\chi_d)$ defined in Section 1.5.1. We recall that for d a fundamental discriminant, $\chi_d(n) = (\frac{d}{n})$ denotes the quadratic character defined by the Kronecker symbol. The associated L-function is defined for $\mathrm{Re}(s) > 1$ by

$$L(s,\chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}, \tag{3.2.59}$$

and has an analytic continuation to $\mathbb{C}$. The L-function satisfies the functional equation

$$L(s,\chi_d) = X_d(s)L(1-s,\chi_d), \tag{3.2.60}$$

where $X_d(s) = |d|^{1/2-s}X(s,a)$ with $a=0$ if $d>0$ and $a=1$ if $d<0$, and

$$X(s,a) = \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1+a-s}{2}\right)\Gamma\left(\frac{s+a}{2}\right)^{-1}. \tag{3.2.61}$$

The moments of moments of the family of quadratic Dirichlet L-functions are defined as

$$\mathrm{MoM}_{L_{\chi_d}}(k,\beta) = \frac{1}{D^*}\sum_{|d|\le D}^{*}\left(\frac{1}{2\pi}\int_0^{2\pi} L(\tfrac{1}{2}+i\theta,\chi_d)^{2\beta}d\theta\right)^k, \tag{3.2.62}$$

where the sum is only over fundamental discriminants and D^* is the number of terms in the sum. As this is a symplectic family of L-functions, the moments of moments are conjectured to behave analogously to the moments of moments of characteristic polynomials over $Sp(2N)$.

Conjecture 3.2.5 (Bailey-Keating [BK21]). *For $k, \beta \in \mathbb{N}$, as $D \to \infty$,*

$$\mathrm{MoM}_{L_{\chi_d}}(k,\beta) = \eta(k,\beta)\mathfrak{c}_{Sp}(k,\beta)(\log D)^{k\beta(2k\beta+1)-k}\left(1+O_{k,\beta}(\log^{-1} D)\right), \quad (3.2.63)$$

where $\mathfrak{c}_{Sp}(k,\beta)$ corresponds to the leading order coefficient in (3.1.13) *and $\eta(k,\beta)$ contains the arithmetic information.*

By adapting the proof of Theorem 3.1.7, we can relatively easily prove that Conjecture 3.2.5 follows from the shifted moment conjecture of [CFK+05]. Assuming $k \in \mathbb{N}$, changing the order of integration and summation gives

$$\mathrm{MoM}_{L_{\chi_d}}(k,\beta) = \frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{1}{D^*}\sum_{|d|\le D}^{*}\prod_{m=1}^{k} L(\tfrac{1}{2}+i\theta_m,\chi_d)^{2\beta}d\theta_1\dots d\theta_k, \quad (3.2.64)$$

and the relevant shifted moment conjecture is the following.

Conjecture 3.2.6 (Conrey et al. [CFK+05]). *Let $k, \beta \in \mathbb{N}$ and let $\underline{\theta} = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$. Then,*

$$\frac{1}{D^*}\sum_{|d|\le D}^{*}\prod_{m=1}^{k} L(\tfrac{1}{2}+i\theta_m,\chi_d)^{2\beta} = \frac{1}{D^*}\sum_{|d|\le D}^{*}\prod_{m=1}^{k} X_d(\tfrac{1}{2}+i\theta_m)^{\beta}Q_{k,\beta}(\log|d|,\underline{\theta}) + O(D^{-\delta}), \quad (3.2.65)$$

for some $\delta > 0$, where

$$Q_{k,\beta}(x,\underline{\theta}) = \frac{(-1)^{k\beta}2^{2k\beta}}{(2\pi i)^{2k\beta}(2k\beta)!}\oint\cdots\oint\frac{G(z_1,\dots,z_{2k\beta})\Delta(z_1^2,\dots,z_{2k\beta}^2)^2\prod_{n=1}^{2k\beta}z_n}{\prod_{n=1}^{2k\beta}\prod_{m=1}^{k}(z_n-i\theta_m)^{2\beta}(z_n+i\theta_m)^{2\beta}} \times e^{\frac{x}{2}\sum_{n=1}^{2k\beta}z_n}dz_1\dots dz_{2k\beta}, \quad (3.2.66)$$

in which the path of integration encloses the poles at $\pm i\theta_m$ for $1 \le m \le k$. Also,

$$G(z_1,\ldots,z_{2k\beta})=A_{k\beta}(z_1,\ldots,z_{2k\beta})\prod_{n=1}^{2k\beta}X(\tfrac{1}{2}+z_n,a)^{-\frac{1}{2}}\prod_{1\le m\le n\le 2k\beta}\zeta(1+z_m+z_n),\tag{3.2.67}$$

where $A_{k\beta}$ is the Euler product, absolutely convergent for $|\mathrm{Re}(z_n)|<1/2$, defined by

$$\begin{aligned}A_{k\beta}(z_1,\ldots,z_{2k\beta})=&\prod_p\prod_{1\le m\le n\le 2k\beta}\left(1-\frac{1}{p^{1+z_m+z_n}}\right)\left(1+\frac{1}{p}\right)^{-1}\\&\times\left(\frac{1}{2}\left(\prod_{n=1}^{2k\beta}\left(1-\frac{1}{p^{1/2+z_n}}\right)^{-1}+\prod_{n=1}^{2k\beta}\left(1+\frac{1}{p^{1/2+z_n}}\right)^{-1}\right)+\frac{1}{p}\right)\end{aligned}\tag{3.2.68}$$

We therefore define

$$\mathrm{MoM}_{Q_{k,\beta}}(D):=\frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{1}{D^*}\sum_{|d|\le D}^{*}\prod_{m=1}^{k}X_d(\tfrac{1}{2}+i\theta_m)^{\beta}Q_{k,\beta}(\log|d|,\underline{\theta})\prod_{j=1}^{k}d\theta_j,\tag{3.2.69}$$

which should approximate $\mathrm{MoM}_{L_{\chi_d}}(k,\beta)$ up to a power saving in D and we can prove the following proposition.

Proposition 3.2.7. *For $k,\beta\in\mathbb{N}$, as $D\to\infty$*

$$\mathrm{MoM}_{Q_{k,\beta}}(D)=A_{k\beta}(0,\ldots,0)\gamma_{Sp}(k,\beta)\left(\frac{\log D}{2}\right)^{k\beta(2k\beta+1)-k}\left(1+O(\log^{-1}D)\right),\tag{3.2.70}$$

where $\gamma_{Sp}(k,\beta)$ is the same coefficient appearing in Theorem 3.1.7.

Proof. As mentioned earlier, the proof follows from modifying the proof of Theorem 3.1.7 and so we will point out the adjustments that need to be made. Comparing the integral $Q_{k,\beta}(x,\underline{\theta})$ with the integral expression for $I_{k\beta}(Sp(2N),\underline{\theta})$ in (3.2.5), we immediately see the similarity on identifying N with $x/2$. In particular, the product of $\zeta(1+z_m+z_n)$ terms replaces the product of $(1-e^{-z_m-z_n})^{-1}$ terms with both having the same analytic structure, namely simple poles at $z_m+z_n=0$. This means that the same analysis we applied to $I_{k\beta}(Sp(2N),\underline{\theta})$ can be applied to $Q_{k,\beta}(x,\underline{\theta})$ to yield an asymptotic formula for $\mathrm{MoM}_{Q_{k,\beta}}(D)$. The difference is that the function $G(z_1,\ldots,z_{2k\beta})$ also contains arithmetic information in the Euler product $A_{k\beta}$ and the $X(s,a)$ factors. However, these factors do not present any additional difficulties. By following the proof of Lemma 3.2.3 and using the facts that $A_{k\beta}$ is analytic in a neighbourhood of zero and $X(s,a)$ is analytic around $s=1/2$ and $X(\frac{1}{2},a)=1$, one can show that

$$\begin{aligned}\mathrm{MoM}_{Q_{k,\beta}}(D) &\sim A_{k\beta}(0,\ldots,0)\gamma_{Sp}(k,\beta)\frac{1}{D^*}\sum_{|d|\le D}^{*}\left(\frac{\log|d|}{2}\right)^{k\beta(2k\beta+1)-k}\\ &= A_{k\beta}(0,\ldots,0)\gamma_{Sp}(k,\beta)\left(\frac{\log D}{2}\right)^{k\beta(2k\beta+1)-k}\left(1+O(\log^{-1}D)\right),\end{aligned} \tag{3.2.71}$$

where $\gamma_{Sp}(k,\beta)$ is as defined in (3.2.58). $\square$

Thus, $\mathrm{MoM}_{Q_{k,\beta}}(D)$ satisfies the asymptotic formula conjectured for $\mathrm{MoM}_{L_{\chi_d}}(k,\beta)$ in Conjecture 3.2.5.

3.3 The special orthogonal group $SO(2N)$

3.3.1 Proof of Theorem 3.1.8

In this section we turn to the orthogonal case and prove Theorem 3.1.8. Here we assume that $k,\beta\in\mathbb{N}$ with k,β not both 1. As in the symplectic case, the eigenvalues of matrices in $SO(2N)$ lie on the unit circle and come in complex conjugate pairs so

$$\overline{\Lambda_X(e^{-i\theta})}=\Lambda_X(e^{i\theta}), \tag{3.3.1}$$

for all $X\in SO(2N)$. Then, as before, we change the order of integration to write

$$\mathrm{MoM}_{SO(2N)}(k,\beta)=\frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}I_{k,\beta}(SO(2N),\theta_1,\ldots,\theta_k)\,d\theta_1\cdots d\theta_k, \tag{3.3.2}$$

where

$$I_{k,\beta}(SO(2N),\underline{\theta}):=\int_{SO(2N)}\prod_{j=1}^{k}\Lambda_X(e^{-i\theta_j})^\beta\Lambda_X(e^{i\theta_j})^\beta\,dX. \tag{3.3.3}$$

In this case, using Theorem 1.1.5 allows us to express $I_{k,\beta},(SO(2N),\underline{\theta})$ as

$$I_{k,\beta}(SO(2N),\underline{\theta}) = \frac{(-1)^{k\beta}2^{2k\beta}}{(2\pi i)^{2k\beta}(2k\beta)!}\oint\cdots\oint \prod_{1\le m<n\le 2k\beta}(1-e^{-z_m-z_n})^{-1}$$
$$\times \frac{\Delta(z_1^2,\ldots,z_{2k\beta}^2)^2\prod_{n=1}^{2k\beta} z_n}{\prod_{n=1}^{2k\beta}\prod_{m=1}^{k}(z_n-i\theta_m)^{2\beta}(z_n+i\theta_m)^{2\beta}} e^{N\sum_{n=1}^{2k\beta} z_n} dz_1\ldots dz_{2k\beta}, \tag{3.3.4}$$

where again the contours enclose the poles at $\pm i\theta_m$ for $1\le m\le k$. We note the similarity between the above expression for $I_{k,\beta}(SO(2N),\underline{\theta})$ and that for $I_{k,\beta}(Sp(2N),\underline{\theta})$ in (3.2.5). Specifically, the only difference is in the product of $(1-e^{-z_m-z_n})^{-1}$ terms; in the symplectic case, the product is over $m\le n$ rather than $m<n$. The proof of Theorem 3.1.8 will therefore mirror that of Theorem 3.1.7 but with this one difference.

First, by decomposing $I_{k,\beta}(SO(2N),\underline{\theta})$ as in (3.2.6), using Lemma 3.2.2 and then following the proof of Lemma 3.2.3, we get that

$$I_{k,\beta}(SO(2N),\underline{\theta}) \sim \sum_{l_1,\ldots,l_k=0}^{2\beta} \frac{(-1)^{k\beta+\sum_{m=1}^{k} l_m} c_{\underline{l}}(k,\beta)}{(2\pi i)^{2k\beta}(2k\beta)!} N^{|\mathcal{A}_{k,\beta;\underline{l}}|} e^{iN\sum_{n=1}^{2k\beta}\mu_n}$$
$$\times \oint_{C_0}\cdots\oint_{C_0} \prod_{\substack{1\le m<n\le 2k\beta\\ \mu_m+\mu_n\neq 0}} \left(1-e^{-\frac{(v_m+v_n)}{N}-i(\mu_m+\mu_n)}\right)^{-1} f(\underline{v};\underline{l}) \prod_{n=1}^{2k\beta} dv_n, \tag{3.3.5}$$

where the μ_n, the set $\mathcal{A}_{k,\beta;\underline{l}}$ and the function $f(\underline{v};\underline{l})$ are as defined in (3.2.17), (3.2.34) and (3.2.41) respectively. We then proceed as in the proof of Proposition 3.2.4 with the change being that we will replace the set $\mathcal{T}_{k,\beta;\underline{l}}$ by

$$\begin{aligned}\widetilde{\mathcal{T}}_{k,\beta;\underline{l}} &:= \{(m,n) : 1\le m<n\le 2k\beta, \mu_m+\mu_n\neq 0\}\\ &= \{(m,n) : 1\le m<n\le 2k\beta\}\setminus \mathcal{A}_{k,\beta;\underline{l}},\end{aligned} \tag{3.3.6}$$

and for $1\le\sigma\le\tau\le k$, we define the subsets

$$\widetilde{\mathcal{U}}^{+}_{\sigma,\tau;\underline{l}} := \{(m,n)\in\widetilde{\mathcal{T}}_{k,\beta;\underline{l}} : \mu_m+\mu_n=\theta_\sigma+\theta_\tau\} \tag{3.3.7}$$

$$\widetilde{\mathcal{U}}^{-}_{\sigma,\tau;\underline{l}} := \{(m,n)\in\widetilde{\mathcal{T}}_{k,\beta;\underline{l}} : \mu_m+\mu_n=-(\theta_\sigma+\theta_\tau)\}, \tag{3.3.8}$$

and

$$\widetilde{\mathcal{V}}^{+}_{\sigma,\tau;\underline{l}} := \{(m,n) \in \widetilde{\mathcal{T}}_{k,\beta;\underline{l}} : \mu_m + \mu_n = \theta_\sigma - \theta_\tau\} \tag{3.3.9}$$

$$\widetilde{\mathcal{V}}^{-}_{\sigma,\tau;\underline{l}} := \{(m,n) \in \widetilde{\mathcal{T}}_{k,\beta;\underline{l}} : \mu_m + \mu_n = -(\theta_\sigma - \theta_\tau)\}. \tag{3.3.10}$$

After making the same change of variables $t_m = N\theta_m$, the product of $(1 - e^{-z_m - z_n})^{-1}$ terms will be

$$\begin{aligned}
&\prod_{\substack{1\le m<n\le 2k\beta \\ \mu_m+\mu_n\neq 0}} \left(1 - e^{-\frac{(v_m+v_n)}{N} - i(\mu_m+\mu_n)}\right)^{-1} \\
&\quad \sim \prod_{1\le\sigma\le\tau\le k} \prod_{(m,n)\in\widetilde{\mathcal{U}}^{+}_{\sigma,\tau;\underline{l}}} \frac{N}{v_m + v_n + i(t_\sigma + t_\tau)} \prod_{(m,n)\in\widetilde{\mathcal{U}}^{-}_{\sigma,\tau;\underline{l}}} \frac{N}{v_m + v_n - i(t_\sigma + t_\tau)} \\
&\qquad \times \prod_{(m,n)\in\widetilde{\mathcal{V}}^{+}_{\sigma,\tau;\underline{l}}} \frac{N}{v_m + v_n + i(t_\sigma - t_\tau)} \prod_{(m,n)\in\widetilde{\mathcal{V}}^{-}_{\sigma,\tau;\underline{l}}} \frac{N}{v_m + v_n - i(t_\sigma - t_\tau)}.
\end{aligned} \tag{3.3.11}$$

The power of N coming from this product is

$$|\widetilde{\mathcal{T}}_{k,\beta;\underline{l}}| = k\beta(2k\beta - 1) - |\mathcal{A}_{k,\beta;\underline{l}}|. \tag{3.3.12}$$

Taking into account this difference and proceeding exactly as in the proof of Proposition 3.2.4, we see that in this case, we will obtain

$$\mathrm{MoM}_{SO(2N)}(k,\beta) \sim \gamma_{SO}(k,\beta) N^{k\beta(2k\beta-1)-k}, \tag{3.3.13}$$

where

$$\gamma_{SO}(k,\beta) := \sum_{l_1,\ldots,l_k=0}^{2\beta} \frac{(-1)^{k\beta + \sum_{m=1}^{k} l_m} c_{\underline{l}}(k,\beta)}{(2\pi)^k (2\pi i)^{2k\beta} (2k\beta)!} \oint_{C_0} \cdots \oint_{C_0} f(\underline{v};\underline{l}) \Omega_{k,\beta}(\underline{v};\underline{l}) \prod_{n=1}^{2k\beta} dv_n, \tag{3.3.14}$$

and

$$\Omega_{k,\beta}(\underline{v};\underline{l}) := \int_0^\infty \cdots \int_0^\infty e^{2i\sum_{m=1}^{k}(l_m - \beta)t_m}$$

$$\times \prod_{1\le\sigma\le\tau\le k} \frac{\prod_{(m,n)\in\widetilde{\mathcal{U}}^{+}_{\sigma,\tau;\underline{l}}}(v_m+v_n+i(t_\sigma+t_\tau))^{-1}\prod_{(m,n)\in\widetilde{\mathcal{U}}^{-}_{\sigma,\tau;\underline{l}}}(v_m+v_n-i(t_\sigma+t_\tau))^{-1}}{\prod_{(m,n)\in\widetilde{\mathcal{V}}^{+}_{\sigma,\tau;\underline{l}}}(v_m+v_n+i(t_\sigma-t_\tau))\prod_{(m,n)\in\widetilde{\mathcal{V}}^{-}_{\sigma,\tau;\underline{l}}}(v_m+v_n-i(t_\sigma-t_\tau))}$$
$$\times\, dt_1\cdots dt_k. \tag{3.3.15}$$

Finally, comparing the asymptotic formula (3.3.13) to the result of Theorem 3.1.6 shows that $\gamma_{SO}(k,\beta)=\mathfrak{c}_{SO}(k,\beta)>0$ which completes the proof of Theorem 3.1.8.

3.3.2 Moments of moments of an orthogonal family of L-functions

For a family of L-functions with orthogonal symmetry, we consider the quadratic twists of an elliptic curve L-function defined in Section 1.5.2. Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor M_E. Recall that the L-function attached to E is defined for $\mathrm{Re}(s)>1$ by

$$L_E(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^{s+1/2}}=\prod_{p|M}\left(1-a_pp^{-s-\frac12}\right)^{-1}\prod_{p\nmid M}\left(1-a_pp^{-s-\frac12}+p^{-2s}\right)^{-1}:=\prod_p\mathcal{L}_p(p^{-s}), \tag{3.3.16}$$

where the a_p are related to the number of points on the reduction of E mod p. Moreover, $L_E(s)$ can be analytically continued to $\mathbb{C}$ and satisfies the functional equation

$$L_E(s)=w_EY(s)L_E(1-s), \tag{3.3.17}$$

where $w_E=\pm1$ is the sign of the functional equation and

$$Y(s)=\left(\frac{\sqrt{M_E}}{2\pi}\right)^{1-2s}\Gamma\left(\frac32-s\right)\Gamma\left(\frac12+s\right)^{-1}. \tag{3.3.18}$$

For d a fundamental discriminant with $(d,M)=1$, the twist of $L_E(s)$ by the quadratic character $\chi_d(n)=(\frac{d}{n})$ is defined for $\mathrm{Re}(s)>1$ by

$$L_E(s,\chi_d)=\sum_{n=1}^{\infty}\frac{a_n\chi_d(n)}{n^{s+1/2}}. \tag{3.3.19}$$

These twisted L-functions can also be analytically continued to $\mathbb{C}$ and they satisfy the functional equation

$$L_E(s,\chi_d)=w_E\chi_d(-M_E)Y_d(s)L_E(1-s,\chi_d), \tag{3.3.20}$$

where $Y_d(s) = |d|^{1-2s}Y(s)$. The set of $L_E(s, \chi_d)$ for which the sign $w_E\chi_d(-M_E)$ of the functional equation equals $+1$ forms a family with even orthogonal symmetry and so we use the special orthogonal group $SO(2N)$ for comparison.

The moments of moments of this family are defined as

$$\mathrm{MoM}_{L_E}(k, \beta) = \frac{1}{D^*} \sideset{}{^*}\sum_{\substack{|d|\le D \\ w_E\chi_d(-M_E)=1}} \left(\frac{1}{2\pi} \int_0^{2\pi} L_E(\tfrac{1}{2} + i\theta, \chi_d)^{2\beta} d\theta \right)^k, \tag{3.3.21}$$

where the sum is only over fundamental discriminants and D^* is the number of terms in the sum. The conjecture made in [BK21] for this family, based on Theorem 3.1.6, is the following.

Conjecture 3.3.1 (Bailey and Keating)**.** *For $k, \beta \in \mathbb{N}$ and k, β not both 1, as $D \to \infty$,*

$$\mathrm{MoM}_{L_E}(k, \beta) = \xi(k, \beta)\mathfrak{c}_{SO}(k, \beta)(\log D)^{k\beta(2k\beta-1)-k}\big(1 + O_{k,\beta}\big(\log^{-1} D\big)\big), \tag{3.3.22}$$

where $\mathfrak{c}_{SO}(k, \beta)$ corresponds to the leading order coefficient in (3.1.15) *and $\xi(k, \beta)$ contains the arithmetic information.*

For the sake of simplicity, we will assume that the conductor M_E is square-free and odd. The sign of the functional equation of the L-function $L_E(s, \chi_d)$

$$w_E\chi_d(-M_E) = w_E\chi_d(-1)\chi_d(M_E) \tag{3.3.23}$$

depends on $\chi_d(-1)$ and $\chi_d(M_E)$. The factor $\chi_d(-1)$ is determined by the sign of d (it is $+1$ if $d > 0$ and -1 if $d < 0$) and with our assumptions on the conductor M_E, the factor $\chi_d(M_E)$ is determined by $d \pmod{M_E}$. Thus, we will restrict our attention to the moments of moments over negative discriminants d with $d \equiv a \pmod{M_E}$ such that $w_E\chi_d(-M_E) = +1$ and define

$$\mathrm{MoM}^-_{L_E}(k, \beta; a) := \frac{1}{D^*} \sideset{}{^*}\sum_{\substack{-D<d<0 \\ d\equiv a \pmod{M_E}}} \left(\frac{1}{2\pi} \int_0^{2\pi} L_E(\tfrac{1}{2} + i\theta, \chi_d)^{2\beta} d\theta \right)^k. \tag{3.3.24}$$

Once again, for $k \in \mathbb{N}$, we may change the order of integration and summation to write

$$\mathrm{MoM}^{-}_{L_E}(k,\beta;a) = \frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{1}{D^*}\sideset{}{^*}\sum_{\substack{-D<d<0\\ d\equiv a \pmod{M_E}}}\prod_{m=1}^{k} L_E(\tfrac{1}{2}+i\theta_m,\chi_d)^{2\beta}d\theta_1\cdots d\theta_k, \tag{3.3.25}$$

and we have the following conjecture from the recipe of [CFK+05] for the shifted moments in the integrand.

Conjecture 3.3.2 (Conrey et al.). *Let $k,\beta\in\mathbb{N}$ and let $\underline{\theta}=(\theta_1,\ldots,\theta_k)\in\mathbb{R}^k$. Then,*

$$\begin{aligned}\frac{1}{D^*}&\sideset{}{^*}\sum_{\substack{-D<d<0\\ d\equiv a \pmod{M_E}}}\prod_{m=1}^{k} L_E(\tfrac{1}{2}+i\theta_m,\chi_d)^{2\beta}\\ &=\frac{1}{D^*}\sideset{}{^*}\sum_{\substack{-D<d<0\\ d\equiv a \pmod{M_E}}}\prod_{m=1}^{k} Y_d(\tfrac{1}{2}+i\theta_m)^{\beta}\Upsilon_{k,\beta}(\log|d|,\underline{\theta})+O(D^{-\delta}),\end{aligned} \tag{3.3.26}$$

for some $\delta>0$, where

$$\begin{aligned}\Upsilon_{k,\beta}(x,\underline{\theta}) = \frac{(-1)^{k\beta}2^{2k\beta}}{(2\pi i)^{2k\beta}(2k\beta)!}&\oint\cdots\oint\frac{H(z_1,\ldots,z_{2k\beta})\Delta(z_1^2,\ldots,z_{2k\beta}^2)^2\prod_{n=1}^{2k\beta}z_n}{\prod_{n=1}^{2k\beta}\prod_{m=1}^{k}(z_n-i\theta_m)^{2\beta}(z_n+i\theta_m)^{2\beta}}\\ &\times e^{x\sum_{n=1}^{2k\beta}z_n}dz_1\ldots dz_{2k\beta},\end{aligned} \tag{3.3.27}$$

in which the path of integration encloses the poles at $\pm i\theta_m$ for $1\le m\le k$. Also,

$$H(z_1,\ldots,z_{2k\beta}) = B_{k\beta}(z_1,\ldots,z_{2k\beta})\prod_{n=1}^{2k\beta}Y(\tfrac{1}{2}+z_n)^{-\frac{1}{2}}\prod_{1\le m<n\le 2k\beta}\zeta(1+z_m+z_n), \tag{3.3.28}$$

where $B_{k\beta,a}$ is the Euler product, absolutely convergent for $\sum_{n=1}^{2k\beta}|z_n|<1/2$, defined by

$$\begin{aligned}B_{k\beta,a}(z_1,\ldots,z_{2k\beta}) = &\prod_{p\nmid M_E}\prod_{1\le m<n\le 2k\beta}\left(1-\frac{1}{p^{1+z_m+z_n}}\right)\\ &\times\left(\frac{1}{2}\left(\prod_{n=1}^{2k\beta}\mathcal{L}_p\left(\frac{1}{p^{1/2+z_n}}\right)+\prod_{n=1}^{2k\beta}\mathcal{L}_p\left(\frac{-1}{p^{1/2+z_n}}\right)\right)+\frac{1}{p}\right)\left(1+\frac{1}{p}\right)^{-1}\end{aligned}$$

$$\times \prod_{p|M_E} \left(\prod_{n=1}^{2k\beta} \mathcal{L}_p\left(\frac{\chi_a(p)}{p^{1/2+z_n}} \right) \right). \tag{3.3.29}$$

Naturally, we define

$$\begin{aligned}\mathrm{MoM}^{-}_{\Upsilon_{k,\beta}}(D;a) := \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{1}{D^*} \sideset{}{^*}\sum_{\substack{-D<d<0 \\ d\equiv a \pmod{M_E}}} \prod_{m=1}^{k} Y_d(\tfrac{1}{2}+i\theta_m)^\beta \\ \times \Upsilon_{k,\beta}(\log|d|, \underline{\theta}) \prod_{j=1}^{k} d\theta_j,\end{aligned} \tag{3.3.30}$$

which should approximate $\mathrm{MoM}^{-}_{L_E}(k,\beta;a)$ up to a power saving in D. Similarly to the symplectic case considered earlier, we can clearly see the similarity between the integral expressions for $\Upsilon_{k,\beta}(x,\underline{\theta})$ above and $I_{k,\beta}(SO(2N),\underline{\theta})$ in (3.3.4) on identifying x with N. Therefore, by following the proof of Theorem 3.1.8 and taking into account the arithmetic factors just as in the case of the quadratic Dirichlet L-functions, one can show that

$$\begin{aligned}\mathrm{MoM}^{-}_{\Upsilon_{k,\beta}}(D;a) &\sim B_{k\beta,a}(0,\dots,0)\gamma_{SO}(k,\beta)\frac{1}{D^*} \sideset{}{^*}\sum_{\substack{-D<d<0 \\ d\equiv a \pmod{M_E}}} (\log|d|)^{k\beta(2k\beta-1)-k} \\ &= B_{k\beta,a}(0,\dots,0)\gamma_{SO}(k,\beta)(\log D)^{k\beta(2k\beta-1)-k}\big(1+O(\log^{-1} D)\big).\end{aligned} \tag{3.3.31}$$

As a consequence, Conjecture 3.3.1 also follows from the shifted moment conjectures of [CFK+05].

Chapter 4

Joint moments of characteristic polynomials of random symplectic and orthogonal matrices

In this chapter we obtain asymptotic formulae for the joint moments

$$\int_{G(2N)} \left(\Lambda_X^{(n_1)}(1)\right)^{k_1} \left(\Lambda_X^{(n_2)}(1)\right)^{k_2} dX \tag{4.0.1}$$

of derivatives of the characteristic polynomials, where $G(2N)$ is one the matrix ensembles $Sp(2N)$, $SO(2N)$ or $O^{-}(2N)$. Our main results give two explicit expressions for the leading order coefficients for each of the matrix ensembles under consideration and are detailed in section 4.2. We use our results to motivate conjectures for the analogous joint moments of derivatives of families of L-functions with symplectic or orthogonal symmetry. We also show that the prediction of our conjectures agree with known results in the function field setting.

4.1 Moments of derivatives of characteristic polynomials

In [CRS06], Conrey, Rubinstein and Snaith considered the problem of obtaining an exact formula for the moments

$$\int_{U(N)} |\Lambda_X'(1)|^{2k} \, dX \tag{4.1.1}$$

of the derivative of the characteristic polynomial over $U(N)$. In particular, one would like to have a formula valid for $k \in \mathbb{C}$ with $\mathrm{Re}(k) > -1/2$. The complex moments of the derivatives are of interest as they can be used to infer information on the zeros of the derivatives via Jensen's formula. Another motivation is the link between random matrix theory and the study of families of L-functions and their value distributions in analytic number theory. Specifically, one can use formulae obtained for characteristic polynomials of the various matrix ensembles to predict formulae for the corresponding quantities for L-functions with the same symmetry type. For results on the radial distribution of the zeros of the derivative of characteristic polynomials and on the horizontal distribution of the zeros of the derivative of the Riemann zeta function, see, for example, [DFF$^+$10, Mez03] and [Sou98, Zha01], respectively.

For the ensemble of random unitary matrices $U(N)$, Conrey, Rubinstein and Snaith [CRS06] proved that for integer $k \geq 1$, as $N \to \infty$,

$$\int_{U(N)} |\Lambda_X'(1)|^{2k}\, dX \sim c_k N^{k^2+2k}, \tag{4.1.2}$$

where

$$c_k = (-1)^{k(k+1)/2} \sum_{h=0}^{k} \binom{k}{h} \left(\frac{d}{dx}\right)^{k+h} \left(e^{-x} x^{-k^2/2} \det_{k\times k} \left(I_{i+j-1}(2\sqrt{x}) \right) \right)\Bigg|_{x=0}, \tag{4.1.3}$$

with $I_n(x)$ denoting the modified Bessel function of the first kind. Also proven in [CRS06] is a similar asymptotic formula for the $2k$-th moment of the derivative of $\mathcal{Z}_X(s)$ at $s = 1$, where recall that $\mathcal{Z}_X(s)$ is equal to the characteristic polynomial $\Lambda_X(s)$ multiplied by a suitable factor so that $\mathcal{Z}_X(e^{-i\theta})$ is real for $\theta \in \mathbb{R}$. As an application, the authors use their result to conjecture asymptotic formulae for the $2k$-th moments of the derivative of the Riemann zeta function and of Hardy's Z-function on the critical line. Forrester and Witte [FW06] have given alternate expressions for the leading order coefficients obtained in [CRS06] in terms of solutions to Painlevé III differential equations.

Significant progress was made towards a formula for the non-integer moments of the derivative by Alvarez [Alv22] who showed that for $k \in \mathbb{N}$, the moments factor as

$$\int_{U(N)} |\Lambda_X'(1)|^{2k}\, dX = \int_{U(N)} |\Lambda_X(1)|^{2k}\, dX \times f(N;k), \tag{4.1.4}$$

where $f(N; k)$ is a polynomial of degree $2k$ expressed as a double multinomial sum of determinants. See also the recent work of Alvarez, Conrey, Rubinstein and Snaith [ACRS24] where they obtain several formulae for the moments

$$\int_{U(N)} |\Lambda'_X(x)|^{2k}\, dX \tag{4.1.5}$$

for both x on the unit circle and general $x \in \mathbb{C}$.

Also of interest are the joint moments of the characteristic polynomial and its derivative. For instance, in the unitary case, one is interested in the quantities

$$\int_{U(N)} |\Lambda'_X(1)|^{2M} |\Lambda_X(1)|^{2k-2M}\, dX, \tag{4.1.6}$$

and

$$F_N(M,k) := \int_{U(N)} |\mathcal{Z}'_X(1)|^{2M} |\mathcal{Z}_X(1)|^{2k-2M}\, dX. \tag{4.1.7}$$

Hughes [Hug01] was able to show that the limit

$$F(M,k) = \lim_{N\to\infty} \frac{1}{N^{k^2+2M}} F_N(M,k) \tag{4.1.8}$$

exists when k and M are integers and conjectured expressions for the limit for all real, suitable k and M. Dehaye [Deh08] gave an alternate proof of Hughes' result with the limit expressed as a certain combinatorial sum. In [Win12], Winn expressed $F_N(h,k)$ in terms of sums over partitions which is also valid for non-integer M of the form $(2m-1)/2$ with $m \in \mathbb{N}$. Hughes' conjecture for the case of real exponents k and M was proven by Assiotis, Keating and Warren in [AKW22] with an explicit expression given for the limit in terms of the expectation of a certain random variable. The characteristic function of this random variable was shown to be connected to a Painlevé III differential equation in the full range of real M and integer k by Assiotis et al. in [ABGS21]. In [AGS22], Assiotis, Gunes and Soor extend the results of [AKW22] to the more general case of the circular Jacobi β ensemble. An asymptotic formula for (4.1.6) when $k \geq M$ are both non-negative integers was obtained by Bailey et al. [BBB$^+$19a]. Basor et al. [BBB$^+$19b] study the joint moments of $\mathcal{Z}_X(s)$, the analogue of Hardy's Z-function, for integer k, M and establish a connection between these and the σ-Painlevé V equation. We note that these joint moments also exhibit a factorisation similar to (4.1.4). Namely, Theorem 4.8 in [Alv22] states that

$$\int_{U(N)} |\Lambda'_X(1)|^{2M} |\Lambda_X(1)|^{2k-2M}\, dX = \int_{U(N)} |\Lambda_X(1)|^{2k}\, dX \times f(N;k,M), \tag{4.1.9}$$

where $f(N;k,M)$ is a polynomial in N of degree $2M$.

More recently, there has been considerable interest in the joint moments

$$\int_{U(N)} |\Lambda_X^{(n_1)}(1)|^{2M} |\Lambda_X^{(n_2)}(1)|^{2k-2M} \, dX \tag{4.1.10}$$

of higher order derivatives. In the case of general n_1, n_2, Barhoumi-Andréani [BA20] gave an asymptotic formula for (4.1.10) for integer k and M with $k \geq M$ and $k \geq 2$, where the leading order coefficient is given in the form of a certain $(k-1)$-fold real integral. Keating and Wei [KW24a] have obtained asymptotic formulae for (4.1.10) and for the joint moments of the n_1-th and n_2-th derivatives of $\mathcal{Z}_X(s)$ for all integers $k \geq M \geq 0$. They give two explicit expressions for the leading order coefficients, one in terms of derivatives of determinants involving the modified Bessel function similarly to (4.1.3), and the other as combinatorial sums involving partitions. They also use their results to motivate conjectures for the joint moments of the n_1-th and n_2-th derivatives of the Riemann zeta function and of the Z-function. The conjectures made in [KW24a] are shown to agree with the known results of Hall [Hal99, Hal04] and Ingham [Ing27]. In [KW24b], Keating and Wei further explore the structure and properties of their leading order coefficients. They establish recursive relations that the coefficients satisfy and also build a connection to a solution of the σ-Painlevé III$'$ equation. Recently, Assiotis, Gunes, Keating and Wei [AGKW24] proved the convergence of the joint moments for the general case of an arbitrary number of derivatives and real exponents. Specifically, the proved that for n_j non-negative integers and h_j positive reals for $j = 1, \ldots, k$, the limit

$$\lim_{N \to \infty} \frac{1}{N^{s^2 + \sum_{j=1}^k h_j n_j}} \int_{U(N)} \prod_{j=1}^k |V_X^{(n_j)}(0)|^{2h_j} \, dX \tag{4.1.11}$$

exists, where $s = \sum_{j=1}^k h_j$ and where they define the characteristic polynomial slightly differently as

$$V_X(\theta) = \det(I - e^{-i\theta} X). \tag{4.1.12}$$

Similarly to [AKW22], the limit is given explicitly in terms of the expectations of certain random variables.

4.1.1 The symplectic and orthogonal case

Extending the results of [CRS06] to the symplectic and orthogonal ensembles, Altuğ et al. [ABP+14] considered the moments of the m-th derivative

$$M_k(G(2N), m) := \int_{G(2N)} \left(\Lambda_X^{(m)}(1)\right)^k \, dX. \tag{4.1.13}$$

They prove asymptotic formulae for $M_k(G(2N), m)$ as $N \to \infty$ for integer $k \geq 1$ when $G(2N) = Sp(2N)$ or $G(2N) = SO(2N)$ and $m = 2$, and when $G(2N) = O^-(2N)$ with $m = 3$. Recall that for $X \in Sp(2N)$ or $SO(2N)$, the characteristic polynomial is of the form

$$\Lambda_X(s) = \prod_{j=1}^{N}(1 - se^{-i\theta_j})(1 - se^{i\theta_j}) = \prod_{j=1}^{N}(1 + s^2 - 2s\cos\theta_j). \tag{4.1.14}$$

A consequence of the form of the characteristic polynomial is that $\Lambda_X'(1)$ may be simply expressed in terms of $\Lambda_X(1)$. Specifically, we have that

$$\Lambda_X'(s) = \sum_{j=1}^{N}(2s - 2\cos\theta_j)\prod_{\substack{k=1\\k\neq j}}^{N}(1 + s^2 - 2s\cos\theta_k), \tag{4.1.15}$$

and so

$$\begin{aligned}\Lambda_X'(1) &= \sum_{j=1}^{N}(2 - 2\cos\theta_j)\prod_{\substack{k=1\\k\neq j}}^{N}(2 - 2\cos\theta_k)\\ &= \sum_{j=1}^{N}\prod_{k=1}^{N}(2 - 2\cos\theta_k)\\ &= N\Lambda_X(1).\end{aligned} \tag{4.1.16}$$

Thus, the moments of the first derivative can be computed exactly using the result of Keating and Snaith [KS00a] on the moments of $\Lambda_X(1)$. Therefore one is interested in the moments of $\Lambda_X''(1)$ and higher order derivatives. If $X \in O^-(2N)$, then $\Lambda_X(1) = 0$ and $\Lambda_X''(1)$ has as simple expression in terms of $\Lambda_X'(1)$. Hence, in this case, it is the moments of $\Lambda_X'''(1)$ and higher derivatives that are of interest.

The leading order coefficients obtained in [ABP$^+$14] are given in terms of derivatives of determinants involving hypergeometric functions. These determinants are shown to satisfy a differential recurrence relation similar to a Toda lattice equation connected to τ-function theory in the study of Painlevé differential equations. An interesting question put forward in [ABP$^+$14] is whether there is a differential equation in the symplectic and orthogonal cases which plays a part analogous to Painlevé III in the unitary setting. Gharakhloo and Witte [GW23] have made promising progress in this direction in their study of $2j-k$ and $j-2k$ bi-orthogonal polynomial systems on the unit circle.

The authors of [ABP$^+$14] also use their results to make conjectures for the asymptotic behaviour of the moments of derivatives at the central point of L-functions with symplectic or orthogonal symmetry. After stating our results in section 4.2, we will extend these to give general conjectures for the joint moments of the derivatives of families of L-functions with these symmetry types.

Finally, one may also consider the characteristic polynomials on the unit circle and study the moments of derivatives of $\tilde{\Lambda}_A(\theta) := \Lambda_X(e^{-i\theta})$. In this case, Gunes [Gun24] has studied the joint moments

$$\int_{Sp(2N)} |\tilde{\Lambda}_A(0)|^{2s-h} |\tilde{\Lambda}_A''(0)|^h \, dX, \tag{4.1.17}$$

and obtained an asymptotic formula as $N \to \infty$ in the range $\alpha(s+\frac{3}{2}) > h \geq 0$, where $\alpha(x)$ denotes the greatest integer strictly less than x. The leading order coefficient for (4.1.17) is given in terms of the expectation of a non-trivial random variable. Moreover, a link between this coefficient and the σ-Painlevé III equation is established and a conjecture for the analogous joint moments of quadratic Dirichlet L-functions is made.

4.2 Statement of results

4.2.1 Notation

For any $w = (w_1, \ldots, w_k) \in \mathbb{C}^k$, the Vandermonde determinant is denoted by

$$\Delta(w) := \det_{k\times k}(w_i^{j-1}) = \prod_{1\leq i<j\leq k} (w_j - w_i), \tag{4.2.1}$$

and we write $w^2 = (w_i^2)_{1\leq i\leq k}$. We will also make use of Vandermonde determinants of differential operators, written as

$$\Delta\left(\frac{d}{dx}\right) := \det_{k\times k}\left(\frac{d^{j-1}}{dx_i^{j-1}}\right) = \prod_{1\leq i<j\leq k}\left(\frac{d}{dx_j} - \frac{d}{dx_i}\right). \tag{4.2.2}$$

Lastly, for $u \in \mathbb{C}$ and $m \in \mathbb{Z}$, we let

$$g_m(u) := \frac{1}{2\pi i}\oint_{|w|=1} \frac{e^{w+u/w^2}}{w^{m+1}} dw$$

$$= \frac{1}{\Gamma(m+1)} {}_0F_2\left(;\frac{m}{2}+1, \frac{m+1}{2}; \frac{u}{4}\right). \tag{4.2.3}$$

These hypergeometric functions will play the role that the modified Bessel function plays in the unitary case. For negative m, say $m = -l$, one should interpret the above expression as the limit as $m \to -l$.

4.2.2 Main results

Our first two theorems give an asymptotic formula for the joint moments of derivatives of characteristic polynomials of matrices over $Sp(2N)$.

Theorem 4.2.1. *Let $0 \le n_1 \le n_2$ be integers and let k_1, k_2 be non-negative integers, not both 0. Set $k = k_1 + k_2$. Then, we have*

$$\begin{aligned}\int_{Sp(2N)} & \left(\Lambda_X^{(n_1)}(1)\right)^{k_1} \left(\Lambda_X^{(n_2)}(1)\right)^{k_2} dX \\ &= b_{k_1,k_2}^{Sp}(n_1,n_2) \cdot (2N)^{k(k+1)/2+k_1n_1+k_2n_2}\left(1+O(N^{-1})\right),\end{aligned} \tag{4.2.4}$$

where

$$\begin{aligned} b_{k_1,k_2}^{Sp}(n_1,n_2) = & \frac{(-1)^{k_1n_1+k_2n_2}}{2^{k(k+1)/2+k_1n_1+k_2n_2}} \sum_{u_1+\cdots+u_P=k_1} \binom{k_1}{u_1\ldots,u_P} \sum_{v_1+\cdots+v_Q=k_2} \binom{k_2}{v_1\ldots,v_Q} \\ & \times \frac{(n_1!)^{k_1}}{\prod_{i=1}^P (\boldsymbol{a}_i!)^{u_i} (\prod_{j=1}^{[n_1/2]} j^{\sum_{i=1}^P u_i a_{i,j}})} \frac{(n_2!)^{k_2}}{\prod_{i=1}^Q (\boldsymbol{b}_i!)^{v_i} (\prod_{j=1}^{[n_2/2]} j^{\sum_{i=1}^Q v_i b_{i,j}})} \\ & \times \sum_{\substack{\sum_{i=1}^k r_{s,i}=W_s \\ s=2,\ldots,[n_2/2]}} \left(\prod_{s=2}^{[n_2/2]} \binom{W_s}{r_{s,1},\ldots,r_{s,k}}\right) \left(\frac{d}{dx}\right)^{W_1} \det_{k\times k}\left(g_{2i-j+2\sum_{s=2}^{[n_2/2]} sr_{s,i}}(x)\right)\Bigg|_{x=0}, \end{aligned} \tag{4.2.5}$$

and, more explicitly,

$$\sum_{\substack{\sum_{i=1}^k r_{s,i}=W_s \\ s=2,\ldots,[n_2/2]}} \left(\prod_{s=2}^{[n_2/2]} \binom{W_s}{r_{s,1},\ldots,r_{s,k}}\right) \left(\frac{d}{dx}\right)^{W_1} \det_{k\times k}\left(g_{2i-j+2\sum_{s=2}^{[n_2/2]} sr_{s,i}}(x)\right)\Bigg|_{x=0} \tag{4.2.6}$$

$$= (-1)^{k(k-1)/2} \sum_{\substack{\sum_{i=1}^k r_{s,i}=W_s \\ s=1,\ldots,[n_2/2]}} \left(\prod_{s=1}^{n} \binom{W_s}{r_{s,1},\ldots,r_{s,k}} \right)$$

$$\times \prod_{j=1}^{k} \frac{1}{(2k+2\sum_{s=1}^{[n_2/2]} sr_{s,j}+1-2j)!} \prod_{1\le i<j\le k} \left(2\sum_{s=1}^{[n_2/2]} sr_{s,j} - 2\sum_{s=1}^{[n_2/2]} sr_{s,i} - 2j + 2i \right). \tag{4.2.7}$$

Here, we define P to be the number of distinct tuples $\mathbf{a}_i := (a_{i,0}, a_{i,1}, \ldots, a_{i,[n_1/2]})$ of integers satisfying

$$a_{i,j} \ge 0 \text{ and } a_{i,0} + 2\sum_{j=1}^{[n_1/2]} j a_{i,j} = n_1,$$

and let $\mathbf{a}_1, \ldots, \mathbf{a}_P$ be these such tuples. In other words, the tuples $\mathbf{a}_i$ correspond to the partitions of n_1 whose parts are all even or equal to 1 and P is the number of these such partitions. Similarly, Q is defined to be the number of distinct tuples $\mathbf{b}_i = (b_{i,0}, b_{i,1}, \ldots, b_{i,[n_2/2]})$ of integers satisfying

$$b_{i,j} \ge 0 \text{ and } b_{i,0} + 2\sum_{j=1}^{[n_2/2]} j b_{i,j} = n_2,$$

and we let $\mathbf{b}_1, \ldots, \mathbf{b}_Q$ be these tuples. We define $\mathbf{a}_i! := \prod_{j=0}^{[n_1/2]} a_{i,j}!$ and $\mathbf{b}_i! := \prod_{j=0}^{[n_2/2]} b_{i,j}!$. Finally, $W_j := \sum_{i=1}^{P} u_i a_{i,j} + \sum_{i=1}^{Q} v_i b_{i,j}$ for $j = 1, \ldots, [n_1/2]$ and $W_j := \sum_{i=1}^{Q} v_i b_{i,j}$ for $j = [n_1/2]+1, \ldots, [n_2/2]$.

Theorem 4.2.2. *Let $0 \le n_1 \le n_2$ be integers and let k_1, k_2 be non-negative integers, not both 0. Set $k = k_1 + k_2$. Then, we have*

$$\int_{Sp(2N)} \left(\Lambda_X^{(n_1)}(1)\right)^{k_1} \left(\Lambda_X^{(n_2)}(1)\right)^{k_2} dX$$
$$= b_{k_1,k_2}^{Sp}(n_1,n_2) \cdot (2N)^{k(k+1)/2+k_1n_1+k_2n_2} \left(1+O(N^{-1})\right), \tag{4.2.8}$$

where

$$b_{k_1,k_2}^{Sp}(n_1,n_2) = \frac{(-1)^{k(k-1)/2+k_1n_1+k_2n_2}}{2^{k(k+1)/2+k_1n_1+k_2n_2}} (n_1!)^{k_1}(n_2!)^{k_2} \sum_{\substack{2\sum_{j=1}^k l_{i,j} \le n_1 \\ i=1,\ldots,k_1}} \sum_{\substack{2\sum_{j=1}^k m_{i,j} \le n_2 \\ i=1,\ldots,k_2}}$$

$$\times \left(\prod_{i=1}^{k_1} \frac{1}{(n_1 - 2\sum_{j=1}^{k} l_{i,j})!} \right) \left(\prod_{i=1}^{k_2} \frac{1}{(n_2 - 2\sum_{j=1}^{k} m_{i,j})!} \right)$$
$$\times \prod_{j=1}^{k} \frac{1}{(2k + V_j - 2j + 1)!} \prod_{1 \le i < j \le k} (V_j - V_i - 2j + 2i). \tag{4.2.9}$$

Here $V_j := 2\sum_{i=1}^{k_1} l_{i,j} + 2\sum_{i=1}^{k_2} m_{i,j}$ *for* $j = 1, \ldots, k$.

Our next two theorems give an asymptotic formula for the joint moments over $SO(2N)$.

Theorem 4.2.3. *With notation as in Theorem 4.2.1, we have*

$$\int_{SO(2N)} \left(\Lambda_X^{(n_1)}(1)\right)^{k_1} \left(\Lambda_X^{(n_2)}(1)\right)^{k_2} dX$$
$$= b_{k_1,k_2}^{SO}(n_1, n_2) \cdot (2N)^{k(k-1)/2 + k_1 n_1 + k_2 n_2} \left(1 + O(N^{-1})\right), \tag{4.2.10}$$

where

$$b_{k_1,k_2}^{SO}(n_1, n_2) = \frac{1}{2^{k(k-3)/2 + k_1 n_1 + k_2 n_2}} \sum_{u_1 + \cdots + u_P = k_1} \binom{k_1}{u_1 \ldots, u_P} \sum_{v_1 + \cdots + v_Q = k_2} \binom{k_2}{v_1 \ldots, v_Q}$$
$$\times \frac{(n_1!)^{k_1}}{\prod_{i=1}^{P} (\boldsymbol{a}_i!)^{u_i} (\prod_{j=1}^{[n_1/2]} j^{\sum_{i=1}^{P} u_i a_{i,j}})} \frac{(n_2!)^{k_2}}{\prod_{i=1}^{Q} (\boldsymbol{b}_i!)^{v_i} (\prod_{j=1}^{[n_2/2]} j^{\sum_{i=1}^{Q} v_i b_{i,j}})}$$
$$\times \sum_{\substack{\sum_{i=1}^{k} r_{s,i} = W_s \\ s = 2, \ldots, [n_2/2]}} \left(\prod_{s=2}^{[n_2/2]} \binom{W_s}{r_{s,1}, \ldots, r_{s,k}} \right) \left(\frac{d}{dx}\right)^{W_1} \det_{k \times k} \left(g_{2i-j-1+2\sum_{s=2}^{[n_2/2]} s r_{s,i}}(x) \right) \Bigg|_{x=0}, \tag{4.2.11}$$

and, more explicitly,

$$\sum_{\substack{\sum_{i=1}^{k} r_{s,i} = W_s \\ s = 2, \ldots, [n_2/2]}} \left(\prod_{s=2}^{[n_2/2]} \binom{W_s}{r_{s,1}, \ldots, r_{s,k}} \right) \left(\frac{d}{dx}\right)^{W_1} \det_{k \times k} \left(g_{2i-j-1+2\sum_{s=2}^{[n_2/2]} s r_{s,i}}(x) \right) \Bigg|_{x=0} \tag{4.2.12}$$
$$= (-1)^{k(k-1)/2} \sum_{\substack{\sum_{i=1}^{k} r_{s,i} = W_s \\ s = 1, \ldots, [n_2/2]}} \left(\prod_{s=1}^{n} \binom{W_s}{r_{s,1}, \ldots, r_{s,k}} \right)$$

$$\times \prod_{j=1}^{k} \frac{1}{(2k + 2\sum_{s=1}^{[n_2/2]} sr_{s,j} - 2j)!} \prod_{1\le i<j\le k} \left(2\sum_{s=1}^{[n_2/2]} sr_{s,j} - 2\sum_{s=1}^{[n_2/2]} sr_{s,i} - 2j + 2i \right). \tag{4.2.13}$$

Theorem 4.2.4. *With notation as in Theorem 4.2.2, we have*

$$\begin{aligned}\int_{SO(2N)} & \left(\Lambda_X^{(n_1)}(1)\right)^{k_1} \left(\Lambda_X^{(n_2)}(1)\right)^{k_2} dX \\ &= b_{k_1,k_2}^{SO}(n_1,n_2) \cdot (2N)^{k(k-1)/2+k_1n_1+k_2n_2} \left(1 + O(N^{-1})\right),\end{aligned} \tag{4.2.14}$$

where

$$\begin{aligned} b_{k_1,k_2}^{SO}(n_1,n_2) = \frac{(-1)^{k(k-1)/2}}{2^{k(k-3)/2+k_1n_1+k_2n_2}} (n_1!)^{k_1} (n_2!)^{k_2} & \sum_{\substack{2\sum_{j=1}^{k} l_{i,j} \le n_1 \\ i=1,\ldots,k_1}} \sum_{\substack{2\sum_{j=1}^{k} m_{i,j} \le n_2 \\ i=1,\ldots,k_2}} \\ \times \left(\prod_{i=1}^{k_1} \frac{1}{(n_1 - 2\sum_{j=1}^{k} l_{i,j})!} \right) & \left(\prod_{i=1}^{k_2} \frac{1}{(n_2 - 2\sum_{j=1}^{k} m_{i,j})!} \right) \\ \times \prod_{j=1}^{k} \frac{1}{(2k + V_j - 2j)!} & \prod_{1\le i<j\le k} (V_j - V_i - 2j + 2i). \end{aligned} \tag{4.2.15}$$

Our final theorem gives an asymptotic formula for the joint moments over $O^{-}(2N)$ with the leading order coefficient expressed in terms of $b_{k_1,k_2}^{Sp}(n_1,n_2)$.

Theorem 4.2.5. *Let $1 \le n_1 \le n_2$ be integers and let k_1, k_2 be non-negative integers, not both 0. Set $k = k_1 + k_2$. Then, we have*

$$\begin{aligned}\int_{O^{-}(2N)} & \left(\Lambda_X^{(n_1)}(1)\right)^{k_1} \left(\Lambda_X^{(n_2)}(1)\right)^{k_2} dX \\ &= b_{k_1,k_2}^{O^-}(n_1,n_2) \cdot (2N)^{k(k+1)/2+k_1(n_1-1)+k_2(n_2-1)} \left(1 + O(N^{-1})\right),\end{aligned} \tag{4.2.16}$$

where

$$b_{k_1,k_2}^{O^-}(n_1,n_2) = (-1)^{k_1(n_1-1)+k_2(n_2-1)}\, 2^k n_1^{k_1} n_2^{k_2}\, b_{k_1,k_2}^{Sp}(n_1-1,n_2-1),$$

with $b_{k_1,k_2}^{Sp}(n_1,n_2)$ as defined in Theorems 4.2.1 and 4.2.2.

Our Theorems 4.2.1 and 4.2.3 exhibit the same structure as the asymptotic formulae obtained in [ABP+14]. Namely, the leading order coefficients are expressed in terms of derivatives of determinants of the hypergeometric functions $g_m(u)$. As mentioned earlier, these determinants were shown to satisfy a differential recurrence relation [ABP+14, Theorem 1.5]. Explicitly, define

$$\mathcal{T}_{k,l}(u) := \det_{k\times k}\big(g_{2i-j+l}(u)\big) \tag{4.2.17}$$

for $k \geq 1$ and $l \in \mathbb{Z}$. Then the differential recurrence relation that these determinants satisfy is the following.

Theorem 4.2.6 (Altuğ et al.)**.** *Let $k \geq 1$ be an integer and $l \in \mathbb{Z}$. Then*

$$\mathcal{T}_{k+1,l}(u)\mathcal{T}_{k-1,l}(u) = 2\big(u\mathcal{T}_{k,l}(u)\mathcal{T}''_{k,l}(u) + \mathcal{T}_{k,l}(u)\mathcal{T}'_{k,l}(u) - u(\mathcal{T}'_{k,l}(u))^2\big). \tag{4.2.18}$$

This recurrence relation allows for the expressions for the leading order coefficients in Theorems 4.2.1 and 4.2.3 to be computed much more quickly as k_1, k_2 get large. However, similarly to the unitary case considered in [KW24a], the formulae given for the leading order coefficients in Theorems 4.2.1 and 4.2.3 may not be computationally efficient when n_1, n_2 are large since one has to compute the tuples $\mathbf{a}_i$ and $\mathbf{b}_i$. This requires computing all of the partitions of n_1 and n_2 whose parts are all even or equal to 1 and this is computationally demanding if n_1, n_2 are large. One also has to compute all the partitions of k_1 and k_2 into P and Q parts respectively. This task also increases quickly in complexity as n_1, n_2 and hence P, Q grow large. We note that by taking $n_1 = 0,\ n_2 = 2$ and $k_1 = 0,\ k_2 = k$ in Theorems 4.2.1 and 4.2.3, we indeed recover the statements of Theorems 1.1 and 1.2 in [ABP+14].

Aside from giving an alternate expression for the leading order coefficients, which are interesting in their own right, one advantage of Theorems 4.2.1 and 4.2.3 is that the formulae are more computationally effective when n_1, n_2 are large and k_1, k_2 are small. This is because the formulae for the coefficients in Theorems 4.2.1 and 4.2.3 require the computation of the even partitions, into k parts, of the integers less than or equal to n_1, n_2. This problem grows quickly in complexity as k_1, k_2 get large and so the formulae of Theorems 4.2.1 and 4.2.3 are preferable in the case that one wants to compute numerical values of the leading coefficients with large n_1, n_2.

Lastly, in the case of the first moment of the n-th derivative of the characteristic polynomial, we obtain the following simple expressions for the leading order coefficient.

Proposition 4.2.7. *For $n \geq 1$ an integer, we have that*

$$b^{Sp}_{0,1}(0,n) = \frac{(-1)^n}{2(n+1)}.$$

Also, for $n \geq 1$, we have

$$b_{0,1}^{SO}(0,n) = 1.$$

4.2.3 Conjectures for moments of derivatives of L-functions

Using the standard random matrix philosophy allows us to make conjectures based on our results for the joint moments of derivatives of L-functions with symmetry type Sp, SO or O^- in the sense of [KS99b]. We give an example conjecture for the family of quadratic Dirichlet L-functions at $s = 1/2$ below. This is an example of a family with symplectic symmetry so we use our results for $Sp(2N)$ as a model.

Conjecture 4.2.8. *Let $\mathcal{D}(X) = \{d \text{ a fundamental discriminant} : |d| < X\}$, and let $L(s, \chi_d)$ be the Dirichlet L-function attached to the quadratic character χ_d. Then, for $0 \leq n_1 \leq n_2$ and $k_1, k_2 \geq 0$ integers with k_1, k_2 not both 0, we have that as $X \to \infty$,*

$$\frac{1}{|\mathcal{D}(X)|} \sum_{d \in \mathcal{D}(X)} L^{(n_1)}(\tfrac{1}{2}, \chi_d)^{k_1} L^{(n_2)}(\tfrac{1}{2}, \chi_d)^{k_2} \sim a_k b_{k_1,k_2}^{Sp}(n_1, n_2)(\log X)^{k(k+1)/2 + k_1 n_1 + k_2 n_2}, \tag{4.2.19}$$

where $k = k_1 + k_2$ and $b_{k_1,k_2}^{Sp}(n_1, n_2)$ is the random matrix theory coefficient defined in Theorems 4.2.1 and 4.2.2. Also,

$$a_k = \prod_p \frac{(1 - 1/p)^{k(k+1)/2}}{1 + 1/p} \left(\frac{(1 - 1/\sqrt{p})^{-k} + (1 + 1/\sqrt{p})^{-k}}{2} + \frac{1}{p} \right), \tag{4.2.20}$$

is an arithmetic factor depending on the family of L-functions. In particular, a_k is the same coefficient appearing in Conjecture 1.5.1 for the moments of $L(\frac{1}{2}, \chi_d)$.

One can naturally use our Theorems 4.2.1-4.2.5 to make analogous conjectures for the joint moments of derivatives at the central point for any family of L-functions with symplectic or orthogonal symmetry.

To the best of our knowledge, there are no results on the moments of derivatives of these quadratic Dirichlet L-functions over number fields. However, the moments of derivatives of the analogous quadratic Dirichlet L-functions over function fields have been investigated. We will discuss the known results in the function field setting in Chapter 7 where we find good agreement with the conjecture based on our results in this chapter.

4.3 Strategy for the proofs

Our strategy is to obtain the joint moments by differentiating the corresponding shifted moments with respect to the shifts. Recall that we define the shifted moments as

$$I(G(2N); z_1, \ldots, z_k) := \int_{G(2N)} \Lambda_X(z_1) \cdots \Lambda_X(z_k) \, dX, \tag{4.3.1}$$

for $G(2N) \in \{Sp(2N),\, SO(2N),\, O^-(2N)\}$. To illustrate the idea, we consider the case of the k-th moment of the n-th derivative. As discussed in [ABP+14], a direct calculation gives

$$\frac{d^n}{d\alpha_1^n} \cdots \frac{d^n}{d\alpha_k^n} I(G(2N); e^{-\alpha_1}, \ldots, e^{-\alpha_k})\big|_{\alpha_j=0} = (-1)^{nk} \int_{G(2N)} \left(\sum_{j=0}^{n} \left\{ {n \atop j} \right\} \Lambda_X^{(j)}(1) \right)^k dX, \tag{4.3.2}$$

where $\left\{ {n \atop j} \right\}$ denotes a Stirling number of the second kind. Then, one can show by induction that the term corresponding to the n-th derivative $\Lambda_X^{(n)}(1)$ gives the leading order term in the sum in (4.3.2) as $N \to \infty$. In other words,

$$\int_{G(2N)} \left(\Lambda_X^{(n)}(1) \right)^k dX = \frac{d^n}{d\alpha_1^n} \cdots \frac{d^n}{d\alpha_k^n} I(G(2N); e^{-\alpha_1}, \ldots, e^{-\alpha_k})\big|_{\alpha_j=0} \left(1 + O(N^{-1})\right). \tag{4.3.3}$$

A similar argument also applies to the joint moments of derivatives and so the starting point for the proofs of Theorems 4.2.1-4.2.5 is the formula

$$\begin{aligned} &\int_{G(2N)} \left(\Lambda_X^{(n_1)}(1) \right)^{k_1} \left(\Lambda_X^{(n_2)}(1) \right)^{k_2} dX \\ &\quad = \prod_{j=1}^{k_1} \left(\frac{d}{d\alpha_j} \right)^{n_1} \prod_{j=k_1+1}^{k} \left(\frac{d}{d\alpha_j} \right)^{n_2} I(G(2N); e^{-\alpha_1}, \ldots, e^{-\alpha_k})\big|_{\alpha_j=0} \left(1 + O(N^{-1})\right). \end{aligned} \tag{4.3.4}$$

We will then use the multiple contour integral expressions of Conrey et al. [CFK+03] for the shifted moments $I(G(2N); e^{-\alpha_1}, \ldots, e^{-\alpha_k})$ and evaluate the derivatives with respect to the shifts α_j. In fact, we will obtain two alternate expressions for the derivatives and in both cases, we will compute the resulting contour integral explicitly. This leads to the two expressions for the leading order coefficients $b^G_{k_1,k_2}(n_1, n_2)$ in Theorems 4.2.1-4.2.5. Lastly, our method of proof can also be applied to more general joint moments of the form

$$\int_{G(2N)} \left(\Lambda_X^{(n_1)}(1)\right)^{k_1} \cdots \left(\Lambda_X^{(n_m)}(1)\right)^{k_m} dX, \tag{4.3.5}$$

for any positive integer m. In particular, the technical lemmas and propositions that we cover in the next section are sufficient to obtain an asymptotic formula in the more general case.

4.4 Preliminary lemmas and propositions

We begin with the following two lemmas concerning Vandermonde determinants of differential operators. The first is quoted from [ABP+14, Lemma 2.8] and follows from the definition.

Lemma 4.4.1. *Let $f_1(x), \ldots, f_k(x)$ be $k-1$ times differentiable. Then*

$$\Delta\left(\frac{d}{dx}\right)\prod_{i=1}^{k} f_i(x_i) = \det_{k\times k}\left(f_i^{(j-1)}(x_i)\right). \tag{4.4.1}$$

Lemma 4.4.2. *Let $f_1(x,y), \ldots, f_k(x,y)$ be $k-1$ times differentiable in x and y. Then*

$$\Delta\left(\frac{d}{dx}\right)\Delta\left(\frac{d}{dy}\right)\prod_{i=1}^{k} f_i(x_i,y_i)\Bigg|_{\substack{x_1=\cdots=x_k=X,\\ y_1=\cdots=y_k=Y}} = \sum_{\mu\in S_k}\det_{k\times k}\left(\frac{d^{i+j-2}}{dX^{i-1}dY^{j-1}} f_{\mu(i)}(X,Y)\right). \tag{4.4.2}$$

In particular, when $f_1 = \cdots = f_k = f$, we have

$$\Delta\left(\frac{d}{dx}\right)\Delta\left(\frac{d}{dy}\right)\prod_{i=1}^{k} f(x_i,y_i)\Bigg|_{\substack{x_1=\cdots=x_k=X,\\ y_1=\cdots=y_k=Y}} = k!\det_{k\times k}\left(\frac{d^{i+j-2}}{dX^{i-1}dY^{j-1}} f(X,Y)\right). \tag{4.4.3}$$

Proof. The case when $f_1 = \cdots = f_k = f$ is the result of [ABP+14, Lemma 2.9] and the proof of the general case follows the same lines. By Lemma 4.4.1, we have

$$\Delta\left(\frac{d}{dx}\right)\prod_{i=1}^{k} f_i(x_i,y_i) = \det_{k\times k}\left(\frac{d^{j-1}}{dx_i^{j-1}} f_i(x_i,y_i)\right) = \sum_{\mu\in S_k}\operatorname{sign}(\mu)\prod_{i=1}^{k}\frac{d^{\mu(i)-1}}{dx_i^{\mu(i)-1}} f_i(x_i,y_i), \tag{4.4.4}$$

where S_k is the set of permutations on $\{1,\ldots,k\}$ and the second equality follows from the Leibniz formula for the determinant. Then, by Lemma 4.4.1 again, we have that

$$\begin{aligned}
&\Delta\left(\frac{d}{dx}\right)\Delta\left(\frac{d}{dy}\right)\prod_{i=1}^{k} f_i(x_i,y_i)\Bigg|_{\substack{x_i=X,\\ y_i=Y}}\\
&=\sum_{\mu\in S_k}\operatorname{sign}(\mu)\,\Delta\left(\frac{d}{dy}\right)\prod_{i=1}^{k}\frac{d^{\mu(i)-1}}{dx_i^{\mu(i)-1}}f_i(x_i,y_i)\Bigg|_{\substack{x_i=X,\\ y_i=Y}}\\
&=\sum_{\mu\in S_k}\operatorname{sign}(\mu)\det_{k\times k}\left(\frac{d^{\mu(i)+j-2}}{dx_i^{\mu(i)-1}dy_i^{j-1}}f_i(x_i,y_i)\right)\Bigg|_{\substack{x_i=X,\\ y_i=Y}}\\
&=\sum_{\mu\in S_k}\operatorname{sign}(\mu)\det_{k\times k}\left(\frac{d^{\mu(i)+j-2}}{dX^{\mu(i)-1}dY^{j-1}}f_i(X,Y)\right)\\
&=\sum_{\mu\in S_k}\det_{k\times k}\left(\frac{d^{i+j-2}}{dX^{i-1}dY^{j-1}}f_{\mu(i)}(X,Y)\right), \qquad (4.4.5)
\end{aligned}$$

where we have interchanged the rows of the matrix to obtain the final line. □

We next express a certain contour integral in terms of the hypergeometric functions $g_m(u)$.

Lemma 4.4.3. *Let $k\in\mathbb{Z}$ and let $n\geq 1$ be an integer. Then, for complex numbers $u_1,\ldots,u_n$, we have*

$$\frac{1}{2\pi i}\oint_{|w|=1}\exp\left(w+\sum_{j=1}^{n}\frac{u_j}{w^{2j}}\right)\frac{dw}{w^{k+1}}=\sum_{m_2,\ldots,m_n=0}^{\infty}\left(\prod_{j=2}^{n}\frac{u_j^{m_j}}{m_j!}\right)g_{k+2\sum_{j=2}^{n}jm_j}(u_1),$$

where $g_m(u)$ is the hypergeometric function defined in (4.2.3).

Proof. We compute the integral by determining the coefficient of w^k in the exponential factor of the integrand and then using the residue theorem. So, let $a_n(k)$ be the coefficient of w^k in $\exp(w+\sum_{j=1}^{n}u_j/w^{2j})$. Then,

$$\begin{aligned}
\exp\left(w+\sum_{j=1}^{n}\frac{u_j}{w^{2j}}\right)&=\exp\left(\frac{u_n}{w^{2n}}\right)\exp\left(w+\sum_{j=1}^{n-1}\frac{u_j}{w^{2j}}\right)\\
&=\left(\sum_{m=0}^{\infty}\frac{u_n^m}{m!}w^{-2nm}\right)\left(\sum_{m=-\infty}^{\infty}a_{n-1}(m)w^m\right).
\end{aligned}$$

From this it follows that

$$\begin{aligned} a_n(k) &= \sum_{m_n=0}^{\infty} \frac{u_n^{m_n}}{m_n!} a_{n-1}(k+2nm_n) \\ &= \sum_{m_2,\ldots,m_n=0}^{\infty} \left(\prod_{j=2}^{n} \frac{u_j^{m_j}}{m_j!}\right) a_1\left(k+2\sum_{j=2}^{n} jm_j\right). \end{aligned}$$

We then see that by definition, $a_1(k+2\sum_{j=2}^{n} jm_j) = g_{k+2\sum_{j=2}^{n} jm_j}(u_1)$ and hence

$$a_n(k) = \sum_{m_2,\ldots,m_n=0}^{\infty} \left(\prod_{j=2}^{n} \frac{u_j^{m_j}}{m_j!}\right) g_{k+2\sum_{j=2}^{n} jm_j}(u_1),$$

as required. □

The next lemma is Lemma 2.5 in [KW24a] which allows us to take higher order derivatives of determinants of functions.

Lemma 4.4.4. *Let $s \geq 0$, $k \geq 1$ be integers and $a_{i,j}(x)$ be s-th differentiable functions of x. Then*

$$\left(\frac{d}{dx}\right)^s \det_{k\times k}\left(a_{i,j}(x)\right) = \sum_{l_1+\cdots+l_k=s} \binom{s}{l_1,\ldots,l_k} \det_{k\times k}\left(a_{i,j}^{(l_i)}(x)\right),$$

where $a_{i,j}^{(l_i)}(x)$ means that we take the l_i-th derivative of $a_{i,j}(x)$.

We also have the following lemma that allows us to explicitly evaluate certain determinants whose entries are reciprocals of the Gamma function.

Lemma 4.4.5. *Let $k \geq 1$ and $m_j \geq 0$ be integers for $j = 1,\ldots,k$. Then, we have*

$$\det_{k\times k}\left(\frac{1}{\Gamma(2k+m_i-2i-j+2)}\right) = \prod_{j=1}^{k} \frac{1}{(2k+m_j-2j)!} \prod_{1\leq i<j\leq k} (m_j - m_i - 2j + 2i).$$

Proof. With our notation, equation (4.13) in [Nor04] can be written as

$$\det_{k\times k}\left(\frac{1}{\Gamma(z_i-j+1)}\right) = \frac{\Delta(z_1,\ldots,z_k)}{\prod_{j=1}^{k}\Gamma(z_j)}.$$

We take $z_i = 2k + m_i - 2i + 1$ for $i = 1,\ldots,k$. Then, we have that

$$\det_{k\times k}\left(\frac{1}{\Gamma(2k+m_i-2i-j+2)}\right)=\prod_{i=1}^{k}\Gamma(2k+m_i-2i+1)^{-1}$$
$$\times\prod_{1\le i<j\le k}(m_j-2j-m_i+2i). \qquad (4.4.6)$$

Since $2k+m_i-2i+1\ge 1$ for $1\le i\le k$, we have that $\Gamma(2k+m_i-2i+1)=(2k+m_i-2i)!$ which completes the proof.

□

Now, for the shifted moments

$$I(G(2N);z_1,\dots,z_k):=\int_{G(2N)}\Lambda_X(z_1)\cdots\Lambda_X(z_k)\,dX$$

of the characteristic polynomials, we will use the multiple contour integral expressions due to Conrey et al. [CFK+03]. In fact, we will use the following approximate versions of their formulae which follow easily from the results of [CFK+03] and the fact that $(1-e^{-x})^{-1}=x^{-1}+O(1)$ for small x.

Lemma 4.4.6 (Corollary 2.4 in [ABP+14]). *Let $\alpha_1,\dots,\alpha_k$ be complex numbers such that $|\alpha_j|\ll 1/N$ for $j=1,2,\dots,k$. Then*

$$I(Sp(2N);e^{-\alpha_1},\dots,e^{-\alpha_k})$$
$$=\frac{(-1)^{k(k-1)/2}}{(2\pi i)^k k!}\oint\cdots\oint_{|w_i|=1}\frac{\Delta(w)\Delta(w^2)\,e^{N\sum_{i=1}^k(w_i-\alpha_i)}}{\prod_{1\le i,j\le k}(w_i^2-\alpha_j^2)}\prod_{i=1}^{k}dw_i\left(1+O(N^{-1})\right). \qquad (4.4.7)$$

Lemma 4.4.7 (Corollary 2.5 in [ABP+14]). *Let $\alpha_1,\dots,\alpha_k$ be complex numbers such that $|\alpha_j|\ll 1/N$ for $j=1,2,\dots,k$. Then*

$$I(SO(2N);e^{-\alpha_1},\dots,e^{-\alpha_k})$$
$$=\frac{(-1)^{k(k-1)/2}2^k}{(2\pi i)^k k!}\oint\cdots\oint_{|w_i|=1}\frac{\Delta(w)\Delta(w^2)(\prod_{i=1}^k w_i)}{\prod_{1\le i,j\le k}(w_i^2-\alpha_j^2)}$$
$$\times e^{N\sum_{i=1}^k(w_i+\alpha_i)}\prod_{i=1}^{k}dw_i\left(1+O(N^{-1})\right). \qquad (4.4.8)$$

Lemma 4.4.8 (Corollary 2.6 in [ABP+14])**.** *Let $\alpha_1, \ldots, \alpha_k$ be complex numbers such that $|\alpha_j| \ll 1/N$ for $j = 1, 2, \ldots, k$. Then*

$$\begin{aligned} I(O^-(2N); e^{-\alpha_1}, \ldots, e^{-\alpha_k}) \\ = \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint \cdots \oint_{|w_i|=1} \frac{\Delta(w)\Delta(w^2)(\prod_{i=1}^k \alpha_i)}{\prod_{1 \le i,j \le k}(w_i^2 - \alpha_j^2)} \\ \times e^{N \sum_{i=1}^k (w_i + \alpha_i)} \prod_{i=1}^k dw_i \left(1 + O(N^{-1})\right). \end{aligned} \tag{4.4.9}$$

As mentioned earlier, we obtain a formula for the joint moments by taking the derivatives of the shifted moments. The next two lemmas give us two expressions for the derivatives of the above contour integral expressions for the shifted moments with respect to the shifts α_j.

Lemma 4.4.9. *Let $n \ge 0$ and $k \ge 1$ be integers. Then*

$$\frac{d^n}{d\alpha^n} \left. \frac{e^{-N\alpha}}{\prod_{i=1}^k (w_i^2 - \alpha^2)} \right|_{\alpha=0} = \left(\prod_{i=1}^k \frac{1}{w_i^2} \right) \sum_{m=0}^n \binom{n}{m} (-N)^{n-m} m! \sum_{\substack{l_1 + \cdots + l_k = m \\ l_j \text{ even}}} \prod_{i=1}^k \frac{1}{w_i^{l_i}}. \tag{4.4.10}$$

Proof. The derivative is computed using the product rule and follows from the proof of Lemma 2.7 in [ABP+14] where we have corrected a typo. Specifically, in our notation, it is written in the proof of [ABP+14, Lemma 2.7] that

$$\frac{d^n}{d\alpha^n} \frac{e^{-N\alpha}}{\prod_{i=1}^k (w_i^2 - \alpha^2)} = \left(\prod_{i=1}^k \frac{1}{w_i^2} \right) \sum_{m=0}^n \binom{n}{m} (-N)^{n-m} e^{-N\alpha} \sum_{\substack{l_1 + \cdots + l_k = m \\ l_j \text{ even}}} \prod_{i=1}^k \frac{l_i!}{w_i^{l_i}}, \tag{4.4.11}$$

where it should read

$$\frac{d^n}{d\alpha^n} \frac{e^{-N\alpha}}{\prod_{i=1}^k (w_i^2 - \alpha^2)} = \left(\prod_{i=1}^k \frac{1}{w_i^2} \right) \sum_{m=0}^n \binom{n}{m} (-N)^{n-m} m!\, e^{-N\alpha} \sum_{\substack{l_1 + \cdots + l_k = m \\ l_j \text{ even}}} \prod_{i=1}^k \frac{1}{w_i^{l_i}}. \tag{4.4.12}$$

□

Lemma 4.4.10. *Let $n \ge 0$ and $k \ge 1$ be integers. Then*

$$\frac{d^n}{d\alpha^n}\left.\frac{e^{-N\alpha}}{\prod_{i=1}^k(w_i^2-\alpha^2)}\right|_{\alpha=0} = \left(\prod_{i=1}^k \frac{1}{w_i^2}\right) \sum_{\substack{m_1+2m_2+\cdots+nm_n=n \\ m_3=m_5=\cdots=0}} \frac{n!}{m_1!\cdots m_n!}(-N)^{m_1} \times \prod_{j=1}^{[n/2]} \left(\frac{1}{j}\sum_{i=1}^k \frac{1}{w_i^{2j}}\right)^{m_{2j}}. \tag{4.4.13}$$

Proof. The proof is similar to that of Lemma 2.1 in [KW24a]. First, we have

$$\frac{d}{d\alpha}\frac{e^{-N\alpha}}{\prod_{i=1}^k(w_i^2-\alpha^2)} = \frac{e^{-N\alpha}}{\prod_{i=1}^k(w_i^2-\alpha^2)}f_1(\alpha), \tag{4.4.14}$$

where

$$f_1(\alpha) = -N + 2\alpha\sum_{i=1}^k \frac{1}{w_i^2-\alpha^2}. \tag{4.4.15}$$

We can then write

$$\frac{d^n}{d\alpha^n}\frac{e^{-N\alpha}}{\prod_{i=1}^k(w_i^2-\alpha^2)} = \frac{e^{-N\alpha}}{\prod_{i=1}^k(w_i^2-\alpha^2)}f_n(\alpha), \tag{4.4.16}$$

where $f_n(\alpha)$ is defined recursively by

$$f_{n+1}(\alpha) = f_n(\alpha)f_1(\alpha) + f_n'(\alpha). \tag{4.4.17}$$

Now, let $g(\alpha)$ be a function such that $g'(\alpha) = f_1(\alpha)$. Then, we have that

$$\frac{d^n}{d\alpha^n}e^{g(\alpha)} = e^{g(\alpha)}f_n(\alpha). \tag{4.4.18}$$

But, by Faà di Bruno's formula, we also have that

$$\begin{aligned}\frac{d^n}{d\alpha^n}e^{g(\alpha)} &= e^{g(\alpha)} \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1!\cdots m_n!}\prod_{j=1}^n\left(\frac{g^{(j)}(\alpha)}{j!}\right)^{m_j} \\ &= e^{g(\alpha)} \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1!\cdots m_n!}\prod_{j=1}^n\left(\frac{f_1^{(j-1)}(\alpha)}{j!}\right)^{m_j}.\end{aligned} \tag{4.4.19}$$

Comparing the above two expressions for $(d/d\alpha)^n e^{g(\alpha)}$, we see that

$$f_n(\alpha) = \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1!\cdots m_n!}\prod_{j=1}^n\left(\frac{f_1^{(j-1)}(\alpha)}{j!}\right)^{m_j}. \tag{4.4.20}$$

One can check that for $j \geq 1$, we have

$$f_1^{(j)}(0) = \begin{cases} 0 & \text{if} \quad j \quad \text{even}, \\ 2j! \sum_{i=1}^k w_i^{-(1+j)} & \text{if} \quad j \quad \text{odd}. \end{cases} \tag{4.4.21}$$

Hence, we have that

$$f_n(0) = \sum_{\substack{m_1+2m_2+\cdots+nm_n=n \\ m_3=m_5=\cdots=0}} \frac{n!}{m_1! \cdots m_n!} (-N)^{m_1} \prod_{j=1}^{[n/2]} \left(\frac{1}{j} \sum_{i=1}^k \frac{1}{w_i^{2j}} \right)^{m_{2j}}. \tag{4.4.22}$$

Evaluating (4.4.16) at $\alpha = 0$ using this expression for $f_n(0)$ yields the desired result. $\square$

In the following two propositions we will compute the main contour integrals that we need to evaluate.

Proposition 4.4.11. *Let $k \geq 1$ and $n \geq 1$ be integers. Also, let $(m_1, \ldots, m_n)$ be a tuple of non-negative integers. Then, we have*

$$\begin{aligned}
&\frac{1}{(2\pi i)^k} \oint \cdots \oint_{|w_i|=1} \Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^k w_i} \prod_{j=1}^n \left(\sum_{i=1}^k \frac{1}{w_i^{2j}} \right)^{m_j} \prod_{i=1}^k \frac{dw_i}{w_i^{2k}} \\
&= (-1)^{k(k-1)/2} k! N^{k(k+1)/2+2\sum_{j=1}^n jm_j} \sum_{\substack{\sum_{i=1}^k r_{s,i}=m_s \\ s=2,\ldots,n}} \left(\prod_{s=2}^n \binom{m_s}{r_{s,1}, \ldots, r_{s,k}} \right) \\
&\qquad \times \left(\frac{d}{du} \right)^{m_1} \det_{k\times k} \left(g_{2i-j+2\sum_{s=2}^n sr_{s,i}}(u) \right) \Bigg|_{u=0},
\end{aligned} \tag{4.4.23}$$

and, more explicitly,

$$\begin{aligned}
&\frac{1}{(2\pi i)^k} \oint \cdots \oint_{|w_i|=1} \Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^k w_i} \prod_{j=1}^n \left(\sum_{i=1}^k \frac{1}{w_i^{2j}} \right)^{m_j} \prod_{i=1}^k \frac{dw_i}{w_i^{2k}} \\
&= k! N^{k(k+1)/2+2\sum_{j=1}^n jm_j} \sum_{\substack{\sum_{i=1}^k r_{s,i}=m_s \\ s=1,\ldots,n}} \left(\prod_{s=1}^n \binom{m_s}{r_{s,1}, \ldots, r_{s,k}} \right)
\end{aligned}$$

$$\times \prod_{j=1}^{k} \frac{1}{(2k+2\sum_{s=1}^{n} sr_{s,j}+1-2j)!} \prod_{1\le i<j\le k} \left(2\sum_{s=1}^{n} sr_{s,j} - 2\sum_{s=1}^{n} sr_{s,i} - 2j + 2i\right). \tag{4.4.24}$$

Also, we have

$$\begin{aligned}
&\frac{1}{(2\pi i)^k} \oint \cdots \oint_{|w_i|=1} \Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^{k} w_i} \prod_{j=1}^{n} \left(\sum_{i=1}^{k} \frac{1}{w_i^{2j}}\right)^{m_j} \prod_{i=1}^{k} \frac{dw_i}{w_i^{2k-1}} \\
&= (-1)^{k(k-1)/2} k! N^{k(k-1)/2+2\sum_{j=1}^{n} jm_j} \sum_{\substack{\sum_{i=1}^{k} r_{s,i}=m_s \\ s=2,\ldots,n}} \left(\prod_{s=2}^{n} \binom{m_s}{r_{s,1},\ldots,r_{s,k}}\right) \\
&\qquad \times \left(\frac{d}{du}\right)^{m_1} \det_{k\times k} \left(g_{2i-j-1+2\sum_{s=2}^{n} sr_{s,i}}(u)\right)\Big|_{u=0},
\end{aligned} \tag{4.4.25}$$

and, more explicitly,

$$\begin{aligned}
&\frac{1}{(2\pi i)^k} \oint \cdots \oint_{|w_i|=1} \Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^{k} w_i} \prod_{j=1}^{n} \left(\sum_{i=1}^{k} \frac{1}{w_i^{2j}}\right)^{m_j} \prod_{i=1}^{k} \frac{dw_i}{w_i^{2k-1}} \\
&= k! N^{k(k-1)/2+2\sum_{j=1}^{n} jm_j} \sum_{\substack{\sum_{i=1}^{k} r_{s,i}=m_s \\ s=1,\ldots,n}} \left(\prod_{s=1}^{n} \binom{m_s}{r_{s,1},\ldots,r_{s,k}}\right) \\
&\qquad \times \prod_{j=1}^{k} \frac{1}{(2k+2\sum_{s=1}^{n} sr_{s,j}-2j)!} \prod_{1\le i<j\le k} \left(2\sum_{s=1}^{n} sr_{s,j} - 2\sum_{s=1}^{n} sr_{s,i} - 2j + 2i\right).
\end{aligned} \tag{4.4.26}$$

Proof. First, note that

$$\Delta(w^2) = \Delta\left(\frac{d}{dX}\right) \exp\left(\sum_{i=1}^{k} w_i^2 X_i\right)\Big|_{X_i=0}, \tag{4.4.27}$$

and

$$\Delta(w)\, e^{N\sum_{i=1}^{k} w_i} = \Delta\left(\frac{d}{dY}\right) \exp\left(\sum_{i=1}^{k} w_i Y_i\right)\Big|_{Y_i=N}. \tag{4.4.28}$$

We may also write

$$\prod_{j=1}^{n}\left(\sum_{i=1}^{k}\frac{1}{w_i^{2j}}\right)^{m_j}=\prod_{j=1}^{n}\left(\frac{d}{dt_j}\right)^{m_j}\exp\left(\sum_{j=1}^{n}t_j\sum_{i=1}^{k}\frac{1}{w_i^{2j}}\right)\Bigg|_{t_j=0}. \tag{4.4.29}$$

Then, we have that

$$\begin{aligned}
&\frac{1}{(2\pi i)^k}\oint\cdots\oint\Delta(w)\Delta(w^2)\,e^{N\sum_{i=1}^{k}w_i}\prod_{j=1}^{n}\left(\sum_{i=1}^{k}\frac{1}{w_i^{2j}}\right)^{m_j}\prod_{i=1}^{k}\frac{dw_i}{w_i^{2k}}\\
&=\Delta\left(\frac{d}{dX}\right)\Delta\left(\frac{d}{dY}\right)\prod_{j=1}^{n}\left(\frac{d}{dt_j}\right)^{m_j}\\
&\qquad\times\frac{1}{(2\pi i)^k}\oint\cdots\oint\exp\left(\sum_{i=1}^{k}\left(w_i^2X_i+w_iY_i+\sum_{j=1}^{n}\frac{t_j}{w_i^{2j}}\right)\right)\prod_{i=1}^{k}\frac{dw_i}{w_i^{2k}}\Bigg|_{\substack{X_i=0,\\Y_i=N,\\t_j=0}}\\
&=\prod_{j=1}^{n}\left(\frac{d}{dt_j}\right)^{m_j}\Delta\left(\frac{d}{dX}\right)\Delta\left(\frac{d}{dY}\right)\\
&\qquad\times\prod_{i=1}^{k}\left(\frac{1}{2\pi i}\oint_{|w|=1}\exp\left(w^2X_i+wY_i+\sum_{j=1}^{n}\frac{t_j}{w^{2j}}\right)\frac{dw}{w^{2k}}\right)\Bigg|_{\substack{X_i=0,\\Y_i=N,\\t_j=0}}\\
&=k!\prod_{j=1}^{n}\left(\frac{d}{dt_j}\right)^{m_j}\\
&\qquad\times\det_{k\times k}\left(\frac{d^{i+j-2}}{dX^{i-1}dY^{j-1}}\frac{1}{2\pi i}\oint_{|w|=1}\exp\left(w^2X+wY+\sum_{l=1}^{n}\frac{t_l}{w^{2l}}\right)\frac{dw}{w^{2k}}\right)\Bigg|_{\substack{X=0,\\Y=N,\\t_l=0}},
\end{aligned} \tag{4.4.30}$$

where the last line is by Lemma 4.4.2. Now,

$$\begin{aligned}
&\frac{d^{i+j-2}}{dX^{i-1}dY^{j-1}}\frac{1}{2\pi i}\oint\exp\left(w^2X+wY+\sum_{l=1}^{n}\frac{t_l}{w^{2l}}\right)\frac{dw}{w^{2k}}\Bigg|_{\substack{X=0,\\Y=N}}\\
&=\frac{1}{2\pi i}\oint\exp\left(wN+\sum_{l=1}^{n}\frac{t_l}{w^{2l}}\right)\frac{dw}{w^{2k-2i-j+3}}\\
&=\frac{N^{2k-2i-j+2}}{2\pi i}\oint\exp\left(w+\sum_{l=1}^{n}\frac{N^{2l}t_l}{w^{2l}}\right)\frac{dw}{w^{2k-2i-j+3}}.
\end{aligned} \tag{4.4.31}$$

Therefore, we have

$$
\begin{aligned}
&\frac{1}{(2\pi i)^k}\oint\cdots\oint \Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^k w_i}\prod_{j=1}^n\left(\sum_{i=1}^k\frac{1}{w_i^{2j}}\right)^{m_j}\prod_{i=1}^k\frac{dw_i}{w_i^{2k}}\\
=&k!\prod_{j=1}^n\left(\frac{d}{dt_j}\right)^{m_j}\det_{k\times k}\left(\frac{N^{2k-2i-j+2}}{2\pi i}\oint\exp\left(w+\sum_{l=1}^n\frac{N^{2l}t_l}{w^{2l}}\right)\frac{dw}{w^{2k-2i-j+3}}\right)\Bigg|_{t_j=0}\\
=&k!N^{k(k+1)/2}\prod_{j=1}^n\left(\frac{d}{dt_j}\right)^{m_j}\det_{k\times k}\left(\frac{1}{2\pi i}\oint\exp\left(w+\sum_{l=1}^n\frac{N^{2l}t_l}{w^{2l}}\right)\frac{dw}{w^{2k-2i-j+3}}\right)\Bigg|_{t_j=0}\\
=&k!N^{k(k+1)/2+\sum_{j=1}^n 2jm_j}\prod_{j=1}^n\left(\frac{d}{du_j}\right)^{m_j}\\
&\quad\times\det_{k\times k}\left(\frac{1}{2\pi i}\oint\exp\left(w+\sum_{l=1}^n\frac{u_l}{w^{2l}}\right)\frac{dw}{w^{2k-2i-j+3}}\right)\Bigg|_{u_j=0}\\
=&(-1)^{k(k-1)/2}k!N^{k(k+1)/2+\sum_{j=1}^n 2jm_j}\prod_{j=1}^n\left(\frac{d}{du_j}\right)^{m_j}\\
&\quad\times\det_{k\times k}\left(\frac{1}{2\pi i}\oint\exp\left(w+\sum_{l=1}^n\frac{u_l}{w^{2l}}\right)\frac{dw}{w^{2i-j+1}}\right)\Bigg|_{u_j=0},
\end{aligned}
\tag{4.4.32}
$$

where we have used the fact that $\det_{k\times k}(N^{-2i-j}a_{i,j}) = N^{-3k(k+1)/2}\det_{k\times k}(a_{i,j})$. Also, the fourth line follows from the change of variables $u_j = N^{2j}t_j$, and in the last line we have interchanged the rows of the matrix by mapping $i \mapsto k+1-i$. Next, by Lemma 4.4.3, the contour integral appearing in the determinant is

$$
\frac{1}{2\pi i}\oint\exp\left(w+\sum_{l=1}^n\frac{u_l}{w^{2l}}\right)\frac{dw}{w^{2i-j+1}} = \sum_{l_2,\ldots,l_n=0}^{\infty}\left(\prod_{s=2}^n\frac{u_s^{l_s}}{l_s!}\right)g_{2i-j+2\sum_{s=2}^n sl_s}(u_1).
\tag{4.4.33}
$$

We use Lemma 4.4.4 to carry out the differentiation of the determinant with respect to $u_2,\ldots,u_n$ which gives us

$$
\begin{aligned}
&\prod_{j=1}^n\left(\frac{d}{du_j}\right)^{m_j}\det_{k\times k}\left(\frac{1}{2\pi i}\oint\exp\left(w+\sum_{l=1}^n\frac{u_l}{w^{2l}}\right)\frac{dw}{w^{2i-j+1}}\right)\Bigg|_{u_j=0}\\
=&\sum_{\substack{\sum_{i=1}^k r_{s,i}=m_s\\ s=2,\ldots,n}}\left(\prod_{s=2}^n\binom{m_s}{r_{s,1},\ldots,r_{s,k}}\right)
\end{aligned}
$$

$$\times \left(\frac{d}{du_1}\right)^{m_1} \det_{k\times k}\left(\prod_{s=2}^{n}\left(\frac{d}{du_s}\right)^{r_{s,i}} \sum_{l_2,\ldots,l_n=0}^{\infty}\left(\prod_{s=2}^{n}\frac{u_s^{l_s}}{l_s!}\right) g_{2i-j+2\sum_{s=2}^{n} sl_s}(u_1)\right)\Bigg|_{u_j=0}$$

$$= \sum_{\substack{\sum_{i=1}^{k} r_{s,i}=m_s\\ s=2,\ldots,n}} \left(\prod_{s=2}^{n}\binom{m_s}{r_{s,1},\ldots,r_{s,k}}\right)\left(\frac{d}{du}\right)^{m_1} \det_{k\times k}\left(g_{2i-j+2\sum_{s=2}^{n} sr_{s,i}}(u)\right)\Big|_{u=0}. \tag{4.4.34}$$

Putting it all together and using (4.4.34) in (4.4.32) yields (4.4.23). We obtain the more explicit expression of (4.4.24) by performing the final derivative with respect to u and computing the resulting determinant. By Lemma 4.4.4 again,

$$\sum_{\substack{\sum_{i=1}^{k} r_{s,i}=m_s\\ s=2,\ldots,n}} \left(\prod_{s=2}^{n}\binom{m_s}{r_{s,1},\ldots,r_{s,k}}\right)\left(\frac{d}{du}\right)^{m_1} \det_{k\times k}\left(g_{2i-j+2\sum_{s=2}^{n} sr_{s,i}}(u)\right)\Big|_{u=0}$$

$$= \sum_{\substack{\sum_{i=1}^{k} r_{s,i}=m_s\\ s=1,\ldots,n}} \left(\prod_{s=1}^{n}\binom{m_s}{r_{s,1},\ldots,r_{s,k}}\right) \det_{k\times k}\left(\left(\frac{d}{du}\right)^{r_{1,i}} g_{2i-j+2\sum_{s=2}^{n} sr_{s,i}}(u)\right)\Bigg|_{u=0}. \tag{4.4.35}$$

By definition, for $j \geq 0$ we have that

$$\left(\frac{d}{du}\right)^{j} g_m(u) = \frac{1}{2\pi i}\oint_{|w|=1}\frac{e^{w+u/w^2}}{w^{m+2j+1}}dw = g_{m+2j}(u), \tag{4.4.36}$$

and

$$g_m(0) = \frac{1}{2\pi i}\oint_{|w|=1}\frac{e^{w}}{w^{m+1}}dw = \frac{1}{\Gamma(m+1)}. \tag{4.4.37}$$

Thus, the sum in (4.4.34) is equal to

$$\sum_{\substack{\sum_{i=1}^{k} r_{s,i}=m_s\\ s=1,\ldots,n}} \left(\prod_{s=1}^{n}\binom{m_s}{r_{s,1},\ldots,r_{s,k}}\right) \det_{k\times k}\left(\frac{1}{\Gamma(2i-j+2\sum_{s=1}^{n} sr_{s,i}+1)}\right). \tag{4.4.38}$$

We evaluate the determinant above by first making the change of variables $i \mapsto k+1-i$ and defining $\tilde{r}_{s,i} = r_{s,k+1-i}$ so that the sum becomes.

$$(-1)^{k(k-1)/2} \sum_{\substack{\sum_{i=1}^k r_{s,i}=m_s \\ s=1,\dots,n}} \left(\prod_{s=1}^{n} \binom{m_s}{r_{s,1},\dots,r_{s,k}} \right)$$
$$\times \det_{k\times k} \left(\frac{1}{\Gamma(2k-2i-j+2\sum_{s=1}^n s\tilde{r}_{s,i}+3)} \right). \tag{4.4.39}$$

Note that since $\sum_{i=1}^k r_{s,i} = \sum_{i=1}^k \tilde{r}_{s,i}$, we may drop the tildes and then apply Lemma 4.4.5 to the determinant. Using the resulting expression for the sum in (4.4.34) yields (4.4.24). The proofs of (4.4.25) and (4.4.26) are similar. $\square$

Proposition 4.4.12. *Let $k \geq 1$ and m_j be integers for $j = 1,\dots,k$. Then, we have*

$$\frac{1}{(2\pi i)^k} \oint \cdots \oint_{|w_i|=1} \frac{\Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^k w_i}}{\prod_{j=1}^k w_j^{2k+m_j}} \prod_{i=1}^k dw_i$$
$$= \sum_{\mu\in S_k} \det_{k\times k} \left(\frac{N^{2k+m_{\mu(i)}-2i-j+2}}{\Gamma(2k+m_{\mu(i)}-2i-j+3)} \right). \tag{4.4.40}$$

Proof. As in the proof of Proposition 4.4.11, we write

$$\Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^k w_i} = \Delta\left(\frac{d}{dX}\right) \Delta\left(\frac{d}{dY}\right) \exp\left(\sum_{i=1}^k w_i^2 X_i + w_i Y_i \right)\Bigg|_{\substack{X_i=0,\\ Y_i=N}}. \tag{4.4.41}$$

Then, we have that

$$\frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{\Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^k w_i}}{\prod_{j=1}^k w_j^{2k+m_j}} \prod_{i=1}^k dw_i$$
$$= \Delta\left(\frac{d}{dX}\right) \Delta\left(\frac{d}{dY}\right) \prod_{i=1}^k f_i(X_i,Y_i)\Bigg|_{\substack{X_i=0,\\ Y_i=N}}, \tag{4.4.42}$$

where

$$f_i(X_i,Y_i) = \frac{1}{2\pi i} \oint_{|w|=1} \frac{e^{(w^2X_i+wY_i)}}{w^{2k+m_i}} dw. \tag{4.4.43}$$

So, by Lemma 4.4.2, we have

$$\frac{1}{(2\pi i)^k}\oint\cdots\oint\frac{\Delta(w)\Delta(w^2)\,e^{N\sum_{i=1}^k w_i}}{\prod_{j=1}^k w_j^{2k+m_j}}\prod_{i=1}^k dw_i$$
$$=\sum_{\mu\in S_k}\det_{k\times k}\left(\frac{d^{i+j-2}}{dX^{i-1}dY^{j-1}}f_{\mu(i)}(X,Y)\right)\Big|_{\substack{X=0,\\ Y=N}}. \tag{4.4.44}$$

Now,

$$\frac{d^{i+j-2}}{dX^{i-1}dY^{j-1}}\,f_{\mu(i)}(X,Y)\Big|_{\substack{X=0,\\ Y=N}}=\frac{1}{2\pi i}\oint_{|w|=1}\frac{e^{Nw}}{w^{2k+m_{\mu(i)}-2i-j+3}}dw$$
$$=\frac{N^{2k+m_{\mu(i)}-2i-j+2}}{\Gamma(2k+m_{\mu(i)}-2i-j+3)}, \tag{4.4.45}$$

and the proposition follows. □

4.5 Proofs of the main results

In this section we will present the proofs of our main results. Following the strategy outlined in Section 4.3, we obtain a formula the joint moments by differentiating the corresponding shifted moments with respect to the shifts. Hence, we begin with the fact that

$$\int_{G(2N)}\left(\Lambda_X^{(n_1)}(1)\right)^{k_1}\left(\Lambda_X^{(n_2)}(1)\right)^{k_2}dX$$
$$=\prod_{j=1}^{k_1}\left(\frac{d}{d\alpha_j}\right)^{n_1}\prod_{j=k_1+1}^{k}\left(\frac{d}{d\alpha_j}\right)^{n_2} I(G(2N);e^{-\alpha_1},\ldots,e^{-\alpha_k})\big|_{\alpha_j=0}\left(1+O(N^{-1})\right), \tag{4.5.1}$$

where $k=k_1+k_2$. Also, the error terms in Lemmas 4.4.6-4.4.8 are uniform in α so we do indeed obtain an asymptotic formula after performing the differentiation.

4.5.1 The symplectic group $Sp(2N)$

Here we consider the symplectic case and prove Theorems 4.2.1 and 4.2.2.

Proof of Theorem 4.2.1. By (4.5.1) and Lemma 4.4.6, we have that

$$\int_{Sp(2N)} \left(\Lambda_X^{(n_1)}(1)\right)^{k_1} \left(\Lambda_X^{(n_2)}(1)\right)^{k_2} dX = \frac{(-1)^{k(k-1)/2}}{k!} J^{Sp}_{k_1,k_2}(n_1,n_2)\left(1+O(N^{-1})\right), \tag{4.5.2}$$

where

$$J^{Sp}_{k_1,k_2}(n_1,n_2) = \prod_{j=1}^{k_1}\left(\frac{d}{d\alpha_j}\right)^{n_1} \prod_{j=k_1+1}^{k}\left(\frac{d}{d\alpha_j}\right)^{n_2} \times \frac{1}{(2\pi i)^k}\oint\cdots\oint_{|w_i|=1} \frac{\Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^k (w_i-\alpha_i)}}{\prod_{1\le i,j\le k}(w_i^2-\alpha_j^2)} \prod_{i=1}^k dw_i \Bigg|_{\alpha_j=0}. \tag{4.5.3}$$

What remains is to evaluate the integral $J^{Sp}_{k_1,k_2}(n_1,n_2)$. We use Lemma 4.4.10 to carry out the differentiation and obtain

$$\begin{aligned} J^{Sp}_{k_1,k_2}(n_1,n_2) &= \frac{1}{(2\pi i)^k}\oint\cdots\oint \Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^k w_i} \\ &\times \left(\sum_{\substack{a_1+2a_2+\cdots+n_1a_{n_1}=n_1\\ a_3=a_5=\cdots=0}} \frac{n_1!}{a_1!\cdots a_{n_1}!}(-N)^{a_1} \prod_{j=1}^{[n_1/2]} \left(\frac{1}{j}\sum_{i=1}^k \frac{1}{w_i^{2j}}\right)^{a_{2j}} \right)^{k_1} \\ &\times \left(\sum_{\substack{b_1+2b_2+\cdots+n_2b_{n_2}=n_2\\ b_3=b_5=\cdots=0}} \frac{n_2!}{b_1!\cdots b_{n_2}!}(-N)^{b_1} \prod_{j=1}^{[n_2/2]} \left(\frac{1}{j}\sum_{i=1}^k \frac{1}{w_i^{2j}}\right)^{b_{2j}} \right)^{k_2} \\ &\times \prod_{i=1}^k \frac{dw_i}{w_i^{2k}}. \end{aligned} \tag{4.5.4}$$

Recall the definition of the tuples $\mathbf{a}_i$ and $\mathbf{b}_i$ defined in the statement of the theorem. Then, we can expand the brackets in the integrand of $J^{Sp}_{k_1,k_2}(n_1,n_2)$ as

$$\begin{aligned} &\left(\sum_{\substack{a_1+2a_2+\cdots+n_1a_{n_1}=n_1\\ a_3=a_5=\cdots=0}} \frac{n_1!}{a_1!\cdots a_{n_1}!}(-N)^{a_1} \prod_{j=1}^{[n_1/2]} \left(\frac{1}{j}\sum_{i=1}^k \frac{1}{w_i^{2j}}\right)^{a_{2j}} \right)^{k_1} \\ &= \sum_{u_1+\cdots+u_P=k_1} \binom{k_1}{u_1,\ldots,u_P} \frac{(n_1!)^{k_1}}{\prod_{i=1}^P (\mathbf{a}_i!)^{u_i}} (-N)^{\sum_{i=1}^P u_i a_{i,0}} \prod_{j=1}^{[n_1/2]} \left(\frac{1}{j}\sum_{l=1}^k \frac{1}{w_l^{2j}}\right)^{\sum_{i=1}^P u_i a_{i,j}}, \end{aligned} \tag{4.5.5}$$

with a similar expression for the bracket to the power of k_2 in the integrand. Using these expansions, our expression for $J^{Sp}_{k_1,k_2}(n_1,n_2)$ becomes

$$\begin{aligned} J^{Sp}_{k_1,k_2}(n_1,n_2) = & \sum_{u_1+\cdots+u_P=k_1} \binom{k_1}{u_1,\ldots,u_P} \frac{(n_1!)^{k_1}(-N)^{\sum_{i=1}^P u_i a_{i,0}}}{\prod_{i=1}^P (\mathbf{a}_i!)^{u_i} (\prod_{j=1}^{[n_1/2]} j^{\sum_{i=1}^P u_i a_{i,j}})} \\ & \times \sum_{v_1+\cdots+v_Q=k_2} \binom{k_2}{v_1,\ldots,v_Q} \frac{(n_2!)^{k_2}(-N)^{\sum_{i=1}^Q v_i b_{i,0}}}{\prod_{i=1}^Q (\mathbf{b}_i!)^{v_i} (\prod_{j=1}^{[n_2/2]} j^{\sum_{i=1}^Q v_i b_{i,j}})} \\ & \times \frac{1}{(2\pi i)^k} \oint \cdots \oint \Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^k w_i} \prod_{j=1}^{[n_2/2]} \left(\sum_{l=1}^k \frac{1}{w_l^{2j}} \right)^{W_j} \prod_{i=1}^k \frac{dw_i}{w_i^{2k}}, \end{aligned} \tag{4.5.6}$$

where $W_j := \sum_{i=1}^P u_i a_{i,j} + \sum_{i=1}^Q v_i b_{i,j}$ for $j = 1,\ldots,[n_1/2]$ and $W_j := \sum_{i=1}^Q v_i b_{i,j}$ for $j = [n_1/2]+1,\ldots,[n_2/2]$. We now apply Proposition 4.4.11 to the contour integral above with $m_j = W_j$. In particular, using (4.4.23) gives us that

$$\begin{aligned} & J^{Sp}_{k_1,k_2}(n_1,n_2) \\ & = (-1)^{k(k-1)/2} k! N^{k(k+1)/2} \sum_{u_1+\cdots+u_P=k_1} \binom{k_1}{u_1,\ldots,u_P} \frac{(n_1!)^{k_1}(-N)^{\sum_{i=1}^P u_i a_{i,0}}}{\prod_{i=1}^P (\mathbf{a}_i!)^{u_i} (\prod_{j=1}^{[n_1/2]} j^{\sum_{i=1}^P u_i a_{i,j}})} \\ & \times \sum_{v_1+\cdots+v_Q=k_2} \binom{k_2}{v_1,\ldots,v_Q} \frac{(n_2!)^{k_2}(-N)^{\sum_{i=1}^Q v_i b_{i,0}}}{\prod_{i=1}^Q (\mathbf{b}_i!)^{v_i} (\prod_{j=1}^{[n_2/2]} j^{\sum_{i=1}^Q v_i b_{i,j}})} \cdot N^{2\sum_{j=1}^{[n_2/2]} jW_j} \\ & \times \sum_{\substack{\sum_{i=1}^k r_{s,i}=W_s \\ s=2,\ldots,[n_2/2]}} \left(\prod_{s=2}^{[n_2/2]} \binom{W_s}{r_{s,1},\ldots,r_{s,k}} \right) \left(\frac{d}{dx}\right)^{W_1} \det_{k\times k}\left(g_{2i-j+2\sum_{s=2}^{[n_2/2]} s r_{s,i}}(x) \right)\Bigg|_{x=0}. \end{aligned} \tag{4.5.7}$$

Using the definition of W_j, we compute the power of N in the summand as

$$\begin{aligned} \sum_{i=1}^P u_i a_{i,0} + \sum_{i=1}^Q v_i b_{i,0} + 2\sum_{j=1}^{[n_2/2]} jW_j = & \sum_{i=1}^P u_i \left(a_{i,0} + 2\sum_{j=1}^{[n_1/2]} j a_{i,j} \right) \\ & + \sum_{i=1}^Q v_i \left(b_{i,0} + 2\sum_{j=1}^{[n_2/2]} j b_{i,j} \right) \end{aligned}$$

$$
\begin{aligned}
&= n_1 \sum_{i=1}^{P} u_i + n_2 \sum_{i=1}^{Q} v_i \\
&= k_1 n_1 + k_2 n_2. \qquad (4.5.8)
\end{aligned}
$$

Also, since $a_{i,0} \equiv n_1 \pmod 2$ and $b_{i,0} \equiv n_2 \pmod 2$ for all i, the factor of (-1) in the summand is

$$
(-1)^{\sum_{i=1}^{P} u_i a_{i,0} + \sum_{i=1}^{Q} v_i b_{i,0}} = (-1)^{n_1 \sum_{i=1}^{P} u_i + n_2 \sum_{i=1}^{Q} v_i} = (-1)^{k_1 n_1 + k_2 n_2}. \qquad (4.5.9)
$$

Combining these two observations with our expression for $J^{Sp}_{k_1,k_2}(n_1, n_2)$ and using (4.5.2) yields the statement of the theorem. $\square$

Proof of Theorem 4.2.2. We begin as in the proof of Theorem 4.2.1 with (4.5.2) and (4.5.3). We use Lemma 4.4.9 for the derivatives in this case which gives us that

$$
\begin{aligned}
J^{Sp}_{k_1,k_2}(n_1, n_2) = \frac{1}{(2\pi i)^k} \oint \cdots \oint_{|w_i|=1} &\Delta(w)\Delta(w^2)\, e^{N \sum_{i=1}^{k} w_i} \\
\times &\left(\sum_{m=0}^{n_1} \binom{n_1}{m} (-N)^{n_1 - m}\, m! \sum_{\substack{l_1 + \cdots + l_k = m \\ l_j \text{ even}}} \prod_{i=1}^{k} \frac{1}{w_i^{l_i}} \right)^{k_1} \\
\times &\left(\sum_{m=0}^{n_2} \binom{n_2}{m} (-N)^{n_2 - m}\, m! \sum_{\substack{l_1 + \cdots + l_k = m \\ l_j \text{ even}}} \prod_{i=1}^{k} \frac{1}{w_i^{l_i}} \right)^{k_2} \prod_{i=1}^{k} \frac{dw_i}{w_i^{2k}}.
\end{aligned}
\qquad (4.5.10)
$$

Rather than expand the brackets in the integrand, we write them as

$$
\begin{aligned}
&\left(\sum_{m=0}^{n_1} \binom{n_1}{m} (-N)^{n_1 - m}\, m! \sum_{\substack{l_1 + \cdots + l_k = m \\ l_j \text{ even}}} \prod_{i=1}^{k} \frac{1}{w_i^{l_i}} \right)^{k_1} \\
&= \left(\sum_{\substack{\sum_{j=1}^{k} l_j \le n_1 \\ l_j \text{ even}}} (-N)^{n_1 - \sum_{j=1}^{k} l_j} \binom{n_1}{\sum_{j=1}^{k} l_j} \left(\sum_{j=1}^{k} l_j \right)! \prod_{j=1}^{k} \frac{1}{w_j^{l_j}} \right)^{k_1}
\end{aligned}
$$

$$= \sum_{\substack{2\sum_{j=1}^{k} l_{i,j} \leq n_1 \\ i=1,\ldots,k_1}} \prod_{i=1}^{k_1} \left((-N)^{n_1 - 2\sum_{j=1}^{k} l_{i,j}} \binom{n_1}{2\sum_{j=1}^{k} l_{i,j}} \left(2\sum_{j=1}^{k} l_{i,j}\right)! \right) \prod_{j=1}^{k} \frac{1}{w_j^{2\sum_{i=1}^{k_1} l_{i,j}}}, \tag{4.5.11}$$

and use a similar expression for the second bracket to the k_2. We then have that

$$\begin{aligned}
&J_{k_1,k_2}^{Sp}(n_1, n_2) \\
&= \sum_{\substack{2\sum_{j=1}^{k} l_{i,j} \leq n_1 \\ i=1,\ldots,k_1}} \prod_{i=1}^{k_1} \left((-N)^{n_1 - 2\sum_{j=1}^{k} l_{i,j}} \binom{n_1}{2\sum_{j=1}^{k} l_{i,j}} \left(2\sum_{j=1}^{k} l_{i,j}\right)! \right) \\
&\quad \times \sum_{\substack{2\sum_{j=1}^{k} m_{i,j} \leq n_2 \\ i=1,\ldots,k_2}} \prod_{i=1}^{k_2} \left((-N)^{n_2 - 2\sum_{j=1}^{k} m_{i,j}} \binom{n_2}{2\sum_{j=1}^{k} m_{i,j}} \left(2\sum_{j=1}^{k} m_{i,j}\right)! \right) \\
&\quad \times \frac{1}{(2\pi i)^k} \oint \cdots \oint_{|w_i|=1} \frac{\Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^{k} w_i}}{\prod_{j=1}^{k} w_j^{2k+2\sum_{i=1}^{k_1} l_{i,j} + 2\sum_{i=1}^{k_2} m_{i,j}}} \prod_{i=1}^{k} dw_i \\
&= \sum_{\substack{2\sum_{j=1}^{k} l_{i,j} \leq n_1 \\ i=1,\ldots,k_1}} \sum_{\substack{2\sum_{j=1}^{k} m_{i,j} \leq n_2 \\ i=1,\ldots,k_2}} (n_1!)^{k_1} (n_2!)^{k_2} (-N)^{k_1 n_1 + k_2 n_2 - 2\sum_{j=1}^{k}(\sum_{i=1}^{k_1} l_{i,j} + \sum_{i=1}^{k_2} m_{i,j})} \\
&\quad \times \left(\prod_{i=1}^{k_1} \frac{1}{(n_1 - 2\sum_{j=1}^{k} l_{i,j})!} \right) \left(\prod_{i=1}^{k_2} \frac{1}{(n_2 - 2\sum_{j=1}^{k} m_{i,j})!} \right) \\
&\quad \times \frac{1}{(2\pi i)^k} \oint \cdots \oint_{|w_i|=1} \frac{\Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^{k} w_i}}{\prod_{j=1}^{k} w_j^{2k+2\sum_{i=1}^{k_1} l_{i,j} + 2\sum_{i=1}^{k_2} m_{i,j}}} \prod_{i=1}^{k} dw_i.
\end{aligned} \tag{4.5.12}$$

We now set $V_j = 2\sum_{i=1}^{k_1} l_{i,j} + 2\sum_{i=1}^{k_2} m_{i,j}$ for $j = 1, \ldots, k$. Then, by Proposition 4.4.12, the contour integral in the last line above is equal to

$$\begin{aligned}
&\frac{1}{(2\pi i)^k} \oint \cdots \oint_{|w_i|=1} \frac{\Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^{k} w_i}}{\prod_{j=1}^{k} w_j^{2k+V_j}} \prod_{i=1}^{k} dw_i \\
&\quad = \sum_{\mu \in S_k} \det_{k\times k} \left(\frac{N^{2k+V_{\mu(i)} - 2i - j + 2}}{\Gamma(2k + V_{\mu(i)} - 2i - j + 3)} \right) \\
&\quad = N^{k(k+1)/2 + \sum_{j=1}^{k} V_j} \sum_{\mu \in S_k} \det_{k\times k} \left(\frac{1}{\Gamma(2k + V_{\mu(i)} - 2i - j + 3)} \right).
\end{aligned} \tag{4.5.13}$$

Hence, our expression for $J_{k_1,k_2}^{Sp}(n_1, n_2)$ becomes

$$J^{Sp}_{k_1,k_2}(n_1,n_2) = (-1)^{k_1n_1+k_2n_2}(n_1!)^{k_1}(n_2!)^{k_2}N^{k(k+1)/2+k_1n_1+k_2n_2} \times \sum_{\mu\in S_k} \sum_{\substack{2\sum_{j=1}^k l_{i,j}\le n_1\\ i=1,\dots,k_1}} \sum_{\substack{2\sum_{j=1}^k m_{i,j}\le n_2\\ i=1,\dots,k_2}} \left(\prod_{i=1}^{k_1}\frac{1}{(n_1-2\sum_{j=1}^k l_{i,j})!}\right) \times \left(\prod_{i=1}^{k_2}\frac{1}{(n_2-2\sum_{j=1}^k m_{i,j})!}\right) \det_{k\times k}\left(\frac{1}{\Gamma(2k+V_{\mu(i)}-2i-j+3)}\right). \tag{4.5.14}$$

By an argument similar to that given at the end of the proof of [KW24a, Theorem 3.5], we have that the sums over $l_{i,j}$ and $m_{i,j}$ do not depend on the choice of permutation μ. Explicitly, given a permutation $\mu\in S_k$, we can make the change of variables $\tilde{l}_{i,j}=l_{i,\mu(j)}$ and $\tilde{m}_{i,j}=m_{i,\mu(j)}$. Then we have that $\sum_{j=1}^k \tilde{l}_{i,j}=\sum_{j=1}^k l_{i,j}$ and $\sum_{j=1}^k \tilde{m}_{i,j}=\sum_{j=1}^k m_{i,j}$. Also, we have

$$V_{\mu(j)} = 2\sum_{i=1}^{k_1} l_{i,\mu(j)} + 2\sum_{i=1}^{k_2} m_{i,\mu(j)} = 2\sum_{i=1}^{k}\tilde{l}_{i,j} + 2\sum_{i=1}^{k}\tilde{m}_{i,j}. \tag{4.5.15}$$

Thus, we may take μ to be the identity and replace the sum over $\mu\in S_k$ by $k!$. To finish, we apply Lemma 4.4.5 to the last determinant which gives us

$$\begin{aligned}
&J^{Sp}_{k_1,k_2}(n_1,n_2)\\
&= (-1)^{k_1n_1+k_2n_2}k!(n_1!)^{k_1}(n_2!)^{k_2}N^{k(k+1)/2+k_1n_1+k_2n_2}\\
&\times \sum_{\substack{2\sum_{j=1}^k l_{i,j}\le n_1\\ i=1,\dots,k_1}} \sum_{\substack{2\sum_{j=1}^k m_{i,j}\le n_2\\ i=1,\dots,k_2}} \left(\prod_{i=1}^{k_1}\frac{1}{(n_1-2\sum_{j=1}^k l_{i,j})!}\right)\left(\prod_{i=1}^{k_2}\frac{1}{(n_2-2\sum_{j=1}^k m_{i,j})!}\right)\\
&\times \prod_{j=1}^{k}\frac{1}{(2k+V_j-2j+1)!}\prod_{1\le i<j\le k}(V_j-V_i-2j+2i).
\end{aligned} \tag{4.5.16}$$

The theorem follows on using this final expression for $J^{Sp}_{k_1,k_2}(n_1,n_2)$ in (4.5.2).

□

4.5.2 The special orthogonal group $SO(2N)$ and $O^-(2N)$

In this section we will give the proofs of Theorems 4.2.3-4.2.5. The proofs are similar to those in the symplectic case covered earlier and so we will briefly discuss any differences.

4.5.2.1 Proof of Theorem 4.2.3

The proof follows the same lines as that of Theorem 4.2.1, the difference being that we now use Lemma 4.4.7 for the shifted moments. We again use Lemma 4.4.10 for the derivatives and now make use of (4.4.25) and (4.4.26) in Proposition 4.4.11 for the resulting contour integral.

4.5.2.2 Proof of Theorem 4.2.4

In this case we follow the proof of 4.2.2, again using Lemma 4.4.7 for the shifted moments. We then use Lemma 4.4.9 for the derivatives and apply Proposition 4.4.12 to the resulting contour integral. We conclude the proof similarly using Lemma 4.4.5.

4.5.2.3 Proof of Theorem 4.2.5

By using (4.5.1) and Lemma 4.4.8, we have that

$$\int_{O^-(2N)} \left(\Lambda_X^{(n_1)}(1)\right)^{k_1} \left(\Lambda_X^{(n_2)}(1)\right)^{k_2} dX = \frac{(-1)^{k(k-1)/2} 2^k}{k!} J^{O^-}_{k_1,k_2}(n_1,n_2)\left(1+O(N^{-1})\right), \tag{4.5.17}$$

where

$$\begin{aligned} J^{O^-}_{k_1,k_2}(n_1,n_2) = &\prod_{j=1}^{k_1}\left(\frac{d}{d\alpha_j}\right)^{n_1} \prod_{j=k_1+1}^{k}\left(\frac{d}{d\alpha_j}\right)^{n_2} \\ &\times \frac{1}{(2\pi i)^k}\oint\cdots\oint \frac{\Delta(w)\Delta(w^2)(\prod_{i=1}^k \alpha_i)\, e^{N\sum_{i=1}^k (w_i+\alpha_i)}}{\prod_{1\le i,j\le k}(w_i^2-\alpha_j^2)} \prod_{i=1}^k dw_i \Bigg|_{\alpha_j=0}. \end{aligned} \tag{4.5.18}$$

Now, for the derivatives, we use the fact that for $n \geq 1$,

$$\frac{d^n}{d\alpha^n}\frac{\alpha e^{N\alpha}}{\prod_{i=1}^k (w_i^2-\alpha^2)}\Bigg|_{\alpha=0} = n\frac{d^{n-1}}{d\alpha^{n-1}}\frac{e^{N\alpha}}{\prod_{i=1}^k (w_i^2-\alpha^2)}\Bigg|_{\alpha=0}. \tag{4.5.19}$$

Hence, we have that

$$J^{O^-}_{k_1,k_2}(n_1,n_2) = n_1^{k_1} n_2^{k_2} \prod_{j=1}^{k_1} \left(\frac{d}{d\alpha_j}\right)^{n_1-1} \prod_{j=k_1+1}^{k} \left(\frac{d}{d\alpha_j}\right)^{n_2-1}$$
$$\times \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{\Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^{k}(w_i+\alpha_i)}}{\prod_{1\le i,j\le k}(w_i^2-\alpha_j^2)} \prod_{i=1}^{k} dw_i \Bigg|_{\alpha_j=0}. \tag{4.5.20}$$

This integral expression is very similar to the expression for $J^{Sp}_{k_1,k_2}(n_1-1,n_2-1)$ given by (4.5.3). Indeed, by making the change of variables $\alpha_j \mapsto -\alpha_j$, we have that

$$J^{O^-}_{k_1,k_2}(n_1,n_2) = (-1)^{k_1(n_1-1)+k_2(n_2-1)} n_1^{k_1} n_2^{k_2} \prod_{j=1}^{k_1} \left(\frac{d}{d\alpha_j}\right)^{n_1-1} \prod_{j=k_1+1}^{k} \left(\frac{d}{d\alpha_j}\right)^{n_2-1}$$
$$\times \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{\Delta(w)\Delta(w^2)\, e^{N\sum_{i=1}^{k}(w_i-\alpha_i)}}{\prod_{1\le i,j\le k}(w_i^2-\alpha_j^2)} \prod_{i=1}^{k} dw_i \Bigg|_{\alpha_j=0}, \tag{4.5.21}$$

from which we see that

$$J^{O^-}_{k_1,k_2}(n_1,n_2) = (-1)^{k_1(n_1-1)+k_2(n_2-1)} n_1^{k_1} n_2^{k_2} J^{Sp}_{k_1,k_2}(n_1-1,n_2-1). \tag{4.5.22}$$

Thus, using (4.5.22) in (4.5.17) and recalling (4.5.2) gives us that

$$\int_{O^-(2N)} \left(\Lambda_X^{(n_1)}(1)\right)^{k_1} \left(\Lambda_X^{(n_2)}(1)\right)^{k_2} dX$$
$$= (-1)^{k_1(n_1-1)+k_2(n_2-1)} n_1^{k_1} n_2^{k_2} 2^k \int_{Sp(2N)} \left(\Lambda_X^{(n_1-1)}(1)\right)^{k_1} \left(\Lambda_X^{(n_2-1)}(1)\right)^{k_2} dX$$
$$\times \left(1+O(N^{-1})\right). \tag{4.5.23}$$

We therefore obtain the statement of the theorem by using our known asymptotic formula for the joint moments over $Sp(2N)$ given in Theorems 4.2.1 and 4.2.2.

4.5.3 Proof of Proposition 4.2.7

Let $n \geq 1$ be an integer. Then, by Theorem 4.2.2, we have that

$$
\begin{aligned}
b^{Sp}_{0,1}(0,n) &= \frac{(-1)^n n!}{2^{n+1}} \sum_{2l\le n} \frac{1}{(n-2l)!(2l+1)!} \\
&= \frac{(-1)^n n!}{2^{n+1}(n+1)!} \sum_{2l\le n} \binom{n+1}{2l+1} \\
&= \frac{(-1)^n}{2^{n+1}(n+1)} \left(\sum_{2l\le n} \binom{n}{2l} + \sum_{2l\le n-1} \binom{n}{2l+1} \right) \\
&= \frac{(-1)^n}{2^{n+1}(n+1)} \sum_{l=0}^{n} \binom{n}{l} \\
&= \frac{(-1)^n}{2(n+1)}, \qquad (4.5.24)
\end{aligned}
$$

where we have used standard properties of the binomial coefficient. In the same manner, by Theorem 4.2.4, we have

$$
\begin{aligned}
b^{SO}_{0,1}(0,n) &= 2^{1-n} n! \sum_{2l\le n} \frac{1}{(n-2l)!(2l)!} \\
&= 2^{1-n} \sum_{2l\le n} \binom{n}{2l} \\
&= 2^{1-n} \left(\sum_{2l\le n-1} \binom{n-1}{2l} + \sum_{2l\le n} \binom{n-1}{2l-1} \right) \\
&= 2^{1-n} \sum_{l=0}^{n-1} \binom{n-1}{l} \\
&= 1. \qquad (4.5.25)
\end{aligned}
$$

4.6 Numerical values for $b^{G}_{k_1,k_2}(n_1,n_2)$

Below we give some numerical values for $b^{Sp}_{k_1,k_2}(n_1,n_2)$ and $b^{SO}_{k_1,k_2}(n_1,n_2)$. Values of $b^{O^-}_{k_1,k_2}(n_1,n_2)$ follow from Theorem 4.2.5 so are omitted. Numerical values for $b^{Sp}_{0,k}(0,2)$ and $b^{SO}_{0,k}(0,2)$ for $k \le 10$ are given in [ABP+14, Section 4].

The following are $b^{Sp}_{0,k}(0,3)$ for $k = 1,\ldots,4$:

$$-\frac{1}{2^3}$$

$$\frac{23}{2^7 \cdot 3 \cdot 5 \cdot 7}$$

$$-\frac{1}{2^8 \cdot 5^2 \cdot 7 \cdot 11}$$

$$\frac{233}{2^{18} \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11}.$$

$b^{Sp}_{0,k}(0,4)$ for $k = 1, \ldots, 4$:

$$\frac{1}{2 \cdot 5}$$

$$\frac{251}{2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11}$$

$$\frac{89 \cdot 13103}{2^9 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17}$$

$$\frac{1627 \cdot 693731}{2^{10} \cdot 3^5 \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23}.$$

We also have $b^{Sp}_{1,1}(n_1, 1)$ for $n_1 = 0, 1$:

$$-\frac{1}{48}, \ \frac{1}{96}.$$

$b^{Sp}_{1,1}(n_1, 2)$ for $n_1 = 0, 1, 2$:

$$\frac{1}{80}, \ -\frac{1}{160}, \ \frac{19}{5040}.$$

$b^{Sp}_{1,1}(n_1, 3)$ for $n_1 = 0, 1, 2, 3$:

$$-\frac{1}{120}, \ \frac{1}{240}, \ -\frac{17}{6720}, \ \frac{23}{13440}.$$

$b^{Sp}_{1,2}(n_1, 1)$ for $n_1 = 0, 1$:

$$\frac{1}{11520}, \ -\frac{1}{23040}.$$

$b^{Sp}_{1,2}(n_1, 2)$ for $n_1 = 0, 1, 2$:

$$\frac{103}{3628800}, \ -\frac{103}{7257600}, \ \frac{487}{59875200}.$$

$b^{Sp}_{1,2}(n_1, 3)$ for $n_1 = 0, 1, 2, 3$:

$$\frac{1}{89600},\ -\frac{1}{179200},\ \frac{19}{5913600},\ -\frac{1}{492800}.$$

The following are $b_{0,k}^{SO}(0,3)$ for $k=1,2,3,4$:

$$1$$

$$\frac{3}{2^2\cdot 5}$$

$$\frac{1}{2^4\cdot 3\cdot 7}$$

$$\frac{1613}{2^9\cdot 3\cdot 5^2\cdot 7^2\cdot 11\cdot 13}.$$

$b_{0,k}^{SO}(0,4)$ for $k=1,2,3,4$:

$$1$$

$$\frac{71}{2\cdot 3^2\cdot 5\cdot 7}$$

$$\frac{23\cdot 2657}{2\cdot 3^3\cdot 5^3\cdot 7^2\cdot 11\cdot 13}$$

$$\frac{7159\cdot 316201}{2^6\cdot 3^5\cdot 5^4\cdot 7^3\cdot 11^2\cdot 13\cdot 17\cdot 19}.$$

We also have $b_{1,1}^{SO}(n_1,1)$ for $n_1=0,1$:

$$1,\ \frac{1}{2}.$$

$b_{1,1}^{SO}(n_1,2)$ for $n_1=0,1,2$:

$$\frac{2}{3},\ \frac{1}{3},\ \frac{7}{30}.$$

$b_{1,1}^{SO}(n_1,3)$ for $n_1=0,1,2,3$:

$$\frac{1}{2},\ \frac{1}{4},\ \frac{11}{60},\ \frac{3}{20}.$$

$b_{1,2}^{SO}(n_1,1)$ for $n_1=0,1$:

$$\frac{1}{12},\ \frac{1}{24}.$$

$b^{SO}_{1,2}(n_1, 2)$ for $n_1 = 0, 1, 2$:

$$\frac{19}{630},\ \frac{19}{1260},\ \frac{26}{2835}.$$

$b^{SO}_{1,2}(n_1, 3)$ for $n_1 = 0, 1, 2, 3$:

$$\frac{23}{1680},\ \frac{23}{3360},\ \frac{43}{10080},\ \frac{1}{336}.$$

Chapter 5

Improving the error in the Ratios Conjecture

5.1 The Ratios Conjecture

As discussed in the introductory chapter, the Ratios Conjecture is a powerful conjecture with applications to a large number of problems in analytic number theory. In this chapter we are concerned with the conjecture for quadratic Dirichlet L-functions in the function field setting. In this case, the Ratios Conjecture was formulated by Andrade and Keating in [AK14] and we state it below.

Conjecture 5.1.1 (The Ratios Conjecture). *For $|\mathrm{Re}(\alpha_j)| < 1/4$ and $1/g \ll \mathrm{Re}(\beta_j) < 1/4$ for $1 \leq j \leq k$, we have, as $g \to \infty$,*

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \frac{\prod_{j=1}^{k} L(\frac{1}{2}+\alpha_j, \chi_D)}{\prod_{j=1}^{k} L(\frac{1}{2}+\beta_j, \chi_D)}$$
$$= \sum_{\epsilon_j \in \{-1,1\}} \prod_{j=1}^{k} q^{g(\epsilon_j \alpha_j - \alpha_j)} Y(\epsilon_1\alpha_1, \dots, \epsilon_k\alpha_k; \gamma) A(\epsilon_1\alpha_1, \dots, \epsilon_k\alpha_k; \gamma) + O(q^{-\delta g}), \quad (5.1.1)$$

for some $\delta > 0$, where

$$Y(\alpha;\gamma) = \frac{\prod_{1 \leq i \leq j \leq k} \zeta_q(1+\alpha_i+\alpha_j) \prod_{1 \leq i < j \leq k} \zeta_q(1+\gamma_i+\beta_j)}{\prod_{1 \leq i,j \leq k} \zeta_q(1+\alpha_i+\beta_j)}, \quad (5.1.2)$$

and

$$A(\alpha;\gamma)=\prod_{P\in\mathcal{P}}\frac{\prod_{1\le i\le j\le k}\left(1-\frac{1}{|P|^{1+\alpha_i+\alpha_j}}\right)\prod_{1\le i<j\le k}\left(1-\frac{1}{|P|^{1+\gamma_i+\beta_j}}\right)}{\prod_{1\le i,j\le k}\left(1-\frac{1}{|P|^{1+\alpha_i+\beta_j}}\right)}$$
$$\times\frac{|P|}{|P|+1}\left(\frac{1}{2}\frac{\prod_{j=1}^{k}\left(1-\frac{1}{|P|^{1/2+\beta_j}}\right)}{\prod_{j=1}^{k}\left(1-\frac{1}{|P|^{1/2+\alpha_j}}\right)}+\frac{1}{2}\frac{\prod_{j=1}^{k}\left(1+\frac{1}{|P|^{1/2+\beta_j}}\right)}{\prod_{j=1}^{k}\left(1+\frac{1}{|P|^{1/2+\alpha_j}}\right)}+\frac{1}{|P|}\right). \tag{5.1.3}$$

Recently, Bui, Florea and Keating [BFK23] have proven the above conjecture for $k\le 3$ in certain ranges of the parameters with the following result.

Theorem 5.1.2 (Bui, Florea and Keating). *Let* $0<\mathrm{Re}(\beta_j)<1/2$ *for* $1\le j\le k$. *Denote* $\alpha=\max\{|\mathrm{Re}(\alpha_1)|,\ldots,|\mathrm{Re}(\alpha_k)|\}$ *and* $\beta=\min\{\mathrm{Re}(\beta_1),\ldots,\mathrm{Re}(\beta_k)\}$. *Then Conjecture 5.1.1 holds for* $1\le k\le 3$ *with the error term* E_k*, where*

$$E_1\ll_\varepsilon\begin{cases}q^{-g\beta(3+2\alpha)+\varepsilon g\beta} & \textit{if } 0\le\mathrm{Re}(\alpha)<1/2 \textit{ and } \beta\gg g^{-1/2+\varepsilon},\\ q^{-g\beta(3-4\alpha)+\varepsilon g\beta} & \textit{if } -1/2<\mathrm{Re}(\alpha)<0 \textit{ and } \beta\gg g^{-1/2+\varepsilon},\end{cases} \tag{5.1.4}$$

and

$$E_2\ll_\varepsilon q^{-g\beta\min\{\frac{1-4\alpha}{1+\beta},\frac{1-2\alpha}{2+\beta}\}+\varepsilon g\beta} \textit{ if } \alpha<1/4 \textit{ and } \beta\gg g^{-1/4+\varepsilon}, \tag{5.1.5}$$

$$E_3\ll_\varepsilon q^{-g\beta\min\{\frac{1/4-4\alpha}{\beta},\frac{1/2-4\alpha}{3+\beta}\}+\varepsilon g\beta} \textit{ if } \alpha<1/16 \textit{ and } \beta\gg g^{-1/6+\varepsilon}. \tag{5.1.6}$$

Our goal in this chapter is to improve the bound on the error term E_2 in Theorem 5.1.2 and increase the range of allowable α. We will do this with the theorem below.

Theorem 5.1.3. *With notation as in Theorem 5.1.2, suppose* $\alpha<1/2$ *and* $\beta\gg g^{-1/4+\varepsilon}$. *Then, the error term* E_2 *satisfies*

$$E_2\ll_\varepsilon\begin{cases}q^{-2g\beta(1+2\min\{\mathrm{Re}(\alpha_1),\mathrm{Re}(\alpha_2)\})+\varepsilon g\beta} & \text{if } \mathrm{Re}(\alpha_1),\mathrm{Re}(\alpha_2)\ge 0,\\ q^{-2g\beta(1-2\max\{|\mathrm{Re}(\alpha_1)|,|\mathrm{Re}(\alpha_2)|\})+\varepsilon g\beta} & \text{if } \mathrm{Re}(\alpha_1),\mathrm{Re}(\alpha_2)<0,\\ q^{-2g\beta(1+2\min\{0,\mathrm{Re}(\alpha_1+\alpha_2)\})+\varepsilon g\beta} & \text{if } \mathrm{Re}(\alpha_i)\ge 0 \text{ and } \mathrm{Re}(\alpha_j)<0 \text{ for } i\ne j.\end{cases} \tag{5.1.7}$$

The purpose of improving the bound on the error term can be seen when considering applications of the Ratios Conjecture such as the one-level density. For instance, it is shown in [CS07] that if one assumes the conjecture in the full ranges, one may obtain a formula for the one-level density of a family of L-functions with no restriction on the support of the Fourier transform of the test function. Since Theorem 5.1.2 does not allow one to take $\mathrm{Re}(\beta_j) \gg 1/g$, when the one-level density of the family of quadratic L-functions is computed in [BFK23, Section 6], the support of the Fourier transform of the test function needs to be restricted. In particular, the restriction on the support is determined by the size of the bound of the error term E_1. The conclusion is that if one cannot take the real parts of the shifts in the denominator to be as small as the conjecture suggests, then it is the size of the error term that dictates how strong a result one may obtain. We will see a concrete example of this idea in Chapter 6 on mollified moments where by using the improved bound for the error E_2 in Theorem 5.1.3, we are able to take a longer mollifier than if we used the bound in Theorem 5.1.2.

5.2 The strategy of Bui, Florea and Keating

In this section we will go over the method of proof of Theorem 5.1.2 in [BFK23]. The strategy used is to first expand the L-functions in the denominator using their Dirichlet series and write

$$\begin{aligned}\frac{1}{|\mathcal{H}_{2g+1}|}&\sum_{D\in\mathcal{H}_{2g+1}}\frac{\prod_{j=1}^{k}L(\frac{1}{2}+\alpha_j,\chi_D)}{\prod_{j=1}^{k}L(\frac{1}{2}+\beta_j,\chi_D)}\\ &=\sum_{f_1,\dots,f_k\in\mathcal{M}}\frac{\prod_{j=1}^{k}\mu(f_j)}{\prod_{j=1}^{k}|f_j|^{1/2+\beta_j}}\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\Big(\prod_{j=1}^{k}L(\tfrac{1}{2}+\alpha_j,\chi_D)\Big)\chi_D\Big(\prod_{j=1}^{k}f_j\Big).\end{aligned} \tag{5.2.1}$$

They then truncate this series and set

$$S_{k,\le X}=\sum_{f_1,\dots,f_k\in\mathcal{M}_{\le X}}\prod_{j=1}^{k}\frac{\mu(f_j)}{|f_j|^{1/2+\beta_j}}\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\Big(\prod_{j=1}^{k}L(\tfrac{1}{2}+\alpha_j,\chi_D)\Big)\chi_D\Big(\prod_{j=1}^{k}f_j\Big), \tag{5.2.2}$$

where X is a parameter to be chosen. Also, $S_{k,>X}$ is defined to be the sum of the terms in (5.2.1) where at least one of the f_j's has degree larger than X. Then, by proving upper bounds for negative moments of these L-functions, it is shown in [BFK23] that

$$S_{k,>X} \ll_\varepsilon q^{-(1-\varepsilon)X\beta}, \tag{5.2.3}$$

for $\beta \gg g^{-1/2k+\varepsilon}$. Also proven in [BFK23] are asymptotic formulae for the twisted, shifted moments appearing in (5.2.1) for $k \leq 3$. Inserting the main terms of these formulae into $S_{k,\leq X}$, extending the sums over f_j to be over all $f_j \in \mathcal{M}$ (which introduces a negligible error) and then performing an Euler product computation yields the main terms as predicted by the Ratios Conjecture.

For the contribution of the error terms in the twisted moment formulae to $S_{k,\leq X}$ in the case $k = 2$ or $k = 3$, Bui, Florea and Keating use their overall bounds for the error and bound the sums over f_j trivially. An optimum value for the parameter X is then chosen to yield the bound on the error terms E_2 and E_3 in Theorem 5.1.2. However, for $k = 1$, they keep the error terms in the first twisted moment explicit and make use of the cancellation coming from the Möbius function. This leads to a better error term and a wider range for the shift parameter α. We will carry out this procedure of more carefully bounding the error term in the case $k = 2$.

5.2.1 Preliminaries

Here we will cover the necessary notation and set out the method of proof of Theorem 5.1.3. For $\mathbf{C} = \{\gamma_1, \dots, \gamma_k\}$, we let

$$\mathbf{C}^- = \{-\gamma : \gamma \in \mathbf{C}\}, \qquad q^{-2g\mathbf{C}} = q^{-2g\sum_{j=1}^k \gamma_j},$$

$$\mu_{\mathbf{C}}(f) = \sum_{f=f_1\cdots f_k} \frac{\mu(f_1)\dots\mu(f_k)}{|f_1|^{\gamma_1}\dots|f_k|^{\gamma_k}} \qquad \text{and} \qquad \tau_{\mathbf{C}}(f) = \sum_{f=f_1\cdots f_k} \frac{1}{|f_1|^{\gamma_1}\dots|f_k|^{\gamma_k}}, \tag{5.2.4}$$

where $\mu(f)$ is the Möbius function on $\mathbb{F}_q[t]$. Note that $|\mu_{\mathbf{C}}(f)| \leq |f|^{-\min\{\mathrm{Re}(\gamma_j)\}}\tau_k(f)$ where $\tau_k(f)$ is the k-fold divisor function. We denote the degree of a polynomial $f \in \mathbb{F}_q[t]$ by $d(f)$.

Below we state the technical lemmas needed in the proof of Theorem 5.1.3. The first is Lemma 2.2 in [Flo17c].

Lemma 5.2.1. *For $f \in \mathcal{M}$ we have*

$$\sum_{D\in\mathcal{H}_{2g+1}} \chi_D(f) = \sum_{C|f^\infty} \sum_{r\in\mathcal{M}_{2g+1-2d(C)}} \chi_f(r) - q\sum_{C|f^\infty} \sum_{r\in\mathcal{M}_{2g-1-2d(C)}} \chi_f(r),$$

where the summations over C are over monic polynomials C whose prime factors *are among the prime factors of f.*

The generalized Gauss sum is defined as

$$G(V,\chi) := \sum_{u \pmod{f}} \chi(u) e\Big(\frac{uV}{f}\Big), \tag{5.2.5}$$

where the exponential was introduced by Hayes in [Hay66]. For $a \in \mathbb{F}_q\big((\frac{1}{x})\big)$, it is defined by

$$e(a) = e^{2\pi i \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_1)/p}, \tag{5.2.6}$$

where a_1 is the coefficient of $1/x$ in the Laurent expansion of a. The next two lemmas are Proposition 3.1 and Lemma 3.2 in [Flo17c].

Lemma 5.2.2. *Let $f \in \mathcal{M}_n$. If n is even then*

$$\sum_{r \in \mathcal{M}_m} \chi_f(r) = \frac{q^m}{|f|}\Bigg(G(0,\chi_f) + q \sum_{V \in \mathcal{M}_{\leq n-m-2}} G(V,\chi_f) - \sum_{V \in \mathcal{M}_{\leq n-m-1}} G(V,\chi_f)\Bigg),$$

otherwise

$$\sum_{r \in \mathcal{M}_m} \chi_f(r) = \frac{q^{m+1/2}}{|f|} \sum_{V \in \mathcal{M}_{n-m-1}} G(V,\chi_f).$$

Lemma 5.2.3. *1. If $(f,h)=1$, then $G(V,\chi_{fh}) = G(V,\chi_f)G(V,\chi_h)$.*

2. Write $V = V_1 P^\alpha$ where $P \nmid V_1$. Then

$$G(V,\chi_{P^j}) = \begin{cases} 0 & \text{if } j \leq \alpha \text{ and } j \text{ odd,} \\ \varphi(P^j) & \text{if } j \leq \alpha \text{ and } j \text{ even,} \\ -|P|^{j-1} & \text{if } j = \alpha+1 \text{ and } j \text{ even,} \\ \chi_P(V_1)|P|^{j-1/2} & \text{if } j = \alpha+1 \text{ and } j \text{ odd,} \\ 0 & \text{if } j \geq \alpha+2. \end{cases}$$

Note that $G(0,\chi_f)$ is non-zero if and only if f is a square, in which case $G(0,\chi_f) = \varphi(f)$ where $\varphi(f)$ is the Euler totient function on $\mathbb{F}_q[t]$.

We will frequently use the following function field version of Perron's formula, the proof of which follows from applying the residue theorem and the geometric series formula.

Lemma 5.2.4 (Perron's formula)**.** *Suppose that the power series*

$$H(u) = \sum_{f \in \mathcal{M}} a(f) u^{d(f)} \tag{5.2.7}$$

converges absolutely for $|u| \leq R < 1$*. Then*

$$\sum_{f \in \mathcal{M}_n} a(f) = \frac{1}{2\pi i} \oint_{|u|=R} \frac{H(u)}{u^{n+1}} \, du, \tag{5.2.8}$$

and

$$\sum_{f \in \mathcal{M}_{\leq n}} a(f) = \frac{1}{2\pi i} \oint_{|u|=R} \frac{H(u)}{(1-u)u^{n+1}} \, du \tag{5.2.9}$$

We also include the final twisted moments formulae and upper bounds for negative moments of the quadratic L-functions proven in [BFK23].

Theorem 5.2.5. *Let* $h = h_1 h_2^2$ *with* $d(h) \ll g$ *and* h_1 *a square-free monic polynomial and let* $\mathbf{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$*. For* $\alpha = \max\{|\mathrm{Re}(\alpha_1)|, \ldots, |\mathrm{Re}(\alpha_k)|\} < 1/2$ *we have*

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \Big(\prod_{j=1}^{k} L(\tfrac{1}{2} + \alpha_j, \chi_D) \Big) \chi_D(h) = \frac{1}{\sqrt{|h_1|}} \sum_{\mathbf{R} \subset \mathbf{A}} q^{-2g\mathbf{R}} \widetilde{\mathcal{S}}_{(\mathbf{A} \setminus \mathbf{R}) \cup \mathbf{R}^-}(h) + \widetilde{E}_k.$$

Here if $\mathbf{C} = \{\gamma_1, \ldots, \gamma_k\}$*, then*

$$\widetilde{\mathcal{S}}_{\mathbf{C}}(h) = \mathcal{A}_{\mathbf{C}}(1) \mathcal{B}_{\mathbf{C}}(h; 1) \prod_{1 \leq i \leq j \leq k} \zeta_q(1 + \gamma_i + \gamma_j),$$

where

$$\mathcal{A}_{\mathbf{C}}(u) = \prod_{P \in \mathcal{P}} \prod_{1 \leq i \leq j \leq k} \left(1 - \frac{u^{2d(P)}}{|P|^{1+\gamma_i+\gamma_j}} \right) \prod_{P \in \mathcal{P}} \left(1 + \left(1 + \frac{1}{|P|} \right)^{-1} \sum_{j=1}^{\infty} \frac{\tau_{\mathbf{C}}(P^{2j})}{|P|^j} u^{2jd(P)} \right) \tag{5.2.10}$$

and

$$\begin{aligned} \mathcal{B}_{\mathbf{C}}(h; u) = \prod_{P | h} & \left(1 + \frac{1}{|P|} + \sum_{j=1}^{\infty} \frac{\tau_{\mathbf{C}}(P^{2j})}{|P|^j} u^{2jd(P)} \right)^{-1} \\ & \times \prod_{P | h_1} \left(\sum_{j=0}^{\infty} \frac{\tau_{\mathbf{C}}(P^{2j+1})}{|P|^j} u^{2jd(P)} \right) \prod_{\substack{P \nmid h_1 \\ P | h_2}} \left(\sum_{j=0}^{\infty} \frac{\tau_{\mathbf{C}}(P^{2j})}{|P|^j} u^{2jd(P)} \right). \end{aligned} \tag{5.2.11}$$

Also,

$$\begin{aligned}\widetilde{E}_1 &= E_{\alpha_1}(h;g) + q^{-2g\alpha_1}E_{-\alpha_1}(h;g-1)\\ &\qquad + O_\varepsilon\big(|h|^{1/2}q^{-(3/2-\alpha)g+\varepsilon g}\big) + O_\varepsilon\big(|h_1|^{1/4}q^{-(3/2-2\alpha)g+\varepsilon g}\big),\end{aligned} \tag{5.2.12}$$

where $E_{\gamma_1}(h;N)$ *is given explicitly in [BFK23, (3.20)] and satisfies*

$$E_{\gamma_1}(h;N) \asymp |h_1|^{1/6}q^{-4g/3-g\mathrm{Re}(\gamma_1)+\varepsilon} + |h_1|^{1/6+\mathrm{Re}(\gamma_1)/3}q^{-4g/3-2g\mathrm{Re}(\gamma_1)/3+\varepsilon}$$

in particular, and

$$\begin{cases}\widetilde{E}_2 \ll_\varepsilon |h|^{1/2}q^{-(1-2\alpha)g+\varepsilon g} + q^{-(1-4\alpha)g+\varepsilon g},\\ \widetilde{E}_3 \ll_\varepsilon |h|^{1/2}q^{-(1/2-4\alpha)g+\varepsilon g} + q^{-(1-6\alpha)g+\varepsilon g} + |h_1|^{-3/4}q^{-(1/4-4\alpha)g+\varepsilon g}.\end{cases} \tag{5.2.13}$$

Theorem 5.2.6. *Let k be a positive integer and $m > 0$ such that $2km > 1$. Let $0 < \mathrm{Re}(\beta_j) < 1/2$ for $1 \le j \le k$. For $\beta = \min\{\mathrm{Re}(\beta_1), \dots, \mathrm{Re}(\beta_k)\} \gg g^{-\frac{1}{2km}+\varepsilon}$, we have*

$$\begin{aligned}\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\prod_{j=1}^{k}\frac{1}{|L(1/2+\beta_j+it_j,\chi_D)|^m}\\ \ll \Big(\frac{1}{\beta}\Big)^{k^2m^2/2}\prod_{j=1}^{k}\min\Big\{\frac{1}{\beta_j},\frac{1}{\overline{t_j}}\Big\}^{-m/2}(\log g)^{km(km+1)/2},\end{aligned}$$

where $\overline{t} = \min\{t \bmod 2\pi, 2\pi - (t \bmod 2\pi)\}$.

5.2.2 Breakdown of proof of Theorem 5.2.5

By the functional equation

$$L(s,\chi_D) = (q^{1-2s})^g L(1-s,\chi_D), \tag{5.2.14}$$

we may write

$$\prod_{j=1}^{k}L(\tfrac{1}{2}+\alpha_j,\chi_D) = q^{-2g\sum_{j=1}^{k}\mathfrak{a}_j\alpha_j}\prod_{j=1}^{k}L(\tfrac{1}{2}+\varepsilon_j\alpha_j,\chi_D), \tag{5.2.15}$$

where $\mathfrak{a}_j = 0$ and $\varepsilon_j = 1$ if $\mathrm{Re}(\alpha_j) \ge 0$, and $\mathfrak{a}_j = 1$ and $\varepsilon_j = -1$ if $\mathrm{Re}(\alpha_j) < 0$. Consequently, in what follows, one may assume that $\mathrm{Re}(\alpha_j) \ge 0$ for every $1 \le j \le k$. Then by the approximate functional equation [BFK23, Lemma 2.1], for $\mathbf{A} = \{\alpha_1, \dots, \alpha_k\}$, we have

$$\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\Big(\prod_{j=1}^{k}L(\tfrac{1}{2}+\alpha_j,\chi_D)\Big)\chi_D(h)=S_{\mathbf{A}}(h;kg)+q^{-2g\mathbf{A}}S_{\mathbf{A}^-}(h;kg-1). \tag{5.2.16}$$

Here, if $\mathbf{C}=\{\gamma_1,\dots,\gamma_k\}$ with $\mathrm{Re}(\gamma_j)\geq 0$ for every $1\leq j\leq k$ or $\mathrm{Re}(\gamma_j)\leq 0$ for every $1\leq j\leq k$, then

$$S_{\mathbf{C}}(h;N)=\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\sum_{d(f)\leq N}\frac{\tau_{\mathbf{C}}(f)\chi_D(fh)}{\sqrt{|f|}}, \tag{5.2.17}$$

for $N\in\{kg,kg-1\}$. Using Lemma 5.2.1, $S_{\mathbf{C}}(h;N)$ is first written as

$$S_{\mathbf{C}}(h;N)=S_{\mathbf{C};1}(h;N)-qS_{\mathbf{C};2}(h;N), \tag{5.2.18}$$

where

$$S_{\mathbf{C};1}(h;N)=\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{d(f)\leq N}\frac{\tau_{\mathbf{C}}(f)}{\sqrt{|f|}}\sum_{C|(fh)^\infty}\sum_{r\in\mathcal{M}_{2g+1-2d(C)}}\chi_{fh}(r) \tag{5.2.19}$$

and

$$S_{\mathbf{C};2}(h;N)=\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{d(f)\leq N}\frac{\tau_{\mathbf{C}}(f)}{\sqrt{|f|}}\sum_{C|(fh)^\infty}\sum_{r\in\mathcal{M}_{2g-1-2d(C)}}\chi_{fh}(r). \tag{5.2.20}$$

Next, we write

$$S_{\mathbf{C};1}(h;N)=S^{\mathrm{e}}_{\mathbf{C};1}(h;N)+S^{\mathrm{o}}_{\mathbf{C};1}(h;N) \tag{5.2.21}$$

according to whether the degree of the product fh is even or odd, respectively. By Lemmas 5.2.2 and 5.2.3, it then follows that

$$S^{\mathrm{e}}_{\mathbf{C};1}(h;N)=M_{\mathbf{C};1}(h;N)+S^{\mathrm{e}}_{\mathbf{C};1}(h;N;V\neq 0), \tag{5.2.22}$$

where

$$M_{\mathbf{C};1}(h;N)=\frac{q}{(q-1)|h|}\sum_{\substack{d(f)\leq N\\ fh=\square}}\frac{\tau_{\mathbf{C}}(f)\varphi(fh)}{|f|^{3/2}}\sum_{\substack{C|(fh)^\infty\\ d(C)\leq g}}\frac{1}{|C|^2}, \tag{5.2.23}$$

$$S^{\mathrm{e}}_{\mathbf{C};1}(h;N;V\neq 0)=\frac{q}{(q-1)|h|}\sum_{\substack{d(f)\leq N\\ d(fh)\ \text{even}}}\frac{\tau_{\mathbf{C}}(f)}{|f|^{3/2}}\sum_{\substack{C|(fh)^{\infty}\\ d(C)\leq g}}\frac{1}{|C|^2}$$
$$\times\Bigg(q\sum_{V\in\mathcal{M}_{\leq d(fh)-2g-3+2d(C)}}G(V,\chi_{fh})-\sum_{V\in\mathcal{M}_{\leq d(fh)-2g-2+2d(C)}}G(V,\chi_{fh})\Bigg),\tag{5.2.24}$$

and

$$S^{\mathrm{o}}_{\mathbf{C};1}(h;N)=\frac{q^{3/2}}{(q-1)|h|}\sum_{\substack{d(f)\leq N\\ d(fh)\ \text{odd}}}\frac{\tau_{\mathbf{C}}(f)}{|f|^{3/2}}\sum_{\substack{C|(fh)^{\infty}\\ d(C)\leq g}}\frac{1}{|C|^2}\sum_{V\in\mathcal{M}_{d(fh)-2g-2+2d(C)}}G(V,\chi_{fh}).\tag{5.2.25}$$

Also, we have

$$S^{\mathrm{e}}_{\mathbf{C};1}(h;N;V\neq 0)=S^{\mathrm{e}}_{\mathbf{C};1}(h;N;V=\square)+S^{\mathrm{e}}_{\mathbf{C};1}(h;N;V\neq\square),\tag{5.2.26}$$

corresponding to whether V is a square or not. We do the same for $S_{\mathbf{C};2}(h;N)$ and define $M_{\mathbf{C};2}(h;N)$, $S^{\mathrm{o}}_{\mathbf{C};2}(h;N)$ and $S^{\mathrm{e}}_{\mathbf{C};2}(h;N;V=\square)$, $S^{\mathrm{e}}_{\mathbf{C};2}(h;N;V\neq\square)$ similarly. Finally, set

$$M_{\mathbf{C}}(h;N)=M_{\mathbf{C};1}(h;N)-qM_{\mathbf{C};2}(h;N)\tag{5.2.27}$$

and

$$S^{\mathrm{e}}_{\mathbf{C}}(h;N;V=\square)=S^{\mathrm{e}}_{\mathbf{C};1}(h;N;V=\square)-qS^{\mathrm{e}}_{\mathbf{C};2}(h;N;V=\square).\tag{5.2.28}$$

For $S^{\mathrm{o}}_{\mathbf{C};1}(h;N)$ and $S^{\mathrm{o}}_{\mathbf{C};2}(h;N)$, the summations over V are over odd degree polynomials, so $V\neq\square$ necessarily. Let

$$S_{\mathbf{C}}(h;N;V\neq\square)=\big(S^{\mathrm{o}}_{\mathbf{C};1}(h;N)-qS^{\mathrm{o}}_{\mathbf{C};2}(h;N)\big)$$
$$+\big(S^{\mathrm{e}}_{\mathbf{C};1}(h;N;V\neq\square)-qS^{\mathrm{e}}_{\mathbf{C};2}(h;N;V\neq\square)\big)\tag{5.2.29}$$

be the total contribution from $V\neq\square$ terms.

The terms $M_{\mathbf{C}}(h;N)$, $S^{\mathrm{e}}_{\mathbf{C}}(h;N;V=\square)$ and $S_{\mathbf{C}}(h;N;V\neq\square)$ are treated in Sections 3.2, 3.3 and 3.4 in [BFK23], respectively. We note that the term $S^{\mathrm{e}}_{\mathbf{C}}(h;N;V=\square)$ contributes to the main terms of the twisted moments for $k\geq 2$.

5.3 Proof of Theorem 5.1.3

5.3.1 Beginning the proof

To prove Theorem 5.1.3, we adopt a slightly different approach to that used by Bui, Florea and Keating for the case $k=1$ and begin by rewriting (5.2.1) as

$$\begin{aligned}&\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\frac{\prod_{j=1}^{k}L(\frac{1}{2}+\alpha_j,\chi_D)}{\prod_{j=1}^{k}L(\frac{1}{2}+\beta_j,\chi_D)}\\&\quad=\sum_{h\in\mathcal{M}}\frac{\mu_{\mathbf{B}}(h)}{|h|^{1/2}}\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\Bigg(\prod_{j=1}^{k}L(\tfrac{1}{2}+\alpha_j,\chi_D)\Bigg)\chi_D(h),\end{aligned}\tag{5.3.1}$$

where $\mathbf{B}=\{\beta_1,\ldots,\beta_k\}$. We then truncate this series at $d(h)\leq X\ll g$ and define

$$\begin{aligned}R_{k,\leq X}&=\sum_{h\in\mathcal{M}_{\leq X}}\frac{\mu_{\mathbf{B}}(h)}{|h|^{1/2}}\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\Bigg(\prod_{j=1}^{k}L(\tfrac{1}{2}+\alpha_j,\chi_D)\Bigg)\chi_D(h)\\&=\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\Bigg(\prod_{j=1}^{k}L(\tfrac{1}{2}+\alpha_j,\chi_D)\Bigg)\sum_{h\in\mathcal{M}_{\leq X}}\frac{\mu_{\mathbf{B}}(h)}{|h|^{1/2}}\chi_D(h),\end{aligned}\tag{5.3.2}$$

and let $R_{k,>X}$ denote the terms in (5.3.1) with $d(h)>X$. Working with a single Dirichlet series rather than a multiple series simplifies the forthcoming argument.

We bound $R_{k,>X}$ similarly to $S_{k,>X}$. Explicitly, we consider the generating series of the sum over h, namely

$$H(u):=\sum_{h\in\mathcal{M}}\frac{\mu_{\mathbf{B}}(h)\chi_D(h)}{|h|^{1/2}}u^{d(h)}=\prod_{j=1}^{k}\mathcal{L}\Bigg(\frac{u}{q^{1/2+\beta_j}},\chi_D\Bigg)^{-1}.\tag{5.3.3}$$

Applying Perron's formula to $H(u)$ gives us

$$\sum_{h\in\mathcal{M}_{\leq X}}\frac{\mu_{\mathbf{B}}(h)\chi_D(h)}{|h|^{1/2}}=\frac{1}{2\pi i}\oint_{|u|=r}\frac{H(u)}{(1-u)u^{X+1}}\,du,\tag{5.3.4}$$

where $r<1$ is such that the series $H(u)$ converges absolutely for $|u|\leq r$. Note that the expression for $H(u)$ as a product of reciprocals of L-functions immediately gives us a meromorphic continuation of $H(u)$ to the complex plane, with poles arising from the zeros of the L-functions. In particular, these poles are on the circles $|u| = q^{\mathrm{Re}(\beta_j)}$ for $j = 1, \dots, k$ and so we have that $H(u)$ is analytic in the region $|u| < q^{\beta}$ where $\beta = \min\{\mathrm{Re}(\beta_1), \dots, \mathrm{Re}(\beta_k)\}$. We may therefore enlarge the contour to $r' = q^{(1-\varepsilon)\beta}$, encountering the simple pole at $u = 1$ only. The residue at $u = 1$ is $-H(1)$ so we then have

$$\sum_{h \in \mathcal{M}_{\leq X}} \frac{\mu_{\mathbf{B}}(h)\chi_D(h)}{|h|^{1/2}} = \frac{1}{2\pi i} \oint_{|u|=r'} \frac{H(u)}{(1-u)u^{X+1}}\, du + H(1). \tag{5.3.5}$$

Denote the set of monic polynomials of degree greater than X by $\mathcal{M}_{>X}$. Then

$$H(1) - \sum_{h \in \mathcal{M}_{\leq X}} \frac{\mu_{\mathbf{B}}(h)\chi_D(h)}{|h|^{1/2}} = \sum_{h \in \mathcal{M}_{>X}} \frac{\mu_{\mathbf{B}}(h)\chi_D(h)}{|h|^{1/2}} \tag{5.3.6}$$

and by definition,

$$R_{k,>X} = \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \left(\prod_{j=1}^{k} L(\tfrac{1}{2} + \alpha_j, \chi_D) \right) \sum_{h \in \mathcal{M}_{>X}} \frac{\mu_{\mathbf{B}}(h)\chi_D(h)}{|h|^{1/2}}. \tag{5.3.7}$$

So, using (5.3.5), we may write

$$\begin{aligned} R_{k,>X} &= \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \left(\prod_{j=1}^{k} L(\tfrac{1}{2} + \alpha_j, \chi_D) \right) \frac{1}{2\pi i} \oint_{|u|=r'} \frac{H(u)}{u^{X+1}(u-1)}\, du \\ &= \frac{1}{2\pi i} \oint_{|u|=r'} \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \frac{\prod_{j=1}^{k} L(\tfrac{1}{2} + \alpha_j, \chi_D)}{\prod_{j=1}^{k} \mathcal{L}\big(\frac{u}{q^{1/2+\beta_j}}, \chi_D\big)} \frac{du}{u^{X+1}(u-1)}. \end{aligned} \tag{5.3.8}$$

Now, using Hölder's inequality, we bound the sum over D in the integrand by

$$\begin{aligned} &\sum_{D \in \mathcal{H}_{2g+1}} \left| \frac{\prod_{j=1}^{k} L(\tfrac{1}{2} + \alpha_j, \chi_D)}{\prod_{j=1}^{k} \mathcal{L}\big(\frac{u}{q^{1/2+\beta_j}}, \chi_D\big)} \right| \\ &\leq \left(\sum_{D \in \mathcal{H}_{2g+1}} \prod_{j=1}^{k} |L(\tfrac{1}{2} + \alpha_j, \chi_D)|^{\frac{1+\varepsilon}{\varepsilon}} \right)^{\frac{\varepsilon}{1+\varepsilon}} \left(\sum_{D \in \mathcal{H}_{2g+1}} \prod_{j=1}^{k} \left| \mathcal{L}\Big(\frac{u}{q^{1/2+\beta_j}}, \chi_D \Big)^{-1} \right|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}}. \end{aligned} \tag{5.3.9}$$

By Corollary 2.8 in [Flo17a], we may bound the shifted moments above by

$$\left(\sum_{D\in\mathcal{H}_{2g+1}}\prod_{j=1}^{k}|L(\tfrac{1}{2}+\alpha_j,\chi_D)|^{\frac{1+\varepsilon}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} \ll q^{\frac{2g\varepsilon}{1+\varepsilon}}g^{\frac{k}{2}\left(\frac{k(1+\varepsilon)}{\varepsilon}+1\right)}. \tag{5.3.10}$$

For the negative moments, observe that for $r=q^{(1-\varepsilon)\beta}$, we have

$$|q^{-\frac{1}{2}-\beta_j}u| = q^{-\frac{1}{2}+(1-\varepsilon)\beta-\mathrm{Re}(\beta_j)} \leq q^{-\frac{1}{2}-\varepsilon\beta}. \tag{5.3.11}$$

So, for $\beta \gg g^{-\frac{1}{2k}+\varepsilon}$, we may use Theorem 5.2.6 to obtain

$$\left(\sum_{D\in\mathcal{H}_{2g+1}}\prod_{j=1}^{k}\left|\mathcal{L}\left(\frac{u}{q^{1/2+\beta_j}},\chi_D\right)^{-1}\right|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \ll q^{\frac{2g}{1+\varepsilon}}\left(\frac{\log g}{\beta}\right)^{\frac{k}{2}(k(1+\varepsilon)+1)}. \tag{5.3.12}$$

By bounding the integral in (5.3.8) using the above two bounds, we then have

$$R_{k,>X} \ll \frac{q^{-(1-\varepsilon)X\beta}}{\beta}g^{\frac{k}{2}\left(\frac{k(1+\varepsilon)}{\varepsilon}+1\right)}\left(\frac{\log g}{\beta}\right)^{\frac{k}{2}(k(1+\varepsilon)+1)}. \tag{5.3.13}$$

Next, just as in [BFK23], applying the twisted moment formulae of Theorem 5.2.5 to (5.3.2), extending the sum over all $h\in\mathcal{M}$ and then writing the main terms as an Euler product yields the formula predicted by the Ratios Conjecture. So, to prove Theorem 5.1.3, we are left to evaluate the contribution of the error term in Theorem 5.2.5 for $k=2$ to $R_{2,\leq X}$. We want to keep the errors terms explicit so we now turn to bounding the errors arising from the terms $M_{\mathbf{C}}(h;N)$, $S^{\mathrm{e}}_{\mathbf{C}}(h;N;V=\square)$ and $S_{\mathbf{C}}(h;N;V\neq\square)$ in the proof of Theorem 5.2.5 given in [BFK23]. We now specialise to the case $k=2$, set $\mathbf{C}=\{\gamma_1,\gamma_2\}$ and $\gamma=\min\{\mathrm{Re}(\gamma_1),\mathrm{Re}(\gamma_2)\}$, and recall that $\beta=\min\{\mathrm{Re}(\beta_1),\mathrm{Re}(\beta_2)\}$.

5.3.2 The error from $M_{\mathbf{C}}(h;N)$

In [BFK23, (3.8)], the main term $M_{\gamma_1,\gamma_2}(h;N)$ is expressed as a contour integral

$$\begin{aligned}M_{\gamma_1,\gamma_2}(h;N) = {}&\frac{1}{\sqrt{|h_1|}}\frac{1}{2\pi i}\oint_{|u|=r}\frac{\mathcal{A}_{\gamma_1,\gamma_2}(u)\mathcal{B}_{\gamma_1,\gamma_2}(h;u)\,du}{u^{N-d(h_1)+1}(1-u)\prod_{1\leq i\leq j\leq 2}(1-q^{-(\gamma_i+\gamma_j)}u^2)}\\ &+O_{\varepsilon}(q^{N/2-N\gamma-2g+\varepsilon g}),\end{aligned} \tag{5.3.14}$$

where $r < 1$ and $h = h_1 h_2^2$ with h_1, h_2 monic and h_1 square-free. Also, the products $\mathcal{A}_{\gamma_1,\gamma_2}(u)$ and $\mathcal{B}_{\gamma_1,\gamma_2}(h;u)$ are given by

$$\begin{aligned}\mathcal{A}_{\gamma_1,\gamma_2}(u) = \prod_P & \left(1 - \frac{u^{2d(P)}}{|P|^{1+\gamma_1+\gamma_2}}\right)\left(1 + \frac{1}{|P|}\right)^{-1} \\ & \times \left(1 + \frac{u^{2d(P)}}{|P|^{1+\gamma_1+\gamma_2}} + \frac{1}{|P|}\left(1 - \frac{u^{2d(P)}}{|P|^{1+2\gamma_1}}\right)\left(1 - \frac{u^{2d(P)}}{|P|^{1+2\gamma_2}}\right)\right), \end{aligned} \tag{5.3.15}$$

and

$$\begin{aligned}\mathcal{B}_{\gamma_1,\gamma_2}(h;u) = \prod_{P|h} & \left(1 + \frac{u^{d(P)}}{|P|^{1+\gamma_1+\gamma_2}} + \frac{1}{|P|}\left(1 - \frac{u^{2d(P)}}{|P|^{1+2\gamma_1}}\right)\left(1 - \frac{u^{2d(P)}}{|P|^{1+2\gamma_2}}\right)\right)^{-1} \\ & \times \prod_{P|h_1}\left(\frac{1}{|P|^{\gamma_1}} + \frac{1}{|P|^{\gamma_2}}\right) \prod_{\substack{P\nmid h_1 \\ P|h_2}}\left(1 + \frac{u^{2d(P)}}{|P|^{1+\gamma_1+\gamma_2}}\right). \end{aligned} \tag{5.3.16}$$

For the contribution of the big O term in (5.3.14) to $R_{2,\leq X}$, we trivially bound the sum over h which gives us an error of size

$$\begin{aligned}\left|\sum_{h\in\mathcal{M}_{\leq X}} \frac{\mu_{\beta_1,\beta_2}(h)}{|h|^{1/2}} O_\varepsilon(q^{N/2-N\gamma-2g+\varepsilon g})\right| & \ll q^{-g-2g\gamma+\varepsilon g} \sum_{h\in\mathcal{M}_{\leq X}} \frac{\tau(h)}{|h|^{1/2+\beta}} \\ & \ll q^{-g-2g\gamma+\varepsilon g} \sum_{n=0}^{X} q^{-n(1/2+\beta)} \sum_{h\in\mathcal{M}_n} \tau(h) \\ & \ll q^{-g-2g\gamma+\varepsilon g} \sum_{n=0}^{X} n q^{n(1/2-\beta)} \\ & \ll q^{X(1/2-\beta)-g-2g\gamma+\varepsilon g}, \end{aligned} \tag{5.3.17}$$

where we have used the fact that $\sum_{f\in\mathcal{M}_n} \tau(f) \ll nq^n$.

From [BFK23, Section 3.2], we have that the Euler product $\mathcal{A}_{\gamma_1,\gamma_2}(u)$ converges absolutely for $|u| < q^{1/2+\gamma}$ and so the contour in (5.3.14) is enlarged to $|u| = q^{1/2+\gamma-\varepsilon}$. The residues from the poles encountered all contribute to the main terms in the twisted moment so we may disregard them here. For the error term, rather than bound the new integral trivially, we keep it as is and introduce the sum over h in $R_{2,\leq X}$. Let $E_{\leq X,\gamma_1,\gamma_2}(N; V = 0)$ be this error term:

$$E_{\le X,\gamma_1,\gamma_2}(N;V=0) = \sum_{h\in\mathcal{M}_{\le X}} \frac{\mu_{\beta_1,\beta_2}(h)}{|h|^{1/2}}\frac{1}{|h_1|^{1/2}} \times \frac{1}{2\pi i}\oint_{|u|=q^{1/2+\gamma-\varepsilon}} \frac{\mathcal{A}_{\gamma_1,\gamma_2}(u)\mathcal{B}_{\gamma_1,\gamma_2}(h;u)\,du}{u^{N-d(h_1)+1}(1-u)\prod_{1\le i\le j\le 2}(1-q^{-(\gamma_i+\gamma_j)}u^2)}. \tag{5.3.18}$$

The generating series for the sum over h is

$$\mathcal{F}_{\gamma_1,\gamma_2}(u,y) = \sum_{h\in\mathcal{M}} \frac{\mu_{\beta_1,\beta_2}(h)}{|h|^{1/2}|h_1|^{1/2}}\mathcal{B}_{\gamma_1,\gamma_2}(h;u)u^{d(h_1)}y^{d(h)}. \tag{5.3.19}$$

The function $\mu_{\beta_1,\beta_2}(h)$ and the product $\mathcal{B}_{\gamma_1,\gamma_2}(h;u)$ are multiplicative functions of h and so the series $\mathcal{F}_{\gamma_1,\gamma_2}(u,y)$ may be expressed as an Euler product. Using the facts that for a prime polynomial P, we have,

$$\mu_{\beta_1,\beta_2}(P) = -\frac{1}{|P|^{\beta_1}} - \frac{1}{|P|^{\beta_2}}, \tag{5.3.20}$$

$$\mu_{\beta_1,\beta_2}(P^2) = \frac{1}{|P|^{\beta_1+\beta_2}}, \tag{5.3.21}$$

and $\mu_{\beta_1,\beta_2}(P^j)=0$ for $j>2$, the generating series has the Euler product

$$\begin{aligned}\mathcal{F}_{\gamma_1,\gamma_2}(u,y) &= \prod_P \Bigg(1 + \frac{\mu_{\beta_1,\beta_2}(P)}{|P|}\mathcal{B}_{\gamma_1,\gamma_2}(P;u)(uy)^{d(P)} \\ &\qquad + \frac{\mu_{\beta_1,\beta_2}(P^2)}{|P|}\mathcal{B}_{\gamma_1,\gamma_2}(P^2;u)y^{2d(P)}\Bigg) \\ &= \prod_P \Bigg(1 - \left(1 + \frac{u^{d(P)}}{|P|^{1+\gamma_1+\gamma_2}} + \frac{1}{|P|}\left(1-\frac{u^{2d(P)}}{|P|^{1+2\gamma_1}}\right)\left(1-\frac{u^{2d(P)}}{|P|^{1+2\gamma_2}}\right)\right)^{-1} \\ &\qquad \times \Bigg(\left(\frac{(uy)^{d(P)}}{|P|^{1+\beta_1}} + \frac{(uy)^{d(P)}}{|P|^{1+\beta_2}}\right)\left(\frac{1}{|P|^{\gamma_1}} + \frac{1}{|P|^{\gamma_2}}\right) \\ &\qquad - \frac{y^{2d(P)}}{|P|^{1+\beta_1+\beta_2}}\left(1 + \frac{u^{2d(P)}}{|P|^{1+\gamma_1+\gamma_2}}\right)\Bigg)\Bigg).\end{aligned} \tag{5.3.22}$$

For $|u| < q^{1/2+\gamma}$, we have

$$\left(1 + \frac{u^{d(P)}}{|P|^{1+\gamma_1+\gamma_2}} + \frac{1}{|P|}\left(1-\frac{u^{2d(P)}}{|P|^{1+2\gamma_1}}\right)\left(1-\frac{u^{2d(P)}}{|P|^{1+2\gamma_2}}\right)\right)^{-1} \ll 1, \tag{5.3.23}$$

as $d(P) \to \infty$. Hence, in this region, we have that

$$\mathcal{F}_{\gamma_1,\gamma_2}(u,y) = \prod_P \left(1 + O\left(\frac{(uy)^{d(P)}}{|P|^{1+\beta+\gamma}}\right) + O\left(\frac{y^{2d(P)}}{|P|^{1+2\beta}}\right) + O\left(\frac{(uy)^{2d(P)}}{|P|^{2+2\beta+2\gamma}}\right)\right). \tag{5.3.24}$$

Here and throughout the chapter, we will use the standard fact that an Euler product of the form

$$\prod_P (1 + a_P) \tag{5.3.25}$$

converges absolutely if and only if the corresponding sum $\sum_P a_P$ does. In particular, this series converges absolutely if $|a_P| \ll |P|^{-1-\varepsilon}$ for some $\varepsilon > 0$. So, in this case we see that $\mathcal{F}_{\gamma_1,\gamma_2}(u,y)$ converges absolutely for $|u| < q^{1/2+\gamma}$, $|y| < q^{\beta}$ and $|uy| < q^{\beta+\gamma}$. Hence, by Perron's formula, we have

$$\begin{aligned} &E_{\leq X,\gamma_1,\gamma_2}(N; V = 0) \\ &= \frac{1}{(2\pi i)^2} \oint_{|y|=r} \oint_{|u|=q^{1/2+\gamma-\varepsilon}} \frac{\mathcal{A}_{\gamma_1,\gamma_2}(u)\mathcal{F}_{\gamma_1,\gamma_2}(u,y)\, du\, dy}{u^{N+1} y^{X+1} (1-u)(1-y) \prod_{1\leq i\leq j\leq 2}(1 - q^{-(\gamma_i+\gamma_j)}u^2)}, \end{aligned} \tag{5.3.26}$$

where we may take $r = q^{-1/2+\beta}$. The products $\mathcal{A}_{\gamma_1,\gamma_2}(u)$ and $\mathcal{F}_{\gamma_1,\gamma_2}(u,y)$ are absolutely convergent and hence uniformly bounded on the region of integration. Hence, bounding the integral using the estimation lemma, we obtain the bound

$$\begin{aligned} E_{\leq X,\gamma_1,\gamma_2}(N; V = 0) &\ll_\varepsilon \oint_{|y|=q^{-1/2+\beta}} \oint_{|u|=q^{1/2+\gamma-\varepsilon}} \frac{du\, dy}{|u|^{N+1}|y|^{X+1}} \\ &\ll_\varepsilon q^{X(1/2-\beta)-N(1/2+\gamma)+\varepsilon g}. \end{aligned} \tag{5.3.27}$$

5.3.3 The error from $S^{\mathrm{e}}_{\mathbf{C}}(h; N; V = \square)$

In Section 3.3.2. of [BFK23], the term $S^{\mathrm{e}}_{\gamma_1,\gamma_2}(h; N; V = \square)$ is expressed as

$$\begin{aligned} S^{\mathrm{e}}_{\gamma_1,\gamma_2}(h; N; V = \square) = -\frac{q}{(q-1)|h_1|^{1/2}} \frac{1}{(2\pi i)^2} \oint_{|u|=q^{-1+\varepsilon}} \oint_{|w|=r_2} \\ \frac{\mathcal{C}_{\gamma_1,\gamma_2}(u,w)\mathcal{D}_{\gamma_1,\gamma_2}(h;u,w)}{(1-u)(1-q^{-\gamma_1}w)(1-q^{-\gamma_2}w)(1-uw^2)(1-q^{-2\gamma_1}uw^2)(1-q^{-2\gamma_2}uw^2)} \end{aligned}$$

$$\times \frac{dwdu}{u^{\left[\frac{d(h_1)}{2}\right]} w^{N-2\left[\frac{d(h_1)+1}{2}\right]+d(h_1)+1}} + O_\varepsilon\big(q^{-g-2g\gamma+\varepsilon g}\big), \tag{5.3.28}$$

where $r_2 < 1$,

$$\begin{aligned}\mathcal{C}_{\gamma_1,\gamma_2}(u,w) = \prod_{P\in\mathcal{P}} \bigg(1 - \frac{w^{d(P)}}{|P|^{1+\gamma_1}}\bigg)\bigg(1 - \frac{w^{d(P)}}{|P|^{1+\gamma_2}}\bigg)\bigg(1 + \frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_1}} \\ + \frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_2}} + \frac{(uw^2)^{d(P)}}{|P|^{1+\gamma_1+\gamma_2}} - \frac{(uw^2)^{d(P)}}{|P|^{2+2\gamma_1}} - \frac{(uw^2)^{d(P)}}{|P|^{2+2\gamma_2}} \\ - \frac{(uw^2)^{d(P)}}{|P|^{2+\gamma_1+\gamma_2}} - \frac{1}{|P|^2 u^{d(P)}} + \frac{w^{2d(P)}}{|P|^{3+2\gamma_1}} + \frac{w^{2d(P)}}{|P|^{3+2\gamma_2}} + \frac{(uw^2)^{2d(P)}}{|P|^{3+2\gamma_1+2\gamma_2}} \\ - \frac{(uw^4)^{d(P)}}{|P|^{4+2\gamma_1+2\gamma_2}}\bigg),\end{aligned} \tag{5.3.29}$$

and $\mathcal{D}_{\gamma_1,\gamma_2}(h;u,w)$ is given by

$$\begin{aligned}&\prod_{P|h}\bigg(1 + \frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_1}} + \frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_2}} + \frac{(uw^2)^{d(P)}}{|P|^{1+\gamma_1+\gamma_2}} - \frac{1}{|P|^2u^{d(P)}} - \frac{(uw^2)^{d(P)}}{|P|^{2+2\gamma_1}} \\ &- \frac{(uw^2)^{d(P)}}{|P|^{2+2\gamma_2}} - \frac{(uw^2)^{d(P)}}{|P|^{2+\gamma_1+\gamma_2}} + \frac{w^{2d(P)}}{|P|^{3+2\gamma_1}} + \frac{w^{2d(P)}}{|P|^{3+2\gamma_2}} + \frac{(uw^2)^{2d(P)}}{|P|^{3+2\gamma_1+2\gamma_2}} - \frac{(uw^4)^{d(P)}}{|P|^{4+2\gamma_1+2\gamma_2}}\bigg)^{-1} \\ &\times \prod_{P|h_1}\bigg(1 - u^{d(P)} + \frac{(uw)^{d(P)}}{|P|^{\gamma_1}} + \frac{(uw)^{d(P)}}{|P|^{\gamma_2}} - \frac{(uw)^{d(P)}}{|P|^{1+\gamma_1}} - \frac{(uw)^{d(P)}}{|P|^{1+\gamma_2}} \\ &\quad + \frac{(uw^2)^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_1+\gamma_2}}\bigg) \\ &\times \prod_{\substack{P\nmid h_1\\ P|h_2}}\bigg(1 - \frac{1}{|P|} + \frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_1}} + \frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_2}} + \frac{(uw^2)^{d(P)}}{|P|^{1+\gamma_1+\gamma_2}} - \frac{(uw^2)^{d(P)}}{|P|^{2+\gamma_1+\gamma_2}}\bigg).\end{aligned} \tag{5.3.30}$$

For the contribution of the big O term in (5.3.28) to $R_{2,\leq X}$, we bound the sum over h trivially which, similarly to (5.3.17), gives us an error of size $\ll_\varepsilon q^{X(1/2-\beta)-g-2g\gamma+\varepsilon g}$.

Next, it is given in [BFK23, Section 3.3.2] that the product $\mathcal{C}_{\gamma_1,\gamma_2}(u,w)$ converges absolutely for $|u| > 1/q$, $|w| < q^{1/2+\gamma}$, $|uw| < q^\gamma$ and $|uw^2| < q^{\mathrm{Re}(\gamma_1+\gamma_2)}$. The w-contour in (5.3.28) is enlarged to $|w| = q^{1/2+\gamma-\varepsilon}$, encountering the simple poles at $w = q^{\gamma_1}$ and $w = q^{\gamma_2}$ arising from the factors $(1-q^{-\gamma_1}w)^{-1}$ and $(1-q^{-\gamma_2}w)^{-1}$ in the integrand. In the proof of Theorem 5.2.5, the residues from these poles both contribute to the main term of the twisted moment so we are only concerned with the error term. For the new integral, we keep it explicit and introduce the sum over

h. Let $E_{\leq X,\gamma_1,\gamma_2}(N;V=\square)$ denote this error term:

$$\begin{aligned}E_{\leq X,\gamma_1,\gamma_2}(N;V=\square) &= -\frac{q}{(q-1)}\sum_{h\in\mathcal{M}_{\leq X}}\frac{\mu_{\beta_1,\beta_2}(h)}{|h|^{1/2}|h_1|^{1/2}}\frac{1}{(2\pi i)^2}\oint_{|u|=q^{-1+\varepsilon}}\oint_{|w|=q^{1/2+\gamma-\varepsilon}}\\ &\times\frac{\mathcal{C}_{\gamma_1,\gamma_2}(u,w)\mathcal{D}_{\gamma_1,\gamma_2}(h;u,w)}{(1-u)(1-q^{-\gamma_1}w)(1-q^{-\gamma_2}w)(1-uw^2)(1-q^{-2\gamma_1}uw^2)(1-q^{-2\gamma_2}uw^2)}\\ &\times\frac{dw\,du}{u^{\left[\frac{d(h_1)}{2}\right]}w^{N-2\left[\frac{d(h_1)+1}{2}\right]+d(h_1)+1}}.\end{aligned}\tag{5.3.31}$$

The generating series for the sum over h is

$$\mathcal{G}(u,w,y)=\sum_{h\in\mathcal{M}}\frac{\mu_{\beta_1,\beta_2}(h)}{|h|^{1/2}|h_1|^{1/2}}\mathcal{D}_{\gamma_1,\gamma_2}(h;u,w)\frac{y^{d(h)}}{u^{d(h_1)/2}},\tag{5.3.32}$$

which, as $\mathcal{D}_{\gamma_1,\gamma_2}(h;u,w)$ is a multiplicative function of h, has the Euler product

$$\begin{aligned}&\prod_P\left(1+\frac{\mu_{\beta_1,\beta_2}(P)y^{d(P)}}{|P|u^{d(P)/2}}\mathcal{D}_{\gamma_1,\gamma_2}(P;u,w)+\frac{\mu_{\beta_1,\beta_2}(P^2)y^{2d(P)}}{|P|}\mathcal{D}_{\gamma_1,\gamma_2}(P^2;u,w)\right)\\ =&\prod_P\Bigg(1-\bigg(1+\frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_1}}+\frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_2}}+\frac{(uw^2)^{d(P)}}{|P|^{1+\gamma_1+\gamma_2}}-\frac{1}{|P|^2u^{d(P)}}\\ &-\frac{(uw^2)^{d(P)}}{|P|^{2+2\gamma_1}}-\frac{(uw^2)^{d(P)}}{|P|^{2+2\gamma_2}}-\frac{(uw^2)^{d(P)}}{|P|^{2+\gamma_1+\gamma_2}}+\frac{w^{2d(P)}}{|P|^{3+2\gamma_1}}+\frac{w^{2d(P)}}{|P|^{3+2\gamma_2}}+\frac{(uw^2)^{2d(P)}}{|P|^{3+2\gamma_1+2\gamma_2}}\\ &-\frac{(uw^4)^{d(P)}}{|P|^{4+2\gamma_1+2\gamma_2}}\bigg)^{-1}\Bigg(\bigg(\frac{y^{d(P)}}{|P|^{1+\beta_1}u^{d(P)/2}}+\frac{y^{d(P)}}{|P|^{1+\beta_2}u^{d(P)/2}}\bigg)\bigg(1-u^{d(P)}+\frac{(uw)^{d(P)}}{|P|^{\gamma_1}}\\ &+\frac{(uw)^{d(P)}}{|P|^{\gamma_2}}-\frac{(uw)^{d(P)}}{|P|^{1+\gamma_1}}-\frac{(uw)^{d(P)}}{|P|^{1+\gamma_2}}+\frac{(uw^2)^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_1+\gamma_2}}\bigg)-\frac{y^{2d(P)}}{|P|^{1+\beta_1+\beta_2}}\bigg(1\\ &-\frac{1}{|P|}+\frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_1}}+\frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_2}}+\frac{(uw^2)^{d(P)}}{|P|^{1+\gamma_1+\gamma_2}}-\frac{(uw^2)^{d(P)}}{|P|^{2+\gamma_1+\gamma_2}}\bigg)\Bigg)\Bigg).\end{aligned}\tag{5.3.33}$$

For $|u|=q^{-1+\varepsilon}$ and $|w|=q^{1/2+\gamma-\varepsilon}$, we have that

$$\begin{aligned}&\bigg(1+\frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_1}}+\frac{w^{d(P)}\big(1-u^{d(P)}\big)}{|P|^{1+\gamma_2}}+\frac{(uw^2)^{d(P)}}{|P|^{1+\gamma_1+\gamma_2}}-\frac{1}{|P|^2u^{d(P)}}-\frac{(uw^2)^{d(P)}}{|P|^{2+2\gamma_1}}\\ &-\frac{(uw^2)^{d(P)}}{|P|^{2+2\gamma_2}}-\frac{(uw^2)^{d(P)}}{|P|^{2+\gamma_1+\gamma_2}}+\frac{w^{2d(P)}}{|P|^{3+2\gamma_1}}+\frac{w^{2d(P)}}{|P|^{3+2\gamma_2}}+\frac{(uw^2)^{2d(P)}}{|P|^{3+2\gamma_1+2\gamma_2}}-\frac{(uw^4)^{d(P)}}{|P|^{4+2\gamma_1+2\gamma_2}}\bigg)^{-1},\end{aligned}\tag{5.3.34}$$

is bounded by $O(1)$ since every term other than the 1 is tending to 0 as $d(P) \to \infty$. Thus, in this region, the local P-factor of $\mathcal{G}(u, w, y)$ is

$$
\begin{aligned}
1 &+ O\bigg(\frac{y^{d(P)}}{|P|^{1+\beta}u^{d(P)/2}}\bigg) + O\bigg(\frac{(u^{1/2}y)^{d(P)}}{|P|^{1+\beta}}\bigg) + O\bigg(\frac{(u^{1/2}wy)^{d(P)}}{|P|^{1+\beta+\gamma}}\bigg) \\
&+ O\bigg(\frac{(u^{1/2}w^2y)^{d(P)}}{|P|^{1+\beta+2\gamma}}\bigg) + O\bigg(\frac{(u^{3/2}w^2y)^{d(P)}}{|P|^{1+\beta+2\gamma}}\bigg) + O\bigg(\frac{y^{2d(P)}}{|P|^{1+2\beta}}\bigg) + O\bigg(\frac{(wy^2)^{d(P)}}{|P|^{2+2\beta+\gamma}}\bigg) \\
&+ O\bigg(\frac{(uwy^2)^{d(P)}}{|P|^{2+2\beta+\gamma}}\bigg) + O\bigg(\frac{(uw^2y^2)^{d(P)}}{|P|^{2+2\beta+2\gamma}}\bigg). \qquad (5.3.35)
\end{aligned}
$$

Similarly to the product $\mathcal{F}_{\gamma_1,\gamma_2}(u, y)$ considered earlier, we have that $\mathcal{G}(u, w, y)$ will converge absolutely if all of the big O terms above are $\ll |P|^{-1-\varepsilon}$. Hence, we see that $\mathcal{G}(u, w, y)$ converges absolutely for $|u| = q^{-1+\varepsilon}$, $|w| = q^{1/2+\gamma-\varepsilon}$, $|y| < q^{\beta}|u^{1/2}|$, $|u^{1/2}y| < q^{\beta}$, $|u^{1/2}wy| < q^{\beta+\gamma}$, $|u^{1/2}w^2y| < q^{\beta+2\gamma}$, $|u^{3/2}w^2y| < q^{\beta+2\gamma}$, $|y| < q^{\beta}$, $|wy^2| < q^{1+2\beta+\gamma}$, $|uwy^2| < q^{1+2\beta+\gamma}$ and $|uw^2y^2| < q^{1+2\beta+2\gamma}$. With $|u| = q^{-1+\varepsilon}$ and $|w| = q^{1/2+\gamma-\varepsilon}$, we may take $|y| = q^{-1/2+\beta-\varepsilon}$ and these conditions are all satisfied.

Using Perron's formula for the sum over h just as before, we therefore write $E_{\leq X,\gamma_1,\gamma_2}(N; V = \square)$ as a triple integral where the integration in y is over the circle $|y| = q^{-1/2+\beta-\varepsilon}$. We bound this integral using the estimation lemma, using the fact that the products $\mathcal{C}_{\gamma_1,\gamma_2}(u, w)$ and $\mathcal{G}(u, w, y)$ are uniformly bounded on the region of integration. This gives us the bound

$$
\begin{aligned}
E_{\leq X,\gamma_1,\gamma_2}(N; V = \square) &\ll_\varepsilon \oint_{|u|=q^{-1+\varepsilon}} \oint_{|w|=q^{1/2+\gamma-\varepsilon}} \oint_{|y|=q^{-1/2+\beta-\varepsilon}} \frac{du\, dw\, dy}{|w|^{N+1}|y|^{X+1}} \\
&\ll_\varepsilon q^{X(1/2-\beta)-N(1/2+\gamma)+\varepsilon g}. \qquad (5.3.36)
\end{aligned}
$$

Remark 5.3.1. *By factoring out the appropriate zeta factors to give an analytic continuation of the generating series $\mathcal{F}_{\gamma_1.\gamma_2}(u, y)$ and $\mathcal{G}(u, w, y)$, one could possibly improve on the error bounds in* (5.3.27) *and* (5.3.36)*. However, we still obtain other errors of size $O(q^{X(1/2-\beta)-g-2g\gamma+\varepsilon g})$ so this does not lead to an improvement of the result of Theorem 5.1.3.*

5.3.4 The error from $S_{\mathbf{C}}(h; N; V \neq \square)$

We let $E_{\leq X}(N; V \neq \square)$ denote the error term in $R_{2,\leq X}$ corresponding to the term $S_{\mathbf{C}}(h; N; V \neq \square)$ after introducing the sum over h. We denote by $E_{\leq X,1}(N; V \neq \square)$ the error term corresponding to $S^{\mathrm{o}}_{\mathrm{C};1}(N; V \neq \square)$. We will bound $E_{\leq X,1}(N; V \neq \square)$ below with the bound for $E_{\leq X}(N; V \neq \square)$ following similarly. Using the integral expression for $S^{\mathrm{o}}_{\mathrm{C};1}(N; V \neq \square)$ given in [BFK23, (3.37)] and introducing the sum over h, we have that $E_{\leq X,1}(N; V \neq \square)$ may be written as

$$E_{\leq X,1}(N;V\neq\square)=\frac{q^{3/2}}{(q-1)}\sum_{h\in\mathcal{M}_{\leq X}}\frac{\mu_{\beta_1,\beta_2}(h)}{|h|^{3/2}}\frac{1}{(2\pi i)^2}\oint_{|u|=q^{-\varepsilon}}\oint_{|w|=q^{1/2+\gamma-\varepsilon}}\sum_{\substack{n\leq N\\ n+d(h)\text{ odd}}}\sum_{j=0}^{g}$$

$$\times\sum_{\substack{r\leq n+d(h)-2g+2j-2\\ r\text{ odd}}}q^{-2j}\sum_{V_2\in\mathcal{M}_{(n+d(h)-r)/2-g+j-1}}\sum_{V_1\in\mathcal{H}_r}\mathcal{H}(V_1;u,w)\mathcal{K}(V,h;u,w)\frac{dw\,du}{w^{n+1}u^{j+1}}. \tag{5.3.37}$$

Here, for $V = V_1V_2^2$ with V_1 square-free, we have that

$$\mathcal{H}(V_1;u,w)=\prod_{P\nmid V_1}\left(1+\sum_{\gamma\in\mathbf{C}}\frac{\chi_{V_1}(P)w^{d(P)}}{|P|^{1+\gamma}}\big(1-u^{d(P)}\big)^{-1}\right), \tag{5.3.38}$$

and

$$\begin{aligned}\mathcal{K}(V,h;u,w)=&\prod_{P|h}\left(\sum_{j=0}^{\infty}\frac{\tau_{\mathbf{C}}(P^j)G(V,\chi_{P^{j+\mathrm{ord}_P(h)}})w^{jd(P)}}{|P|^{3j/2}}\right)\big(1-u^{d(P)}\big)^{-1}\\&\times\prod_{\substack{P\nmid h\\ P|V}}\left(1+\sum_{j=1}^{\infty}\frac{\tau_{\mathbf{C}}(P^j)G(V,\chi_{P^j})w^{jd(P)}}{|P|^{3j/2}}\big(1-u^{d(P)}\big)^{-1}\right)\\&\times\prod_{\substack{P\nmid V_1\\ P|hV_2}}\left(1+\sum_{\gamma\in\mathbf{C}}\frac{\chi_{V_1}(P)w^{d(P)}}{|P|^{1+\gamma}}\big(1-u^{d(P)}\big)^{-1}\right)^{-1}.\end{aligned} \tag{5.3.39}$$

It is given in Section 3.4 of [BFK23] that $\mathcal{H}(V_1; u, w)$ is convergent for $|u| < 1$ and $|w| < q^{1/2+\gamma-\varepsilon}$.

We consider the generating series of the sum over h in (5.3.37) as follows. For the product $\mathcal{K}(V, h; u, w)$, we let $A_P(V; u, w)$ be the local Euler factor over primes $P|h$, we let $B_P(V; u, w)$ denote the second Euler factor over primes $P \nmid h$ and $P|V$, and let $C_P(V; u, w)^{-1}$ denote the final Euler factor over primes $P \nmid V_1$ and $P|hV_2$. By Lemma 5.2.3, if $P \nmid V$ and $\mathrm{ord}_P(h) = 1$, then $G(V, \chi_{P^{j+\mathrm{ord}_P(h)}}) = 0$ for $j \geq 1$. Thus, in this case,

$$A_P(V;u,w)=\chi_P(V)|P|^{1/2}\big(1-u^{d(P)}\big)^{-1}=\chi_{V_1}(P)|P|^{1/2}\big(1-u^{d(P)}\big)^{-1}, \tag{5.3.40}$$

where we have used the fact that $q \equiv 1 \pmod 4$ so $\chi_P(V_1) = \chi_{V_1}(P)$. Similarly, if $P \nmid V$ and $\mathrm{ord}_P(h) = 2$, then $A_P(V; u, w) = 0$. Consequently, we have

$$
\begin{aligned}
&\sum_{h \in \mathcal{M}} \frac{\mu_{\beta_1,\beta_2}(h)}{|h|^{3/2}} \mathcal{K}(V, h; u, w) y^{d(h)} \\
&= \prod_{P|V_1} \left(B_P(V; u, w) + \frac{\mu_{\beta_1,\beta_2}(P)}{|P|^{3/2}} A_P(V; u, w) y^{d(P)} + \frac{\mu_{\beta_1,\beta_2}(P^2)}{|P|^3} A_P(V; u, w) y^{2d(P)} \right) \\
&\quad \times \prod_{\substack{P \nmid V_1 \\ P | V_2}} C_P(V; u, w)^{-1} \Bigg(B_P(V; u, w) + \frac{\mu_{\beta_1,\beta_2}(P)}{|P|^{3/2}} A_P(V; u, w) y^{d(P)} \\
&\qquad\qquad + \frac{\mu_{\beta_1,\beta_2}(P^2)}{|P|^3} A_P(V; u, w) y^{2d(P)} \Bigg) \\
&\quad \times \prod_{P \nmid V} \left(1 + \frac{\mu_{\beta_1,\beta_2}(P)}{|P|^{3/2}} A_P(V; u, w) C_P(V; u, w)^{-1} y^{d(P)} \right) \\
&= \prod_{P|V} B_P(V; u, w) \prod_{\substack{P \nmid V_1 \\ P | V_2}} C_P(V; u, w)^{-1} \\
&\quad \times \prod_{P|V} \left(1 - \left(\frac{y^{d(P)}}{|P|^{3/2+\beta_1}} + \frac{y^{d(P)}}{|P|^{3/2+\beta_2}} - \frac{y^{2d(P)}}{|P|^{3+\beta_1+\beta_2}} \right) A_P(V; u, w) B_P(V; u, w)^{-1} \right) \\
&\quad \times \prod_{P \nmid V} \left(1 - \left(\frac{\chi_{V_1}(P) y^{d(P)}}{|P|^{1+\beta_1}} + \frac{\chi_{V_1}(P) y^{d(P)}}{|P|^{1+\beta_2}} \right) C_P(V; u, w)^{-1} \left(1 - u^{d(P)}\right)^{-1} \right).
\end{aligned}
\tag{5.3.41}
$$

Using this and observing that $\mathcal{H}(V_1; u, w) = \prod_{P \nmid V_1} C_P(V; u, w)$, we further write

$$
\begin{aligned}
&\mathcal{H}(V_1; u, w) \sum_{h \in \mathcal{M}} \frac{\mu_{\beta_1,\beta_2}(h)}{|h|^{3/2}} \mathcal{K}(V, h; u, w) y^{d(h)} \\
=& \prod_{P|V} \left(B_P(V; u, w) - \left(\frac{y^{d(P)}}{|P|^{3/2+\beta_1}} + \frac{y^{d(P)}}{|P|^{3/2+\beta_2}} - \frac{y^{2d(P)}}{|P|^{3+\beta_1+\beta_2}} \right) A_P(V; u, w) \right) \\
&\times \prod_{P \nmid V_1} \left(C_P(V; u, w) - \left(\frac{\chi_{V_1}(P) y^{d(P)}}{|P|^{1+\beta_1}} + \frac{\chi_{V_1}(P) y^{d(P)}}{|P|^{1+\beta_2}} \right) \left(1 - u^{d(P)}\right)^{-1} \right) \\
&\times \prod_{\substack{P \nmid V_1 \\ P | V_2}} \left(C_P(V; u, w) - \left(\frac{\chi_{V_1}(P) y^{d(P)}}{|P|^{1+\beta_1}} + \frac{\chi_{V_1}(P) y^{d(P)}}{|P|^{1+\beta_2}} \right) \left(1 - u^{d(P)}\right)^{-1} \right)^{-1} \\
:=& \mathcal{J}(V_1; u, w) \prod_{P|V_1} E_P(V; u, w) \prod_{\substack{P \nmid V_1 \\ P | V_2}} F_P(V; u, w),
\end{aligned}
\tag{5.3.42}
$$

where

$$E_P(V;u,w) = \left(B_P(V;u,w) - \left(\frac{y^{d(P)}}{|P|^{3/2+\beta_1}} + \frac{y^{d(P)}}{|P|^{3/2+\beta_2}} - \frac{y^{2d(P)}}{|P|^{3+\beta_1+\beta_2}}\right)A_P(V;u,w)\right), \tag{5.3.43}$$

$$\begin{aligned} F_P(V;u,w) =& \left(B_P(V;u,w) - \left(\frac{y^{d(P)}}{|P|^{3/2+\beta_1}} + \frac{y^{d(P)}}{|P|^{3/2+\beta_2}} - \frac{y^{2d(P)}}{|P|^{3+\beta_1+\beta_2}}\right)A_P(V;u,w)\right) \\ &\times \left(C_P(V;u,w) - \left(\frac{\chi_{V_1}(P)y^{d(P)}}{|P|^{1+\beta_1}} + \frac{\chi_{V_1}(P)y^{d(P)}}{|P|^{1+\beta_2}}\right)\left(1-u^{d(P)}\right)^{-1}\right)^{-1}, \end{aligned} \tag{5.3.44}$$

and

$$\begin{aligned} \mathcal{J}(V_1;u,w) &= \prod_{P\nmid V_1}\left(C_P(V;u,w) - \left(\frac{\chi_{V_1}(P)y^{d(P)}}{|P|^{1+\beta_1}} + \frac{\chi_{V_1}(P)y^{d(P)}}{|P|^{1+\beta_2}}\right)\left(1-u^{d(P)}\right)^{-1}\right) \\ &= \prod_{P\nmid V_1}\left(1 + \left(\frac{\chi_{V_1}(P)w^{d(P)}}{|P|^{1+\gamma_1}} + \frac{\chi_{V_1}(P)w^{d(P)}}{|P|^{1+\gamma_2}} - \frac{\chi_{V_1}(P)y^{d(P)}}{|P|^{1+\beta_1}} \right.\right. \\ &\qquad \left.\left. - \frac{\chi_{V_1}(P)y^{d(P)}}{|P|^{1+\beta_2}}\right)\left(1-u^{d(P)}\right)^{-1}\right). \end{aligned} \tag{5.3.45}$$

Now, following the argument in [BFK23, Section 4.1], let i be minimal such that $|u^i w| < q^\gamma$ and $|u^i y| < q^\beta$. Such an i exists since $|u| = q^{-\varepsilon} < 1$. By writing $(1-u^{d(P)})^{-1} = \sum_{j=0}^\infty u^{jd(P)}$ and factoring out the appropriate L-functions corresponding to the zero and polar terms of the product $\mathcal{J}(V_1;u,w)$, we may write

$$\mathcal{J}(V_1;u,w) = \prod_{j=1}^{2}\left(\frac{\mathcal{L}\left(\frac{w}{q^{1+\gamma_j}},\chi_{V_1}\right)\mathcal{L}\left(\frac{uw}{q^{1+\gamma_j}},\chi_{V_1}\right)\cdots\mathcal{L}\left(\frac{u^{i-1}w}{q^{1+\gamma_j}},\chi_{V_1}\right)}{\mathcal{L}\left(\frac{y}{q^{1+\beta_j}},\chi_{V_1}\right)\mathcal{L}\left(\frac{uy}{q^{1+\beta_j}},\chi_{V_1}\right)\cdots\mathcal{L}\left(\frac{u^{i-1}y}{q^{1+\beta_j}},\chi_{V_1}\right)}\right)\mathcal{U}(V_1;u,w), \tag{5.3.46}$$

where the product $\mathcal{U}(V_1;u,w)$ is absolutely convergent and hence analytic in the region $|u| < q^{-\varepsilon}$, $|w| < q^{1/2+\gamma}$ and $|y| < q^{1/2+\beta}$. Note that the L-functions in the above expression for $\mathcal{J}(V_1;u,w)$ do not have any poles or zeros in this region as they satisfy the Riemann hypothesis. We refer to the expository article [Alb24] for further details on the factorisation method for obtaining a meromorphic continuation of an Euler product. Using Perron's formula for the sum over h, we then have

$$
\begin{aligned}
&E_{\leq X,1}(N; V \neq \Box) \\
&= \frac{q^{3/2}}{(q-1)} \frac{1}{(2\pi i)^3} \oint_{|u|=q^{-\varepsilon}} \oint_{|w|=r_1} \oint_{|y|=r_2} \sum_{m \leq X} \sum_{\substack{n \leq N \\ n+m \text{ odd}}} \sum_{j=0}^{g} \sum_{\substack{r \leq n+m-2g+2j-2 \\ r \text{ odd}}} q^{-2j} \\
&\times \sum_{V_2 \in \mathcal{M}_{\frac{(n+m-r)}{2}-g+j-1}} \sum_{V_1 \in \mathcal{H}_r} \mathcal{J}(V_1; u, w) \prod_{P|V_1} E_P(V; u, w) \prod_{\substack{P \nmid V_1 \\ P|V_2}} F_P(V; u, w) \frac{dy\, dw\, du}{u^{j+1} w^{n+1} y^{m+1}},
\end{aligned}
\tag{5.3.47}
$$

where $r_1 = q^{1/2+\gamma-\varepsilon}$ and we may take $|y| = q^{1/2+\beta-\varepsilon}$.

We are now ready to bound $E_{\leq X,1}(N; V \neq \Box)$. Following the argument in [BFK23, Section 4.1], we first apply the Cauchy-Schwarz inequality to get

$$
\begin{aligned}
&\left| \sum_{V_1 \in \mathcal{H}_r} \mathcal{J}(V_1; u, w) \prod_{P|V_1} E_P(V; u, w) \prod_{\substack{P \nmid V_1 \\ P|V_2}} F_P(V; u, w) \right| \\
&\qquad \ll \left(\sum_{V_1 \in \mathcal{H}_r} |\mathcal{J}(V_1; u, w)|^2 \right)^{1/2} \left(\sum_{V_1 \in \mathcal{H}_r} \prod_{P|V_1} |E_P(V; u, w)|^2 \prod_{\substack{P \nmid V_1 \\ P|V_2}} |F_P(V; u, w)|^2 \right)^{1/2}.
\end{aligned}
\tag{5.3.48}
$$

The factors $E_P(V; u, w)$ and $F_P(V; u, w)$ may be bounded trivially and for the L-functions in the numerator of $\mathcal{J}(V_1; u, w)$ in (5.3.46), we use the Lindelöf bound [BCD+17, Theorem 20]:

$$
\mathcal{L}(u, \chi_{V_1}) \ll |V_1|^{\epsilon}, \tag{5.3.49}
$$

which is valid for $|u| < q^{-1/2}$. Then, for the L-functions in the denominator, we use Theorem 5.2.6 to bound the sum over V_1. It follows that

$$
\sum_{V_1 \in \mathcal{H}_r} \mathcal{J}(V_1; u, w) \prod_{P|V_1} E_P(V; u, w) \prod_{\substack{P \nmid V_1 \\ P|V_2}} F_P(V; u, w) \ll_{\varepsilon} q^{r+\varepsilon r}. \tag{5.3.50}
$$

We bound the integral expression for $E_{\leq X,1}(N; V \neq \Box)$ using absolute values and using the above bound for the sum over V_1, we have

$$
\begin{aligned}
&E_{\leq X,1}(N;V\neq\Box)\\
&\quad\ll \oint_{|u|=q^{-\varepsilon}}\oint_{|w|=q^{1/2+\gamma-\varepsilon}}\oint_{|y|=q^{1/2+\beta-\varepsilon}}\sum_{m\leq X}\sum_{\substack{n\leq N\\ n+m\text{ odd}}}\sum_{j=0}^{g}\sum_{\substack{r\leq n+m-2g+2j-2\\ r\text{ odd}}}q^{-2j}\\
&\qquad\times\sum_{V_2\in\mathcal{M}_{(n+m-r)/2-g+j-1}}q^{r+\varepsilon r}\frac{dy\,dw\,du}{|u|^{j+1}|w|^{n+1}|y|^{m+1}}.
\end{aligned}
\tag{5.3.51}
$$

Applying the triangle inequality to the remaining sums, we bound the integrand on the contours of integration by

$$
\begin{aligned}
&\sum_{m\leq X}q^{-m(1/2+\beta-\varepsilon)}\sum_{\substack{n\leq N\\ n+m\text{ odd}}}q^{-n(1/2+\gamma-\varepsilon)}\sum_{j=0}^{g}q^{\varepsilon j-2j}\sum_{\substack{r\leq n+m-2g+2j-2\\ r\text{ odd}}}q^{r+\varepsilon r}\sum_{V_2\in\mathcal{M}_{\frac{(n+m-r)}{2}-g+j-1}}1\\
&\ll_\varepsilon\sum_{m\leq X}q^{-m(1/2+\beta-\varepsilon)}\sum_{\substack{n\leq N\\ n+m\text{ odd}}}q^{-n(1/2+\gamma-\varepsilon)}\sum_{j=0}^{g}q^{\varepsilon j-2j}\sum_{\substack{r\leq n+m-2g+2j-2\\ r\text{ odd}}}q^{(n+m+r)/2-g+j+\varepsilon r}\\
&\ll_\varepsilon\sum_{m\leq X}q^{-m(1/2+\beta-\varepsilon)}\sum_{\substack{n\leq N\\ n+m\text{ odd}}}q^{-n(1/2+\gamma-\varepsilon)}\sum_{j=0}^{g}q^{n+m-2g+\varepsilon(n+m-2g+3j)}\\
&\ll_\varepsilon\sum_{m\leq X}q^{-m(1/2+\beta-\varepsilon)}\sum_{\substack{n\leq N\\ n+m\text{ odd}}}q^{n(1/2-\gamma)+m-2g+\varepsilon(2n+m+g)}\\
&\ll_\varepsilon\sum_{m\leq X}q^{m(1/2-\beta)+N(1/2-\gamma)-2g+\varepsilon(2N+2m+g)}\\
&\ll_\varepsilon q^{X(1/2-\beta)+N(1/2-\gamma)-2g+\varepsilon(2N+2X+g)}\\
&\ll_\varepsilon q^{X(1/2-\beta)+N(1/2-\gamma)-2g+\varepsilon g}.
\end{aligned}
\tag{5.3.52}
$$

The length of the contours is of size 1 as $g\to\infty$ so by the estimation lemma, we therefore have the bound

$$
E_{\leq X,1}(N;V\neq\Box)\ll_\varepsilon q^{X(1/2-\beta)+N(1/2-\gamma)-2g+\varepsilon g}, \tag{5.3.53}
$$

The other terms in $E_{\leq X}(N;V\neq\Box)$ may be bounded similarly. Consequently, for $N\in\{2g,2g-1\}$, we have that

$$
E_{\leq X}(N;V\neq\Box)\ll_\varepsilon q^{X(1/2-\beta)-g-2g\gamma+\varepsilon g}. \tag{5.3.54}
$$

5.3.5 Completing the proof

To finish the proof of Theorem 5.1.3, we collect the bounds for the error terms in (5.3.17), (5.3.27), (5.3.36) and (5.3.54) and choose a suitable value for X. First, if $\mathrm{Re}(\alpha_1), \mathrm{Re}(\alpha_2) \geq 0$, the contribution of the error in $S_{\mathbf{A}}(h; 2g)$ to $R_{2,\leq X}$ will be

$$\ll_\varepsilon q^{X(1/2-\beta)-g-2g\min\{\mathrm{Re}(\alpha_1),\mathrm{Re}(\alpha_2)\}+\varepsilon g} \tag{5.3.55}$$

Similarly, the contribution of the error in the second piece of the approximate functional equation $q^{-2g\mathbf{A}} S_{\mathbf{A}^-}(h; 2g-1)$ will be

$$\begin{aligned} &\ll_\varepsilon q^{-2g\mathrm{Re}(\alpha_1+\alpha_2)+X(1/2-\beta)-g-2g\min\{-\mathrm{Re}(\alpha_1),-\mathrm{Re}(\alpha_2)\}+\varepsilon g} \\ &\ll_\varepsilon q^{X(1/2-\beta)-g-2g\min\{\mathrm{Re}(\alpha_1),\mathrm{Re}(\alpha_2)\}+\varepsilon g}. \end{aligned} \tag{5.3.56}$$

Recalling the bound for $R_{2,>X}$ in (5.3.13), we choose $X = \frac{2g(1+2\min\{\mathrm{Re}(\alpha_1),\mathrm{Re}(\alpha_2)\}-\varepsilon)}{(1-2\beta\varepsilon)}$ which proves Theorem 5.1.3 in this case with the error

$$E_2 \ll_\varepsilon q^{-2g\beta(1+2\min\{\mathrm{Re}(\alpha_1),\mathrm{Re}(\alpha_2)\})+\varepsilon g\beta}. \tag{5.3.57}$$

If $\mathrm{Re}(\alpha_1), \mathrm{Re}(\alpha_2) < 0$, then by additionally using (5.2.15), we have that the error in $R_{2,\leq X}$ will be

$$\begin{aligned} &\ll_\varepsilon q^{-2g\mathrm{Re}(\alpha_1+\alpha_2)} q^{X(1/2-\beta)-g-2g\min\{-\mathrm{Re}(\alpha_1),-\mathrm{Re}(\alpha_2)\}+\varepsilon g} \\ &\ll_\varepsilon q^{X(1/2-\beta)-g+2g\max\{|\mathrm{Re}(\alpha_1)|,|\mathrm{Re}(\alpha_2)|\}+\varepsilon g}. \end{aligned} \tag{5.3.58}$$

In this case we take $X = \frac{2g(1-2\max\{|\mathrm{Re}(\alpha_1)|,|\mathrm{Re}(\alpha_2)|\}-\varepsilon)}{(1-2\beta\varepsilon)}$ and obtain the bound

$$E_2 \ll_\varepsilon q^{-2g\beta(1-2\max\{|\mathrm{Re}(\alpha_1)|,|\mathrm{Re}(\alpha_2)|\})+\varepsilon g\beta}. \tag{5.3.59}$$

Finally, if $\mathrm{Re}(\alpha_1) < 0$ and $\mathrm{Re}(\alpha_2) \geq 0$, the error in $R_{2,\leq X}$ will be

$$\begin{aligned} &\ll_\varepsilon q^{-2g\mathrm{Re}(\alpha_1)} q^{X(1/2-\beta)-g-2g\min\{-\mathrm{Re}(\alpha_1),\mathrm{Re}(\alpha_2)\}+\varepsilon g} \\ &\ll_\varepsilon q^{X(1/2-\beta)-g-2g\min\{0,\mathrm{Re}(\alpha_1+\alpha_2)\}+\varepsilon g}. \end{aligned} \tag{5.3.60}$$

Choosing $X = \frac{2g(1+2\min\{0,\mathrm{Re}(\alpha_1+\alpha_2)\}-\varepsilon)}{(1-2\beta\varepsilon)}$ then finishes the proof with the bound

$$E_2 \ll_\varepsilon q^{-2g\beta(1+2\min\{0,\mathrm{Re}(\alpha_1+\alpha_2)\})+\varepsilon g\beta}. \tag{5.3.61}$$

The case $\mathrm{Re}(\alpha_1) \geq 0$ and $\mathrm{Re}(\alpha_2) < 0$ is similar.

Chapter 6

Mollified moments of quadratic Dirichlet L-functions over function fields

6.1 The mollification method

Mollifying L-functions is a very useful technique for obtaining information about their zeros. We demonstrate the general idea in the case of the Riemann zeta function. Recall Conjecture 1.4.6 which states that the $2k$-th moment of zeta should satisfy

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt \sim a_\zeta(k)c_\zeta(k)(\log T)^{k^2}, \tag{6.1.1}$$

for a certain arithmetic factor $a_\zeta(k)$ and a precise value for the constant $c_\zeta(k)$ put forward by Keating and Snaith [KS00b]. The purpose of a mollifier is to dampen the large values of $\zeta(\frac{1}{2}+it)$ and thus save the factor of $(\log T)^{k^2}$ in the $2k$-th moment. More specifically, the mollifier is a Dirichlet polynomial that should approximate $\zeta(\frac{1}{2}+it)^{-1}$. Since, for $\mathrm{Re}(s) > 1/2$ assuming the Riemann Hypothesis, the reciprocal of the zeta function has the Dirichlet series

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \tag{6.1.2}$$

the natural choice of mollifier stems from truncating this series. The most common choice of mollifier for the zeta function is

$$M_N(s,P) = \sum_{n\le N} \frac{\mu(n)}{n^s} P\left(\frac{\log(N/n)}{\log N}\right), \tag{6.1.3}$$

were $P(x)$ is a polynomial satisfying $P(0) = 0$. The mollified moments should then satisfy

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)M_N(\tfrac{1}{2}+it,P)|^{2k}\,dt \asymp 1, \tag{6.1.4}$$

as $T \to \infty$.

Mollifying the zeta function was first performed successfully by Selberg [Sel42] who showed that a positive (but small) proportion of the non-trivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = 1/2$. Later, Levinson [Lev74] showed how mollifying the second moment of zeta can yield a lower bound on the proportion of non-trivial zeros on the critical line. In particular, with $P(x) = x$ and the length of the mollifier taken to be $N = T^\theta$ with $0 < \theta < 1/2$, Levinson proved that

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)M_{T^\theta}(\tfrac{1}{2}+it,P)|^{2}\,dt \sim 1+\frac{1}{\theta}. \tag{6.1.5}$$

He used this result to infer that at least $1/3$ of the non-trivial zeros of $\zeta(s)$ are in fact on the critical line. Subsequently, Conrey [Con89] considered the case of a general polynomial P and showed that

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)M_{T^\theta}(\tfrac{1}{2}+it,P)|^{2}\,dt \sim |P(1)|^2+\frac{1}{\theta}\int_0^1 P'(t)^2\,dt \tag{6.1.6}$$

for $\theta < 4/7$. From this Conrey was able to deduce that the proportion of zeros on the line is at least $2/5$.

The case of the zeta function highlights the key point when it comes to working with mollifiers. If one wants to obtain stronger results then one needs to increase the length of the mollifier but this is in general very difficult. There have been slight improvements in the proportion of zeros on the line by taking different (but not longer) mollifiers to ≥ 0.4105 in [BCY11] and ≥ 0.417293 in [PRZZ20].

Farmer [Far93] conjectured that (6.1.5) holds for all $\theta > 0$ and this is the so called $\theta = \infty$ conjecture. Farmer also proved that this conjecture would imply that 100% of the non-trivial zeros are on the critical line. It has also been shown by Radziwiłł [Rad12] that $M_{T^\theta}(s, P)$ is in a sense the optimal mollifier of length T^θ and that Levinson's method can give a proportion of 100% only if one can take θ arbitrarily large. In [BG17], Bettin and Gonek showed that the $\theta = \infty$ conjecture in fact implies the Riemann hypothesis and more specifically that if one can obtain the bound

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)M_{T^\theta}(\tfrac{1}{2}+it,P)|^2\,dt \ll_\varepsilon T^\varepsilon \tag{6.1.7}$$

for all $\varepsilon > 0$ and some $\theta > 0$, then $\zeta(s)$ has no zeros in the half plane $\mathrm{Re}(s) > \frac{1}{2}+\frac{1}{2\theta}$.

The method of mollifying may also be applied to derivatives of L-functions as well. For instance, it is known that RH implies that all the zeros of $\xi^{(k)}(s)$, where $\xi(s)$ is Riemann's ξ-function, lie on the line $\mathrm{Re}(s) = 1/2$. By considering the second mollified moment of $\xi^{(k)}(s)$, Conrey [Con83] showed that the proportion of zeros of $\xi^{(k)}(s)$ on the critical line is $1 + O(k^{-2})$ as $k \to \infty$. Importantly, this result does not require an arbitrarily long mollifier to be used.

6.1.1 Non-vanishing of L-functions

In [Cho65], Chowla conjectured that $L(\frac{1}{2},\chi) \neq 0$ for all Dirichlet L-functions. This conjecture remains open and it is now widely believed that an L-function should not vanish at the central point unless there is some specific reason for it. For example, the sign of the functional equation may be -1 or there may be some arithmetic reason why the L-function vanishes.

Another application of the mollifier method is to obtain non-vanishing results for L-functions at the central point. Here we consider the family of quadratic Dirichlet L-functions to illustrate the setup. Asymptotic formulae for the first and second moments were obtained by Jutila [Jut81] who showed that as $X \to \infty$,

$$\sideset{}{^*}\sum_{0<d\le X} L(\tfrac{1}{2},\chi_d) \sim c_1 X\log X \tag{6.1.8}$$

and

$$\sideset{}{^*}\sum_{0<d\le X} L(\tfrac{1}{2},\chi_d)^2 \sim c_2 X(\log X)^3, \tag{6.1.9}$$

for certain constants c_0, c_1 and where the sum is only over fundamental discriminants. By the Cauchy-Schwarz inequality, one then has that

$$\sideset{}{^*}\sum_{\substack{0<d\le X\\ L(1/2,\chi_d)\neq 0}} 1 \ge \frac{\left(\sideset{}{^*}\sum_{0<d\le X} L(\frac{1}{2},\chi_d)\right)^2}{\sideset{}{^*}\sum_{0<d\le X} L(\frac{1}{2},\chi_d)^2} \gg \frac{X}{\log X}. \tag{6.1.10}$$

This shows that the proportion of discriminants $0 < d \le X$ such that $L(\frac{1}{2},\chi_d) \neq 0$ is $\gg (\log X)^{-1}$. Letting $X \to \infty$, the proportion of $L(\frac{1}{2},\chi_d) \neq 0$ is then at least 0%.

To obtain a positive proportion of non-vanishing for $L(\frac{1}{2},\chi_d)$, the mollifier method can be applied. Similarly to the case of the Riemann zeta function, the mollifier is a Dirichlet polynomial that should approximate $L(\frac{1}{2}, \chi_d)^{-1}$. This is the approach employed by Soundararajan in [Sou00]. He takes a mollifier roughly of the form

$$M_N(\chi_d) = \sum_{n \le N} \frac{\mu(n)\chi_d(n)}{n^{1/2}}, \tag{6.1.11}$$

where $N = X^\theta$. For $\theta < 1/2$, Soundararajan shows that the mollified first and second moments satisfy

$$\sideset{}{^*}\sum_{0<d\le X} |L(\tfrac{1}{2}, \chi_d) M_{X^\theta}(\chi_d)| \sim c_1' X, \tag{6.1.12}$$

and

$$\sideset{}{^*}\sum_{0<d\le X} |L(\tfrac{1}{2}, \chi_d) M_{X^\theta}(\chi_d)|^2 \sim c_2' X. \tag{6.1.13}$$

for some c_0', c_1'. Applying the Cauchy-Schwarz inequality here then gives

$$\sideset{}{^*}\sum_{\substack{0<d\le X \\ L(1/2,\chi_d)\neq 0}} 1 \ge \frac{\left(\sideset{}{^*}\sum_{0<d\le X} |L(\frac{1}{2}, \chi_d) M_{X^\theta}(\chi_d)|\right)^2}{\sideset{}{^*}\sum_{0<d\le X} |L(\frac{1}{2}, \chi_d) M_{X^\theta}(\chi_d)|^2} \asymp X. \tag{6.1.14}$$

This yields a positive and computable proportion of non-vanishing depending on θ. The proportion tends to 1 as $\theta \to \infty$ so again, one wants to take the mollifier to be as long as possible. With $\theta < 1/2$, Soundararajan gets that at least 87.5% of $L(\frac{1}{2}, \chi_d) \neq 0$. Assuming GRH, Özlük and Snyder [ÖC99] computed the one-level density of zeros of this family when the support of the Fourier transform of the test function is in $(-2, 2)$. Their result implies that at least $15/16 = 93.75\%$ of $L(s, \chi_d)$ do not vanish at $s = 1/2$. We note that while Soundararajan's non-vanishing proportion is smaller, the approach using the mollified moments does not require GRH to be assumed.

A similar approach can be used to yield non-vanishing results on the derivatives of L-functions at the central point. In [KMV00], Kowalski, Michel and VanderKam gave asymptotic formulae for the mollified first and second moments of the derivatives $\xi^{(k)}(\frac{1}{2}, f)$ where $\xi(s, f)$ is the completed L-function attached to a primitive Hecke eigenform f of weight 2. This is an orthogonal family of L-functions so half of the family have an even functional equation and the other half odd. Consequently, the strongest result one can hope to obtain is that 50% of $\xi^{(k)}(\frac{1}{2}, f) \neq 0$ for any k. The key result of [KMV00] is that this proportion can be shown to tend to 50% as

$k \to \infty$, regardless of the length of the mollifier. This is analogous to the result of Conrey [Con83] on the proportion of zeros of $\xi^{(k)}(s)$ on the critical line. Using similar techniques, Michel and VanderKam [MV00] computed the mollified moments of the derivatives $\xi^{(k)}(\frac{1}{2}, \chi)$ where now $\xi(s, \chi)$ is a completed Dirichlet L-function attached to the character χ. They show that as $k \to \infty$, the proportion of non-vanishing of $\xi^{(k)}(\frac{1}{2}, \chi)$ approaches 1/2 if using the "standard" mollifier and 2/3 if they use a two-piece mollifier.

In this chapter our goal is to compute mollified moments of the family of quadratic Dirichlet L-functions $L(s, \chi_D)$ in the function field setting. We will prove asymptotic formulae for the first and second mollified moments with a sufficiently short mollifier and show that this restriction on the mollifier may be removed if one assumes the Ratios Conjecture. As an application, we obtain a non-vanishing result on the derivatives of the completed L-functions $\xi(s, \chi_D)$. Our results are therefore a function field analogue of those of [Sou00] and [KMV00, MV00].

6.2 Background and statement of results

6.2.1 The Ratios Conjecture

It was shown by Conrey and Snaith [CS07] that one may compute mollified moments of L-functions using formulae for the averages of ratios of the L-functions. In particular, if one assumes the Ratios Conjecture for the L-functions in question, one may obtain the mollified moments for a mollifier of arbitrarily long length. This then allows the strongest possible results on proportions of zeros on the critical line or non-vanishing at the central point to be obtained.

Our approach to compute the mollified moments for the family of quadratic Dirichlet L-functions in the function field setting will be based on that demonstrated in [CS07] and also the ideas given in [You10]. Our full results will also be conditional on the Ratios Conjecture but we obtain partial, unconditional results using the the work of Bui, Florea and Keating [BFK23]. They prove the Ratios Conjecture for the family of quadratic Dirichlet L-functions over function fields in certain ranges of the parameters which allows us to make our results unconditional for sufficiently short mollifiers.

Below we recall the Ratios Conjecture for the family $L(s, \chi_D)$, formulated by Andrade and Keating in [AK14], that we will make use of subsequently.

Conjecture 6.2.1 (The Ratios Conjecture)**.** *For* $|\mathrm{Re}(\alpha_j)| < 1/4$ *and* $1/g \ll \mathrm{Re}(\beta_j) < 1/4$ *for* $1 \leq j \leq k$*, we have, as* $g \to \infty$*,*

$$\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\frac{\prod_{j=1}^{k}L(\frac{1}{2}+\alpha_j,\chi_D)}{\prod_{j=1}^{k}L(\frac{1}{2}+\beta_j,\chi_D)}$$
$$=\sum_{\epsilon_j\in\{-1,1\}}\prod_{j=1}^{k}q^{g(\epsilon_j\alpha_j-\alpha_j)}Y(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k;\beta)A(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k;\beta)+O(q^{-\delta g}) \quad (6.2.1)$$

for some $\delta>0$, where

$$Y(\alpha;\beta)=\frac{\prod_{1\leq i\leq j\leq k}\zeta_q(1+\alpha_i+\alpha_j)\prod_{1\leq i<j\leq k}\zeta_q(1+\beta_i+\beta_j)}{\prod_{1\leq i,j\leq k}\zeta_q(1+\alpha_i+\beta_j)}, \quad (6.2.2)$$

and

$$A(\alpha;\beta)=\prod_{P\in\mathcal{P}}\frac{\prod_{1\leq i\leq j\leq k}\left(1-\frac{1}{|P|^{1+\alpha_i+\alpha_j}}\right)\prod_{1\leq i<j\leq k}\left(1-\frac{1}{|P|^{1+\beta_i+\beta_j}}\right)}{\prod_{1\leq i,j\leq k}\left(1-\frac{1}{|P|^{1+\alpha_i+\beta_j}}\right)}$$
$$\times\frac{|P|}{|P|+1}\left(\frac{1}{2}\frac{\prod_{j=1}^{k}\left(1-\frac{1}{|P|^{1/2+\beta_j}}\right)}{\prod_{j=1}^{k}\left(1-\frac{1}{|P|^{1/2+\alpha_j}}\right)}+\frac{1}{2}\frac{\prod_{j=1}^{k}\left(1+\frac{1}{|P|^{1/2+\beta_j}}\right)}{\prod_{j=1}^{k}\left(1+\frac{1}{|P|^{1/2+\alpha_j}}\right)}+\frac{1}{|P|}\right). \quad (6.2.3)$$

The conditions on the real parts of the shifts α_j and β_j are there to ensure that the Euler products appearing in the expected asymptotic formula are convergent. However, the conjecture can be extended to a wider range, so long as the Euler products are convergent. As mentioned in Chapter 5, the above conjecture has recently been proven for certain ranges of the parameters when $1\leq k\leq 3$ in [BFK23]. We recall their result below.

Theorem 6.2.2 (Bui, Florea and Keating)**.** *Let $0<\mathrm{Re}(\beta_j)<1/2$ for $1\leq j\leq k$. Denote $\alpha=\max\{|\mathrm{Re}(\alpha_1)|,\ldots,|\mathrm{Re}(\alpha_k)|\}$ and $\beta=\min\{\mathrm{Re}(\beta_1),\ldots,\mathrm{Re}(\beta_k)\}$. Then Conjecture 6.2.1 holds for $1\leq k\leq 3$ with the error term E_k, where*

$$E_1\ll_\varepsilon\begin{cases}q^{-g\beta(3+2\alpha)+\varepsilon g\beta} & \text{if } 0\leq \mathrm{Re}(\alpha)<1/2 \text{ and } \beta\gg g^{-1/2+\varepsilon},\\ q^{-g\beta(3-4\alpha)+\varepsilon g\beta} & \text{if } -1/2<\mathrm{Re}(\alpha)<0 \text{ and } \beta\gg g^{-1/2+\varepsilon},\end{cases} \quad (6.2.4)$$

and

$$E_2\ll_\varepsilon q^{-g\beta\min\{\frac{1-4\alpha}{1+\beta},\frac{1-2\alpha}{2+\beta}\}+\varepsilon g\beta} \text{ if } \alpha<1/4 \text{ and } \beta\gg g^{-1/4+\varepsilon}, \quad (6.2.5)$$

$$E_3 \ll_\varepsilon q^{-g\beta \min\{\frac{1/4-4\alpha}{\beta}, \frac{1/2-4\alpha}{3+\beta}\}+\varepsilon g\beta} \text{ if } \alpha < 1/16 \text{ and } \beta \gg g^{-1/6+\varepsilon}. \tag{6.2.6}$$

We will make use of Theorem 6.2.2 in the cases $k = 1$ and $k = 2$. We will see in the proofs of our main results that because Theorem 6.2.2 does not allow us to take $\beta \gg 1/g$ with a power saving error as in the full conjecture, it is the size of the bound on the error term that determines how long a mollifier we may take. This was part of the motivation for improving the bound on the error E_2 in Theorem 6.2.2 in Chapter 5 where we obtained the following.

Theorem 6.2.3. *With notation as in Theorem 6.2.2, suppose $\alpha < 1/2$ and $\beta \gg g^{-1/4+\varepsilon}$. Then, the error term E_2 satisfies*

$$E_2 \ll_\varepsilon \begin{cases} q^{-2g\beta(1+2\min\{\mathrm{Re}(\alpha_1),\mathrm{Re}(\alpha_2)\})+\varepsilon g\beta} & \text{if } \mathrm{Re}(\alpha_1), \mathrm{Re}(\alpha_2) \geq 0, \\ q^{-2g\beta(1-2\max\{|\mathrm{Re}(\alpha_1)|,|\mathrm{Re}(\alpha_2)|\})+\varepsilon g\beta} & \text{if } \mathrm{Re}(\alpha_1), \mathrm{Re}(\alpha_2) < 0, \\ q^{-2g\beta(1+2\min\{0,\mathrm{Re}(\alpha_1+\alpha_2)\})+\varepsilon g\beta} & \text{if } \mathrm{Re}(\alpha_i) \geq 0 \text{ and } \mathrm{Re}(\alpha_j) < 0 \text{ for } i \neq j. \end{cases} \tag{6.2.7}$$

6.2.2 The mollification

For our choice of mollifier, we take the standard notion of a mollifier in the number field setting and translate it to the function field setting in the natural way. The mollifier is a Dirichlet polynomial whose purpose is to approximate $L(\frac{1}{2}, \chi_D)^{-1}$ so we begin with the Dirichlet series

$$\frac{1}{L(s, \chi_D)} = \sum_{f \in \mathcal{M}} \frac{\mu(f)\chi_D(f)}{|f|^s}, \tag{6.2.8}$$

which is convergent for $\mathrm{Re}(s) > 1/2$. Truncating this series and multiplying by a smoothing function leads us to the mollifier

$$M_y(\chi_D, P) = \sum_{\substack{f \in \mathcal{M} \\ |f| \leq y}} \frac{\mu(f)\chi_D(f)}{|f|^{1/2}} P\left(\frac{\log(y/|f|)}{\log y}\right), \tag{6.2.9}$$

where P is a polynomial satisfying $P(0) = 0$ and $y = (q^{2g})^\theta$ for $\theta > 0$. Provided that the mollifier is not too long, i.e. θ is sufficiently small, we will prove asymptotic formulae for the first and second mollified moments

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} L(\tfrac{1}{2}, \chi_D) M_y(\chi_D, P), \tag{6.2.10}$$

and

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} |L(\tfrac{1}{2}, \chi_D) M_y(\chi_D, P)|^2. \tag{6.2.11}$$

In fact, we obtain much more general results on mollified moments of linear combinations of the derivatives $\xi^{(k)}(\frac{1}{2}, \chi_D)$ where we recall that $\xi(s, \chi_D) = q^{(2s-1)g/2} L(s, \chi_D)$ is the completed L-function. These are described by our main results below.

Theorem 6.2.4. *Let Q be an even polynomial and P a polynomial satisfying $P(0) = 0$. Then for $\theta < 3/2$, we have as $g \to \infty$,*

$$\begin{aligned} Q\left(\frac{1}{g \log q} \frac{d}{d\alpha}\right) \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \xi(\tfrac{1}{2} + \alpha, \chi_D) M_y(\chi_D, P) \Big|_{\alpha=0} \\ = P(1)Q(1) + \frac{1}{2\theta} P'(1) \int_0^1 Q(t) dt + O(1/g). \end{aligned} \tag{6.2.12}$$

Assuming Conjecture 6.2.1, the result holds for any $\theta > 0$.

Theorem 6.2.5. *Let Q_1, Q_2 be even polynomials and P_1, P_2 polynomials satisfying $P_j(0) = P_j'(0) = 0$ for $j = 1, 2$. Then for $\theta < 1/2$, we have as $g \to \infty$,*

$$\begin{aligned} & Q_1\left(\frac{1}{g \log q} \frac{d}{d\alpha}\right) Q_2\left(\frac{1}{g \log q} \frac{d}{d\beta}\right) \\ & \quad \times \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \xi(\tfrac{1}{2} + \alpha, \chi_D) \xi(\tfrac{1}{2} + \beta, \chi_D) M_y(\chi_D, P_1) M_y(\chi_D, P_2) \Big|_{\alpha=\beta=0} \\ = & \frac{1}{8\theta} \int_0^1 \int_0^1 \left(\frac{1}{\theta} P_1''(r) \tilde{Q}_1(u) - 4\theta P_1(r) Q_1'(u)\right) \left(\frac{1}{\theta} P_2''(r) \tilde{Q}_2(u) - 4\theta P_2(r) Q_2'(u)\right) du dr \\ & \quad + \frac{1}{4} \left(\frac{1}{\theta} P_1'(1) \tilde{Q}_1(1) + 2P_1(1) Q_1(1)\right) \left(\frac{1}{\theta} P_2'(1) \tilde{Q}_2(1) + 2P_2(1) Q_2(1)\right) + O(1/g), \end{aligned} \tag{6.2.13}$$

where

$$\tilde{Q}(u) = \int_0^u Q(t)\, dt. \tag{6.2.14}$$

Assuming Conjecture 6.2.1, the result holds for any $\theta > 0$.

Remark 6.2.6. *1. We only consider even polynomials Q in Theorems 6.2.4 and 6.2.5 since by the functional equation*

$$\xi(s, \chi_D) = \xi(1 - s, \chi_D), \tag{6.2.15}$$

we have that $\xi^{(k)}(\frac{1}{2}, \chi_D) = 0$ if k is odd.

2. *Theorem 6.2.5 is the function field analogue of Theorem 5.2 in [CS07] which gives a formula for the mollified second moment of quadratic Dirichlet L-functions assuming the Ratios Conjecture. Our result additionally shows that the formula holds unconditionally for $\theta < 1/2$.*

3. *Asymptotics for the mollified moments in* (6.2.10) *and* (6.2.11) *can be recovered by taking $Q(x) = Q_1(x) = Q_2(x) = 1$ in Theorems 6.2.4 and 6.2.5. In particular, we find that*

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} L(\tfrac{1}{2}, \chi_D) M_y(\chi_D, P) \sim P(1) + \frac{1}{2\theta} P'(1), \tag{6.2.16}$$

and

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} |L(\tfrac{1}{2}, \chi_D) M_y(\chi_D, P)|^2 \sim \left(P(1) + \frac{1}{2\theta} P'(1)\right)^2 + \frac{1}{24\theta^3} \int_0^1 P''(r)^2 \, dr. \tag{6.2.17}$$

The conditions on θ in Theorems 6.2.4 and 6.2.5 are necessary to ensure that the error we obtain does not dominate the main term. We will show explicitly how we are led to these conditions in the proofs of the theorems. Similarly, the condition that the polynomial P in the mollifier satisfies $P(0) = P'(0) = 0$ in Theorem 6.2.5 is also needed to bound an error term suitably. Again, we will point out where this is necessary in the proof.

6.2.3 Applications to non-vanishing

As an application of our results on the mollified moments, we are able to obtain non-vanishing results on the derivatives $\xi^{(2k)}(\frac{1}{2}, \chi_D)$. To do this we first introduce some notation. For $Q(x) = \sum_{n \geq 0} a_n x^n$ an even polynomial, we define the differential operator

$$\widehat{Q} = Q\left(\frac{1}{g\log q}\frac{d}{ds}\right) = \sum_{n\geq 0}\frac{a_n}{(g\log q)^n}\frac{d^n}{ds^n}. \tag{6.2.18}$$

Then we define the first and second mollified moments

$$\mathcal{S}_1(P,Q) = \frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\widehat{Q}\big(\xi(s,\chi_D)\big)(\tfrac{1}{2})M_y(\chi_D,P), \tag{6.2.19}$$

and

$$\mathcal{S}_2(P,Q) = \frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}|\widehat{Q}\big(\xi(s,\chi_D)\big)(\tfrac{1}{2})M_y(\chi_D,P)|^2. \tag{6.2.20}$$

By Theorem 6.2.4, we have that

$$\mathcal{S}_1(P,Q) = P(1)Q(1) + \frac{1}{2\theta}P'(1)\tilde{Q}(1) + O(1/g). \tag{6.2.21}$$

Similarly, taking $Q = Q_1 = Q_2$ in Theorem 6.2.5, for a polynomial P with $P(0) = P'(0) = 0$, we have

$$\begin{aligned}\mathcal{S}_2(P,Q) = &\left(P(1)Q(1) + \frac{1}{2\theta}P'(1)\tilde{Q}(1)\right)^2 \\ &+ \frac{1}{8\theta}\int_0^1\int_0^1\left(\frac{1}{\theta}P''(r)\tilde{Q}(u) - 4\theta P(r)Q'(u)\right)^2 du\,dr + O(1/g).\end{aligned} \tag{6.2.22}$$

Now, by applying the Cauchy-Schwarz inequality in the usual way, we have that

$$\frac{1}{|\mathcal{H}_{2g+1}|}|\{D\in\mathcal{H}_{2g+1} : \widehat{Q}\big(\xi(s,\chi_D)\big)(\tfrac{1}{2})\neq 0\}| \geq \frac{\mathcal{S}_1(P,Q)^2}{\mathcal{S}_2(P,Q)}. \tag{6.2.23}$$

Using the above two expressions for $\mathcal{S}_1(P,Q)$ and $\mathcal{S}_2(P,Q)$ on the right-hand side of this inequality then gives us the following general non-vanishing result.

Theorem 6.2.7. *For Q an even polynomial, we have that as $g\to\infty$,*

$$\frac{1}{|\mathcal{H}_{2g+1}|}|\{D\in\mathcal{H}_{2g+1} : \widehat{Q}\big(\xi(s,\chi_D)\big)(\tfrac{1}{2})\neq 0\}| \geq \sup_{P,\theta}\frac{1}{1+\mathcal{R}(P,Q)} + o(1), \tag{6.2.24}$$

where the supremum is over all polynomials P with $P(0) = P'(0) = 0$ and real numbers $0<\theta<1/2$. Also, the ratio $\mathcal{R}(P,Q)$ is given by

$$\mathcal{R}(P,Q) = \frac{(2\theta)^{-1}\int_0^1\int_0^1 \left(P''(r)\tilde{Q}(u) - 4\theta^2 P(r)Q'(u)\right)^2 du\, dr}{\left(2\theta P(1)Q(1) + P'(1)\tilde{Q}(1)\right)^2}. \tag{6.2.25}$$

Remark 6.2.8. *1. The ratio $\mathcal{R}(P,Q)$ is non-negative and so the non-vanishing proportion is at most 1 as expected.*

2. The ratio $\mathcal{R}(P,Q)$ above has a very similar expression to the analogous ratio appearing in [KMV00]. In particular, if we set $\Delta = 2\theta$, replace $\tilde{Q}$ by Q and Q by Q', we get exactly the expression given in [KMV00, (31)].

3. By the continuity of $\mathcal{R}(P,Q)$ in P and θ, when determining the value of the supremum above, one may take $\theta = 1/2$ and allow P to range over all power series $P(x) = a_2x^2 + a_3x^3 + \ldots$ such that this series and the series for everything up to and including $P''(x)^2$ are absolutely convergent on $[0,1]$.

Finally, we examine in more detail the non-vanishing of $\xi^{(2k)}(\frac{1}{2}, \chi_D)$. We denote the proportion of non-vanishing by

$$p_{2k} = \liminf_{g\to\infty} \frac{|\{D \in \mathcal{H}_{2g+1} : \xi^{(2k)}(\frac{1}{2}, \chi_D) \neq 0\}|}{|\mathcal{H}_{2g+1}|}. \tag{6.2.26}$$

By taking $Q(x) = x^{2k}$ in Theorem 6.2.7 and determining the optimal choice of P to minimise $\mathcal{R}(P, x^{2k})$, we will prove the following.

Theorem 6.2.9. *For all $k \geq 0$, we have that $p_{2k} > 0$. Moreover, we have $p_{2k} \geq \pi_{2k}$, where π_{2k} satisfies*

$$\pi_{2k} = 1 - \frac{1}{16(2k)^2} + O(k^{-3}), \tag{6.2.27}$$

as $k \to \infty$. In particular, we have

$$p_0 \geq 0.875,\ p_2 \geq 0.9895,\ p_4 \geq 0.9971,\ p_6 \geq 0.9986 \text{ and } p_8 \geq 0.9991. \tag{6.2.28}$$

For the family of quadratic Dirichlet L-functions $L(s,\chi_D)$ over function fields, it is known due to a result of Li [Li18] that Chowla's conjecture does not hold. In fact, Li's result implies that $L(\frac{1}{2}, \chi_D) = 0$ infinitely often. However, by the density conjectures of Katz and Sarnak [KS99b], it is predicted that we should have $L(\frac{1}{2}, \chi_D) \neq 0$ for almost all, i.e. 100%, of discriminants D. Bui and Florea [BF18], by computing the one-level density of zeros, have shown that the proportion of non-vanishing of

$L(\frac{1}{2}, \chi_D)$ is $> 94\%$. In [ELS20], Ellenberg, Li and Shusterman obtained an upper bound, depending on q, on the proportion of $L(s, \chi_D)$ which vanish at $s = 1/2$. Their bound tends to 0 as $q \to \infty$ and thus improves upon the bound given in [BF18] for sufficiently large q.

Our lower bound on this proportion p_0 of 0.875 is therefore not an improvement but shows a clear analogy between our results and those of Soundararajan in [Sou00]. Our Theorem 6.2.9 is also a function field analogue of [KMV00, Theorem 1.2] and [MV00, Theorem 1] and shows that L-functions $L(s, \chi_D)$ with a high order of vanishing at $s = 1/2$ are rare.

6.3 The mollified first moment

In this section we will prove Theorem 6.2.4. We begin with the shifted mollified first moment

$$\mathcal{M}(\alpha, P) := \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} L(\tfrac{1}{2} + \alpha, \chi_D) M_y(\chi_D, P), \tag{6.3.1}$$

Proposition 6.3.1. *For $\theta < 3/2$, we have*

$$\mathcal{M}(\alpha, P) = \frac{1 + q^{-2g\alpha}}{2} P(1) + \frac{1 - q^{-2g\alpha}}{2\alpha \log y} P'(1) + O(1/g), \tag{6.3.2}$$

uniformly for $\alpha \ll 1/g$. Assuming Conjecture 6.2.1, the result holds for any $\theta > 0$.

Proof. We begin by writing the mollifier in the form of a contour integral by using the fact that for $|f| \leq y$ and $n \in \mathbb{N}$,

$$\left(\frac{\log(y/|f|)}{\log y} \right)^n = \frac{n!}{(\log y)^n} \frac{1}{2\pi i} \int_{(c)} \left(\frac{y}{|f|} \right)^z \frac{dz}{z^{n+1}}, \tag{6.3.3}$$

where $c > 0$. This can be seen by moving the contour to $-\infty$ and computing the residue of the pole at $z = 0$. Therefore, by writing $P(x) = \sum_{n \geq 1} p_n x^n$, we have

$$M_y(\chi_D, P) = \sum_{n \geq 1} \frac{p_n n!}{\log^n y} \sum_{\substack{f \in \mathcal{M} \\ |f| \leq y}} \frac{\mu(f) \chi_D(f)}{|f|^{1/2}} \frac{1}{2\pi i} \int_{(c)} \left(\frac{y}{|f|} \right)^z \frac{dz}{z^{n+1}}. \tag{6.3.4}$$

Now, if $|f| > y$, by moving the contour far to the right we see that the contour integral above vanishes. Thus, we may drop the condition that $|f| \leq y$ from the sum over f. Then, since $\mathrm{Re}(z) > 0$, the resulting series converges and we have

$$M_y(\chi_D, P) = \sum_{n \geq 1} \frac{p_n n!}{\log^n y} \frac{1}{2\pi i} \int_{(c)} \frac{y^z}{z^{n+1}} \frac{dz}{L(\frac{1}{2} + z, \chi_D)}. \tag{6.3.5}$$

Using this expression for the mollifier in (6.3.1) yields

$$\mathcal{M}(\alpha, P) = \sum_{n\geq 1} \frac{p_n n!}{\log^n y} \frac{1}{2\pi i} \int_{(c)} \frac{y^z}{z^{n+1}} \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D\in\mathcal{H}_{2g+1}} \frac{L(\frac{1}{2}+\alpha, \chi_D)}{L(\frac{1}{2}+z, \chi_D)} \, dz. \tag{6.3.6}$$

By Theorem 6.2.2, for $|\mathrm{Re}(\alpha)| < 1/2$ and $g^{-1/2+\varepsilon} \ll c < 1/2$, the mean value of the ratio of L-functions appearing in the integrand is given by

$$\begin{aligned}\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D\in\mathcal{H}_{2g+1}} \frac{L(\frac{1}{2}+\alpha, \chi_D)}{L(\frac{1}{2}+z, \chi_D)} &= \frac{\zeta_q(1+2\alpha)}{\zeta_q(1+\alpha+z)} A(\alpha; z) + q^{-2g\alpha} \frac{\zeta_q(1-2\alpha)}{\zeta_q(1-\alpha+z)} A(-\alpha; z) \\ &\quad + O(q^{-gc(3-4|\mathrm{Re}(\alpha)|)+\varepsilon cg}),\end{aligned} \tag{6.3.7}$$

where $A(\alpha; z)$ is the Euler product defined in (6.2.3). As $y = q^{2g\theta}$ and $\alpha \ll 1/g$, by bounding the integral by absolute values, the contribution of the error term above to (6.3.6) will be

$$\ll_\varepsilon q^{cg(2\theta-3+\varepsilon)}. \tag{6.3.8}$$

This error is $\ll_\varepsilon q^{-\varepsilon g}$ if and only if $\theta < 3/2$. On the other hand, if we assume the Ratios Conjecture, then (6.3.7) holds for $1/g \ll c < 1/2$ with an error that is $O(q^{-\varepsilon g})$ uniformly. In this case, we may take $c \asymp 1/g$ and then the contribution of the error term to (6.3.6) will be $\ll_\varepsilon q^{-\varepsilon g}$ for any $\theta > 0$. Thus, unconditionally for $\theta < 3/2$ and conditionally for any $\theta > 0$, we may write

$$\mathcal{M}(\alpha, P) = I(\alpha, P) + q^{-2g\alpha} I(-\alpha, P) + O(q^{-\varepsilon g}), \tag{6.3.9}$$

where

$$I(\alpha, P) = \zeta_q(1+2\alpha) \sum_{n\geq 1} \frac{p_n n!}{\log^n y} J_\alpha(y), \tag{6.3.10}$$

with

$$J_\alpha(y) = \frac{1}{2\pi i} \int_{(c)} \frac{y^z}{z^{n+1}} \frac{A(\alpha; z)}{\zeta_q(1+\alpha+z)} \, dz. \tag{6.3.11}$$

By moving the contour to $\mathrm{Re}(z) = -\delta$, where $\delta > 0$ is sufficiently small so that the Euler product is absolutely convergent, we have that $J_\alpha(y)$ is given by the residue at $z = 0$ plus the new integral along the line $\mathrm{Re}(z) = -\delta$. We write the residue at $z = 0$ as an integral where the contour is a circle around zero and we bound the new integral by absolute values to obtain

$$\left|\frac{1}{2\pi i}\int_{(-\delta)}\frac{y^z}{z^{n+1}}\frac{A(\alpha;z)}{\zeta_q(1+\alpha+z)}\,dz\right| \ll y^{-\delta}\int_{(-\delta)}\frac{1}{|z|^{n+1}}\,dz \ll y^{-\delta}\int_{-\infty}^{\infty}\frac{1}{|t|^{n+1}}\,dt, \tag{6.3.12}$$

where we have used that fact that the product $A(\alpha; z)$ is absolutely convergent on the line $\mathrm{Re}(z) = -\delta$ and so the factor $A(\alpha; z)\zeta_q(1 + \alpha + z)^{-1}$ is uniformly bounded. As $n \geq 1$, the last integral over t converges and so the integral over $\mathrm{Re}(z) = -\delta$ is bounded by $y^{-\delta} \ll q^{-\varepsilon g}$. We therefore have that

$$J_\alpha(y) = \frac{1}{2\pi i}\oint\frac{y^z}{z^{n+1}}\frac{A(\alpha;z)}{\zeta_q(1+\alpha+z)}\,dz + O(q^{-\varepsilon g}), \tag{6.3.13}$$

where the contour is now a circle of radius $\asymp 1/g$ around zero. On this circular contour, for $\alpha, z \ll 1/g$, we use the Taylor expansion

$$\frac{A(\alpha;z)}{\zeta_q(1+\alpha+z)} = (\alpha+z)A(0;0)\log q + O(g^{-2}). \tag{6.3.14}$$

From the definition in (6.2.3), we have that $A(0; 0) = 1$ which gives us

$$J_\alpha(y) = \frac{\log q}{2\pi i}\oint\frac{y^z}{z^{n+1}}\left((\alpha+z)+O(g^{-2})\right)\,dz = \frac{\log q}{2\pi i}\oint\frac{y^z}{z^{n+1}}(\alpha+z)\,dz + O(g^{n-2}), \tag{6.3.15}$$

where we have used the estimation lemma to bound the integral of the error term by

$$\left|\frac{g^{-2}}{2\pi i}\oint\frac{y^z}{z^{n+1}}\,dz\right| \ll g^{n-2}. \tag{6.3.16}$$

Next, by computing the residue at $z = 0$, we have that for $n \geq 1$,

$$\frac{1}{2\pi i}\oint\frac{y^z}{z^{n+1}}dz = \frac{\log^n y}{n!}. \tag{6.3.17}$$

Therefore

$$\sum_{n\geq 1}\frac{p_n n!}{\log^n y}\frac{1}{2\pi i}\oint\frac{y^z}{z^{n+1}}dz = \sum_{n\geq 1}p_n = P(1) \tag{6.3.18}$$

and

$$\sum_{n\geq 1}\frac{p_n n!}{\log^n y}\frac{1}{2\pi i}\oint\frac{y^z}{z^{n}}dz = \frac{1}{\log y}\sum_{n\geq 1}np_n = \frac{1}{\log y}P'(1). \tag{6.3.19}$$

Therefore, by combining (6.3.10), (6.3.15), (6.3.18) and (6.3.19), we have

$$I(\alpha;P) = \zeta_q(1+2\alpha)\log q\left(\alpha P(1) + \frac{1}{\log y}P'(1) + O(g^{-2})\right). \tag{6.3.20}$$

Lastly, for $\alpha \asymp 1/g$, we have the Laurent expansion

$$\zeta_q(1+2\alpha) = \frac{1}{2\alpha\log q} + O(1) = O(g), \tag{6.3.21}$$

and so we have

$$I(\alpha;P) = \frac{1}{2}P(1) + \frac{1}{2\alpha\log y}P'(1) + O(g^{-1}), \tag{6.3.22}$$

uniformly on any fixed annulus $|\alpha| \asymp 1/g$. Plugging this back into (6.3.9) yields

$$\mathcal{M}(\alpha;P) = \frac{1+q^{-2g\alpha}}{2}P(1) + \frac{1-q^{-2g\alpha}}{2\alpha\log y}P'(1) + O(g^{-1}), \tag{6.3.23}$$

with the assumption that $\alpha \asymp 1/g$. But since $\mathcal{M}(\alpha;P)$ and the main term above are holomorphic for $\alpha \ll 1/g$, the error term is also holomorphic in this region. By the maximum modulus principle, the bound on the error term also holds uniformly for $\alpha \ll 1/g$ which completes the proof. □

6.3.1 Proof of Theorem 6.2.4

We define

$$\mathcal{N}(\alpha;P) = \frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}} \xi(\tfrac{1}{2}+\alpha,\chi_D)M_y(\chi_D,P). \tag{6.3.24}$$

By definition, $\xi(\frac{1}{2}+\alpha,\chi_D) = q^{g\alpha}L(\frac{1}{2}+\alpha,\chi_D)$ and so by Proposition 6.3.1, we have

$$\mathcal{N}(\alpha;P) = \frac{q^{g\alpha}+q^{-g\alpha}}{2}P(1) + \frac{q^{g\alpha}-q^{-g\alpha}}{2\alpha\log y}P'(1) + O(g^{-1}), \tag{6.3.25}$$

uniformly for $\alpha \ll 1/g$. We now write $\alpha = \frac{a}{g\log q}$. Then, as $y = (q^{2g})^{\theta}$, the above can be rewritten as

$$\mathcal{N}\left(\frac{a}{g\log q};P\right) = P(1)\cosh a + P'(1)\frac{\sinh a}{2a\theta} + O(g^{-1}). \tag{6.3.26}$$

For an even polynomial Q, we have that

$$Q\left(\frac{d}{da}\right)\cosh a\big|_{a=0} = Q(1), \tag{6.3.27}$$

and

$$Q\left(\frac{d}{da}\right)\frac{\sinh a}{a}\bigg|_{a=0}=Q\left(\frac{d}{da}\right)\int_0^1\cosh at\,dt\bigg|_{a=0}=\int_0^1 Q(t)\,dt. \tag{6.3.28}$$

Thus, we have

$$Q\left(\frac{d}{da}\right)\mathcal{N}\left(\frac{a}{g\log q};P\right)\bigg|_{a=0}=P(1)Q(1)+\frac{1}{2\theta}P'(1)\int_0^1 Q(t)\,dt+O(g^{-1}). \tag{6.3.29}$$

By noting that $\frac{d}{da}=\frac{1}{g\log q}\frac{d}{d\alpha}$, we see that this is precisely the statement of Theorem 6.2.4.

6.4 The mollified second moment

In this section we will prove Theorem 6.2.5 on the mollified second moment. Similarly to the mollified first moment, we will first obtain a formula for the shifted mollified moment

$$\mathcal{M}(\alpha,\beta;P_1,P_2):=\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}L(\tfrac{1}{2}+\alpha,\chi_D)L(\tfrac{1}{2}+\beta,\chi_D)M_y(\chi_D,P_1)M_y(\chi_D,P_2). \tag{6.4.1}$$

Proposition 6.4.1. *For polynomials P_1,P_2 satisfying $P_j(0)=P_j'(0)=0$ for $j=1,2$, and for $\theta<1/2$, we have*

$$\begin{aligned}
&\mathcal{M}(\alpha,\beta;P_1,P_2)\\
&=\frac{1}{4}\left(\frac{\alpha\beta(1-q^{-2g(\alpha+\beta)})}{\alpha+\beta}+\frac{\alpha\beta(q^{-2g\alpha}-q^{-2g\beta})}{\alpha-\beta}\right)\log y\int_0^1 P_1(r)P_2(r)dr\\
&\quad+\frac{1}{4}(1+q^{-2g\alpha})(1+q^{-2g\beta})\int_0^1\big(P_1(r)P_2'(r)+P_1'(r)P_2(r)\big)dr\\
&\quad+\frac{1}{4}\left(\frac{(1+q^{-2g\alpha})(1-q^{-2g\beta})}{\beta}+\frac{(1-q^{-2g\alpha})(1+q^{-2g\beta})}{\alpha}\right)\frac{1}{\log y}\int_0^1 P_1'(r)P_2'(r)dr\\
&\quad+\frac{1}{4}\left(\frac{1-q^{-2g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{-2g\beta}-q^{-2g\alpha}}{\alpha-\beta}\right)\frac{1}{\log y}\int_0^1\big(P_1(r)P_2''(r)+P_1''(r)P_2(r)\big)dr\\
&\quad+\frac{(1-q^{-2g\alpha})(1-q^{-2g\beta})}{4\alpha\beta}\frac{1}{\log^2 y}\int_0^1\big(P_1'(r)P_2''(r)+P_1''(r)P_2'(r)\big)dr
\end{aligned}$$

$$+\frac{1}{4\alpha\beta}\left(\frac{1-q^{-2g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{-2g\alpha}-q^{-2g\beta}}{\alpha-\beta}\right)\frac{1}{\log^3 y}\int_0^1 P_1''(r)P_2''(r)dr+O(g^{-1}) \tag{6.4.2}$$

uniformly for $\alpha,\beta \ll 1/g$. Assuming Conjecture 6.2.1, the result holds for all $\theta>0$.

Proof. We write $P_1(x)=\sum_{m\geq 2} p_{1,m}x^m$ and $P_2(x)=\sum_{n\geq 2} p_{2,n}x^n$ and use (6.3.5) which gives us

$$\mathcal{M}(\alpha,\beta;P_1,P_2)=\sum_{m,n\geq 2}\frac{p_{1,m}m!p_{2,n}n!}{\log^{m+n} y}\frac{1}{(2\pi i)^2}\int_{(c)}\int_{(c)}\frac{y^{w+z}}{w^{m+1}z^{n+1}}$$
$$\times\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\frac{L(\frac{1}{2}+\alpha,\chi_D)L(\frac{1}{2}+\beta,\chi_D)}{L(\frac{1}{2}+w,\chi_D)L(\frac{1}{2}+z,\chi_D)}\,dw\,dz, \tag{6.4.3}$$

where $c>0$. For $\alpha,\beta\ll 1/g$ and $g^{-1/4+\varepsilon}\ll c<1/2$, Proposition 6.2.3 gives us that the average of the ratio of L-functions in the integrand is given by Conjecture 6.2.1 with an error that is $\ll_\varepsilon q^{-2cg+\varepsilon cg}$. By bounding the integral with absolute values, the contribution of this error to (6.4.3) will be

$$\ll_\varepsilon y^{2c}q^{-2cg+\varepsilon cg}\ll_\varepsilon q^{2cg(2\theta-1+\varepsilon/2)}. \tag{6.4.4}$$

For $\theta<1/2$, we may bound this error by $O(q^{-\varepsilon g})$ for sufficiently small $\varepsilon>0$. Assuming Conjecture 6.2.1, the error in the ratio of L-functions is $\ll_\varepsilon q^{-\varepsilon g}$ and we may take $c\asymp 1/g$. This allows us to bound the error in (6.4.3) by $O(q^{-\varepsilon g})$ for any $\theta>0$. In either case, by inserting the main terms of the Ratios Conjecture into (6.4.3), we write

$$\mathcal{M}(\alpha,\beta;P_1,P_2)=I(\alpha,\beta)+q^{-2g\alpha}I(-\alpha,\beta)+q^{-2g\beta}I(\alpha,-\beta)+q^{-2g(\alpha+\beta)}I(-\alpha,-\beta)$$
$$+O(q^{-\varepsilon g}), \tag{6.4.5}$$

where

$$I(\alpha,\beta)=\zeta_q(1+2\alpha)\zeta_q(1+2\beta)\zeta_q(1+\alpha+\beta)\sum_{m,n\geq 2}\frac{p_{1,m}m!p_{2,n}n!}{(\log y)^{m+n}}J_{\alpha,\beta}(y), \tag{6.4.6}$$

with

$$J_{\alpha,\beta}(y) = \frac{1}{(2\pi i)^2} \int_{(c)} \int_{(c)} \frac{y^{w+z}}{w^{m+1} z^{n+1}} \times \frac{\zeta_q(1+w+z) A(\alpha,\beta;w,z)}{\zeta_q(1+\alpha+w)\zeta_q(1+\alpha+z)\zeta_q(1+\beta+w)\zeta_q(1+\beta+z)} \, dw \, dz. \tag{6.4.7}$$

To evaluate the contour integral $J_{\alpha,\beta}(y)$, we would like to move the contours to the left of 0 but the factor of $\zeta_q(1+w+z)$ with its pole at $w+z=0$ makes this somewhat tricky. To deal with this, we use the fact that for $\mathrm{Re}(w+z) > 0$, we may write

$$\frac{y^{w+z}}{(w+z)} = \int_0^y u^{w+z} \frac{du}{u}. \tag{6.4.8}$$

Thus, we have

$$J_{\alpha,\beta}(y) = \frac{1}{(2\pi i)^2} \int_1^y \int_{(c)} \int_{(c)} \frac{u^{w+z}}{w^{m+1} z^{n+1}} \times \frac{(w+z)\zeta_q(1+w+z) A(\alpha,\beta;w,z)}{\zeta_q(1+\alpha+w)\zeta_q(1+\alpha+z)\zeta_q(1+\beta+w)\zeta_q(1+\beta+z)} \, dw \, dz \frac{du}{u}, \tag{6.4.9}$$

with the integration in u only over $1 \leq u \leq y$ since if $u < 1$, we can move the contours far to the right to see that the integrals in w and z vanish. As $(w+z)\zeta_q(1+w+z)$ is analytic at $w+z=0$, the poles of the integrand are now at $w=0$ or $z=0$ only. So, by moving the contours to $\mathrm{Re}(w) = \mathrm{Re}(z) = -\delta$ where $\delta > 0$ is sufficiently small to ensure that the Euler product is absolutely convergent, we get that $J_{\alpha,\beta}(y)$ is given by the residue at $w = z = 0$ plus the new integral over the lines $\mathrm{Re}(w) = \mathrm{Re}(z) = -\delta$. The residue at $w = z = 0$ may be written as an integral with the contours being circles around zero and the new integral may be bounded by absolute values similarly to (6.3.12). Using the fact that the Euler product and zeta factors are uniformly bounded on the lines of integration, we have

$$\begin{aligned} &\Bigg| \int_1^y \int_{(-\delta)} \int_{(-\delta)} \frac{u^{w+z}}{w^{m+1} z^{n+1}} \\ &\qquad \times \frac{(w+z)\zeta_q(1+w+z) A(\alpha,\beta;w,z)}{\zeta_q(1+\alpha+w)\zeta_q(1+\alpha+z)\zeta_q(1+\beta+w)\zeta_q(1+\beta+z)} \, dw \, dz \frac{du}{u} \Bigg| \\ &\ll \int_1^y u^{-2\delta} \int_{(-\delta)} \int_{(-\delta)} \frac{|w+z|}{|w|^{m+1}|z|^{n+1}} \, dw \, dz \frac{du}{u} \end{aligned}$$

$$\ll \int_1^y u^{-2\delta} \int_{(-\delta)} \int_{(-\delta)} \left(\frac{1}{|w|^m |z|^{n+1}} + \frac{1}{|w|^{m+1}|z|^n} \right) dw\, dz \frac{du}{u}$$
$$\ll \int_1^y u^{-2\delta} \frac{du}{u} \ll 1. \tag{6.4.10}$$

In the second to last line above, we have used the fact that the integrals over w and z are convergent since $m, n \geq 2$. This is why we need the condition that $P_j(0) = P_j'(0) = 0$.

We express the residue at $w = z = 0$ as an integral with the contours being circles of radius $\asymp 1/g$ around the zero. Then, for $w, z, \alpha, \beta \ll 1/g$, we approximate the integrand on these contours via the Taylor expansion

$$\frac{(w+z)\zeta_q(1+w+z)A(\alpha,\beta;w,z)}{\zeta_q(1+\alpha+w)\zeta_q(1+\alpha+z)\zeta_q(1+\beta+w)\zeta_q(1+\beta+z)}$$
$$= (\alpha+w)(\alpha+z)(\beta+w)(\beta+z)A(0,0;0,0)\log^3 q + O(g^{-5}). \tag{6.4.11}$$

Since $A(0,0;0,0) = 1$ as can be seen by the definition, this gives us

$$J_{\alpha,\beta}(y) = \frac{\log^3 q}{(2\pi i)^2} \int_1^y \oint \oint \frac{u^{w+z}}{w^{m+1}z^{n+1}}$$
$$\times \left((\alpha+w)(\alpha+z)(\beta+w)(\beta+z) + O(g^{-5}) \right) dw\, dz \frac{du}{u}$$
$$= \frac{\log^3 q}{(2\pi i)^2} \int_1^y \left(\oint \frac{u^w}{w^{m+1}} (\alpha+w)(\beta+w)\, dw \right) \left(\oint \frac{u^z}{z^{n+1}} (\alpha+z)(\beta+z)\, dz \right) \frac{du}{u}$$
$$+ O(g^{m+n-4}), \tag{6.4.12}$$

where we again use the estimation lemma to bound the integral of the error term by

$$\left| \frac{g^{-5}}{(2\pi i)^3} \int_1^y \oint \oint \frac{u^{w+z}}{w^{m+1}z^{n+1}} dw\, dz \frac{du}{u} \right| \ll g^{m+n-5} \int_1^y u^{1/g} \frac{du}{u} \ll g^{m+n-4}. \tag{6.4.13}$$

By generalising (6.3.18) and (6.3.19), for $i \in \{0,1\}$ and $j \in \{0,1,2\}$, we have that

$$\sum_{n \geq 1} \frac{p_{i,n} n!}{\log^n y} \frac{1}{2\pi i} \oint \frac{y^z}{z^{n+1-j}} = \frac{1}{\log^j y} P_i^{(j)}(1). \tag{6.4.14}$$

Hence, by combining (6.4.6), (6.4.12) and (6.4.14), we have

$$
\begin{aligned}
I(\alpha,\beta) &= \zeta_q(1+2\alpha)\zeta_q(1+2\beta)\zeta_q(1+\alpha+\beta)\log^3 q \\
&\times \Bigg[\int_1^y \left(\alpha\beta P_1\left(\frac{\log u}{\log y}\right) + \frac{\alpha+\beta}{\log y} P_1'\left(\frac{\log u}{\log y}\right) + \frac{1}{\log^2 y} P_1''\left(\frac{\log u}{\log y}\right)\right) \\
&\times \left(\alpha\beta P_2\left(\frac{\log u}{\log y}\right) + \frac{\alpha+\beta}{\log y} P_2'\left(\frac{\log u}{\log y}\right) + \frac{1}{\log^2 y} P_2''\left(\frac{\log u}{\log y}\right)\right) \frac{du}{u} + O(g^{-4}) \Bigg]
\end{aligned} \tag{6.4.15}
$$

By making the change of variables $u = y^r$, the integral in the above becomes

$$
\begin{aligned}
\log y \int_0^1 &\left(\alpha\beta P_1(r) + \frac{\alpha+\beta}{\log y} P_1'(r) + \frac{1}{\log^2 y} P_1''(r) \right) \\
&\times \left(\alpha\beta P_2(r) + \frac{\alpha+\beta}{\log y} P_2'(r) + \frac{1}{\log^2 y} P_2''(r) \right) dr.
\end{aligned} \tag{6.4.16}
$$

Now, for $\alpha, \beta \asymp 1/g$ and $|\alpha+\beta| \gg 1/g$, we have the Laurent expansion

$$
\zeta_q(1+2\alpha)\zeta_q(1+2\beta)\zeta_q(1+\alpha+\beta) = \frac{1}{4\alpha\beta(\alpha+\beta)\log^3 q} + O(g^2) = O(g^3). \tag{6.4.17}
$$

Hence, we have that

$$
\begin{aligned}
I(\alpha,\beta) = \frac{\log y}{4\alpha\beta(\alpha+\beta)} \int_0^1 &\left(\alpha\beta P_1(r) + \frac{\alpha+\beta}{\log y} P_1'(r) + \frac{1}{\log^2 y} P_1''(r) \right) \\
&\times \left(\alpha\beta P_2(r) + \frac{\alpha+\beta}{\log y} P_2'(r) + \frac{1}{\log^2 y} P_2''(r) \right) dr + O(g^{-1}),
\end{aligned} \tag{6.4.18}
$$

uniformly on any fixed annuli such that $\alpha, \beta \asymp 1/g$ and $|\alpha+\beta| \gg 1/g$. Returning to (6.4.5) with this formula and collecting similar terms gives us

$$
\begin{aligned}
&\mathcal{M}(\alpha,\beta;P_1,P_2) \\
&= \frac{1}{4}\left(\frac{\alpha\beta(1-q^{-2g(\alpha+\beta)})}{\alpha+\beta} + \frac{\alpha\beta(q^{-2g\alpha}-q^{-2g\beta})}{\alpha-\beta} \right) \log y \int_0^1 P_1(r)P_2(r)dr \\
&\quad + \frac{1}{4}(1+q^{-2g\alpha})(1+q^{-2g\beta}) \int_0^1 \big(P_1(r)P_2'(r) + P_1'(r)P_2(r) \big) dr
\end{aligned}
$$

$$+\frac{1}{4}\left(\frac{(1+q^{-2g\alpha})(1-q^{-2g\beta})}{\beta}+\frac{(1-q^{-2g\alpha})(1+q^{-2g\beta})}{\alpha}\right)\frac{1}{\log y}\int_0^1 P_1'(r)P_2'(r)dr$$
$$+\frac{1}{4}\left(\frac{1-q^{-2g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{-2g\beta}-q^{-2g\alpha}}{\alpha-\beta}\right)\frac{1}{\log y}\int_0^1 \big(P_1(r)P_2''(r)+P_1''(r)P_2(r)\big)dr$$
$$+\frac{(1-q^{-2g\alpha})(1-q^{-2g\beta})}{4\alpha\beta}\frac{1}{\log^2 y}\int_0^1 \big(P_1'(r)P_2''(r)+P_1''(r)P_2'(r)\big)dr$$
$$+\frac{1}{4\alpha\beta}\left(\frac{1-q^{-2g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{-2g\alpha}-q^{-2g\beta}}{\alpha-\beta}\right)\frac{1}{\log^3 y}\int_0^1 P_1''(r)P_2''(r)dr+O(g^{-1}). \tag{6.4.19}$$

Finally, since both $\mathcal{M}(\alpha,\beta,P_1,P_2)$ and the main term on the right-hand side above are holomorphic for $\alpha,\beta\ll 1/g$, the error term is also holomorphic in this region. By the maximum modulus principle, the above formula holds uniformly for $\alpha,\beta\ll 1/g$ which completes the proof of the proposition.

□

6.4.1 Proof of Theorem 6.2.5

We define

$$\mathcal{N}(\alpha,\beta;P_1,P_2)=\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\xi(\tfrac{1}{2}+\alpha,\chi_D)\xi(\tfrac{1}{2}+\beta,\chi_D)M_y(\chi_D,P_1)M_y(\chi_D,P_2). \tag{6.4.20}$$

By the definition of the completed L-function, we have $\xi(\frac{1}{2}+\alpha,\chi_D)\xi(\frac{1}{2}+\beta,\chi_D)=q^{g(\alpha+\beta)}L(\frac{1}{2}+\alpha,\chi_D)L(\frac{1}{2}+\beta,\chi_D)$. Therefore, by Proposition 6.4.1, we have that

$$\mathcal{N}(\alpha,\beta;P_1,P_2)$$
$$=\frac{\log y}{4}\left(\frac{\alpha\beta(q^{g(\alpha+\beta)}-q^{-g(\alpha+\beta)})}{\alpha+\beta}+\frac{\alpha\beta(q^{g(\beta-\alpha)}-q^{g(\alpha-\beta)})}{\alpha-\beta}\right)\int_0^1 P_1(r)P_2(r)dr$$
$$+\frac{1}{4}(q^{g\alpha}+q^{-g\alpha})(q^{g\beta}+q^{-g\beta})\int_0^1 \big(P_1(r)P_2'(r)+P_1'(r)P_2(r)\big)dr$$
$$\frac{1}{4\log y}\left(\frac{(q^{g\alpha}+q^{-g\alpha})(q^{g\beta}-q^{-g\beta})}{\beta}+\frac{(q^{g\beta}-q^{-g\alpha})(q^{g\beta}+q^{-g\beta})}{\alpha}\right)\int_0^1 P_1'(r)P_2'(r)dr$$
$$+\frac{1}{4\log y}\left(\frac{q^{g(\alpha+\beta)}-q^{-g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{g(\alpha-\beta)}-q^{g(\beta-\alpha)}}{\alpha-\beta}\right)\int_0^1 \big(P_1(r)P_2''(r)+P_1''(r)P_2(r)\big)dr$$
$$+\frac{(q^{g\alpha}-q^{-g\alpha})(q^{g\beta}-q^{-g\beta})}{4\alpha\beta}\frac{1}{\log^2 y}\int_0^1 \big(P_1'(r)P_2''(r)+P_1''(r)P_2'(r)\big)dr$$

$$+\frac{1}{4\alpha\beta}\left(\frac{q^{g(\alpha+\beta)}-q^{-g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{g(\beta-\alpha)}-q^{g(\alpha-\beta)}}{\alpha-\beta}\right)\frac{1}{\log^3 y}\int_0^1 P_1''(r)P_2''(r)dr+O(g^{-1}), \tag{6.4.21}$$

uniformly for $\alpha,\beta\ll 1/g$. Next, we scale the variables by writing $\alpha=\frac{a}{g\log q}$ and $\beta=\frac{b}{g\log q}$. Then, as $y=q^{2g\theta}$, the above can be rewritten as

$$\begin{aligned}\mathcal{N}(\alpha,\beta;P_1,P_2)=&\frac{1}{8\theta^3}\int_0^1\frac{\sinh au}{a}\frac{\sinh bu}{b}du\int_0^1 P_1''(r)P_2''(r)dr\\&+\frac{1}{4\theta^2}\frac{\sinh a}{a}\frac{\sinh b}{b}\int_0^1\big(P_1'(r)P_2''(r)+P_1''(r)P_2'(r)\big)dr\\&+\frac{1}{2\theta}\int_0^1\cosh au\cosh bu\,du\int_0^1\big(P_1(r)P_2''(r)+P_1''(r)P_2(r)\big)dr\\&+\frac{1}{2\theta}\left(\frac{\sinh a\cosh b}{a}+\frac{\sinh b\cosh a}{b}\right)\int_0^1 P_1'(r)P_2'(r)dr\\&+\cosh a\cosh b\int_0^1\big(P_1(r)P_2'(r)+P_1'(r)P_2(r)\big)dr\\&+2\theta ab\int_0^1\sinh au\sinh bu\,du\int_0^1 P_1(r)P_2(r)dr+O(g^{-1}).\end{aligned} \tag{6.4.22}$$

We now also define

$$\mathcal{N}(Q_1,Q_2;P_1,P_2):=Q_1\left(\frac{d}{da}\right)Q_2\left(\frac{d}{db}\right)\mathcal{N}\left(\frac{a}{g\log q},\frac{b}{g\log q},P_1,P_2\right)\bigg|_{a=b=0}. \tag{6.4.23}$$

For an even polynomial Q, we have

$$Q\left(\frac{d}{da}\right)\cosh a\big|_{a=0}=Q(1), \tag{6.4.24}$$

$$Q\left(\frac{d}{da}\right)\frac{\sinh au}{a}\bigg|_{a=0}=Q\left(\frac{d}{da}\right)\int_0^u\cosh at\,dt\bigg|_{a=0}=\int_0^u Q(t)\,dt, \tag{6.4.25}$$

and

$$Q\left(\frac{d}{da}\right)a\sinh au\big|_{a=0}=Q\left(\frac{d}{da}\right)\frac{d}{dt}\cosh at\bigg|_{a=0}=\frac{d}{dt}Q(t)=Q'(t). \tag{6.4.26}$$

Using the above formulae, we have

$$\begin{aligned}\mathcal{N}(Q_1, Q_2; P_1, P_2) = &\frac{1}{8\theta^3}\int_0^1 \tilde{Q}_1(u)\tilde{Q}_2(u)du \int_0^1 P_1''(r)P_2''(r)dr \\ &+ \frac{1}{4\theta^2}\tilde{Q}_1(1)\tilde{Q}_2(1)\int_0^1 \big(P_1'(r)P_2''(r) + P_1''(r)P_2'(r)\big)dr \\ &+ \frac{1}{2\theta}\int_0^1 Q_1(u)Q_2(u)du \int_0^1 \big(P_1(r)P_2''(r) + P_1''(r)P_2(r)\big)dr \\ &+ \frac{1}{2\theta}\Big(Q_1(1)\tilde{Q}_2(1) + \tilde{Q}_1(1)Q_2(1)\Big)\int_0^1 P_1'(r)P_2'(r)dr \\ &+ Q_1(1)Q_2(1)\int_0^1 \big(P_1(r)P_2'(r) + P_1'(r)P_2(r)\big)dr \\ &+ 2\theta\int_0^1 Q_1'(u)Q_2'(u)du \int_0^1 P_1(r)P_2(r)dr + O(g^{-1}), \quad (6.4.27)\end{aligned}$$

where

$$\tilde{Q}(u) = \int_0^u Q(t)\,dt. \tag{6.4.28}$$

To complete the proof we write the above expression for $\mathcal{N}(P_1, P_2; Q_1, Q_2)$ in the more compact form given in the statement of Theorem 6.2.5. This requires the following identities which all follow from integration by parts:

$$\int_0^1 \big(P_1'(r)P_2(r) + P_1(r)P_2'(r)\big)\,dr = P_1(1)P_2(1), \tag{6.4.29}$$

$$\int_0^1 \big(P_1''(r)P_2'(r) + P_1'(r)P_2''(r)\big)\,dr = P_1'(1)P_2'(1), \tag{6.4.30}$$

$$\int_0^1 P_1''(r)P_2(r)\,dr = P_1'(1)P_2(1) - \int_0^1 P_1'(r)P_2'(r)\,dr, \tag{6.4.31}$$

and

$$\int_0^1 \tilde{Q}_1(u)Q_2'(u)\,du = \tilde{Q}(1)Q_2(1) - \int_0^1 Q_1(u)Q_2(u)\,du. \tag{6.4.32}$$

where the last identity uses the fact that $\tilde{Q}(0) = 0$. Using these identities, it follows that

$$\mathcal{N}(Q_1, Q_2; P_1; P_2)$$

$$
\begin{aligned}
=&2\theta \int_0^1 \int_0^1 P_1(r)Q_1'(u)P_2(r)Q_2'(u)\, du\, dr + P_1(1)P_2(1)Q_1(1)Q_2(1) \\
&+ \frac{1}{2\theta}\big(P_1(1)Q_1(1)P_2'(1)\tilde{Q}_2(1) + P_1'(1)\tilde{Q}_1(1)P_2(1)Q_2(1)\big) \\
&- \frac{1}{2\theta}\Big(\int_0^1 \int_0^1 \big(P_1(r)Q_1'(u)P_2''(r)\tilde{Q}_2(u) + P_1''(r)\tilde{Q}_1(u)P_2(r)Q_2'(u)\big)\, du\, dr\Big) \\
&+ \frac{1}{4\theta^2}\tilde{Q}_1(1)\tilde{Q}_2(1)P_1'(1)P_2'(1) + \frac{1}{8\theta^3}\int_0^1 \int_0^1 P_1''(r)\tilde{Q}_1(u)P_2''(r)\tilde{Q}_2(u)\, du\, dr + O(g^{-1}).
\end{aligned}
\tag{6.4.33}
$$

The expression on the right may then be factorised which yields the statement of Theorem 6.2.5.

6.5 Non-vanishing of $\xi^{(2k)}(\frac{1}{2}, \chi_D)$

In this section we will prove Theorem 6.2.9. The strategy is to take $Q(x) = x^{2k}$ in Theorem 6.2.7 and then find the optimal polynomial $P = P_{2k}$ to minimise the ratio $\mathcal{R}(P, x^{2k})$. Although we are ultimately taking $Q(x) = x^{2k}$, much of this optimisation works more generally so we will often leave expressions in terms of Q. A similar optimisation process was carried out in Section 7 of [KMV00].

For any polynomial Q, we denote

$$
I(Q) = \int_0^1 Q(t)\, dt. \tag{6.5.1}
$$

Recalling the expression for the ratio $\mathcal{R}(P, Q)$ given in Theorem 6.2.7, we see that we need to minimize

$$
\mathcal{R}(P, Q) = \frac{\int_0^1 \big((2\theta)^{-1}I(\tilde{Q}^2)P''(r)^2 - 4\theta I(\tilde{Q}Q')P(r)P''(r) + 8\theta^3 I(Q'^2)P(r)^2\big)\, dr}{\big(2\theta P(1)Q(1) + P'(1)\tilde{Q}(1)\big)^2} \tag{6.5.2}
$$

over all Taylor series $P(x) = a_2x^2 + a_3x^3 + \dots$ such that this series and the series for everything up to and including $P''(x)^2$ are absolutely convergent on $[0, 1]$.

We proceed by first assuming that there does exist a P which minimises $\mathcal{R}(P, Q)$. Then, since we are at a minimum, the derivative of $\mathcal{R}(P + \varepsilon f, Q)$ with respect to ε must be zero at $\varepsilon = 0$ for any f which satisfies the same conditions as P. By directly taking the derivative of $\mathcal{R}(P + \varepsilon f, Q)$ with respect to ε at $\varepsilon = 0$, setting this equal to zero and then collecting like terms, we find that for all such f, P must satisfy

$$
\begin{aligned}
\big(2\theta f(1)Q(1) &+ f'(1)\tilde{Q}(1)\big)\big(2\theta P(1)Q(1) + P'(1)\tilde{Q}(1)\big)\mathcal{R}(P,Q) \\
&= \int_0^1 f''(r)\big((2\theta)^{-1}I(\tilde{Q}^2)P''(r) - 2\theta I(\tilde{Q}Q')P(r)\big)\,dr \\
&\quad + \int_0^1 f(r)\big(8\theta^3 I(Q'^2)P(r) - 2\theta I(\tilde{Q}Q')P''(r)\big)\,dr. \qquad (6.5.3)
\end{aligned}
$$

Note that $\mathcal{R}(P,Q)$ is invariant if we scale P by a constant. Hence, by scaling P suitably, we may assume that $\big(2\theta P(1)Q(1) + P'(1)\tilde{Q}(1)\big)\mathcal{R}(P,Q) = 1$ in the above. Next, let $\Pi(x)$ be a function with absolutely convergent Taylor series on $[0,1]$ such that $\Pi''(x) = P(x)$. In particular, this implies that the coefficient of x^2 and x^3 in $\Pi(x)$ are zero. Then, using integration by parts twice, we may write

$$
\begin{aligned}
&\int_0^1 f(r)\big(8\theta^3 I(Q'^2)P(r) - 2\theta I(\tilde{Q}Q')P''(r)\big)\,dr \\
&= f(1)\big(8\theta^3 I(Q'^2)\Pi'(1) - 2\theta I(\tilde{Q}Q')\Pi'''(1)\big) \\
&\quad + f'(1)\big(2\theta I(\tilde{Q}Q')\Pi''(1) - 8\theta^3 I(Q'^2)\Pi(1)\big) \\
&\quad + \int_0^1 f''(r)\big(8\theta^3 I(Q'^2)\Pi(r) - 2\theta I(\tilde{Q}Q')\Pi''(r)\big)\,dr, \qquad (6.5.4)
\end{aligned}
$$

where we have used the fact that $f(0) = f'(0) = 0$. Plugging this back into (6.5.3) shows us that P must satisfy

$$
\begin{aligned}
&2\theta f(1)Q(1) + f'(1)\tilde{Q}(1) \\
&= f(1)\big(8\theta^3 I(Q'^2)\Pi'(1) - 2\theta I(\tilde{Q}Q')\Pi'''(1)\big) \\
&\quad + f'(1)\big(2\theta I(\tilde{Q}Q')\Pi''(1) - 8\theta^3 I(Q'^2)\Pi(1)\big) \\
&\quad + \int_0^1 f''(r)\big((2\theta)^{-1}I(\tilde{Q}^2)\Pi''''(r) - 4\theta I(\tilde{Q}Q')\Pi''(r) + 8\theta^3 I(Q'^2)\Pi(r)\big)\,dr. \quad (6.5.5)
\end{aligned}
$$

Now, we consider the expression

$$
\frac{1}{2\theta}I(\tilde{Q}^2)\Pi''''(x) - 4\theta I(\tilde{Q}Q')\Pi''(x) + 8\theta^3 I(Q'^2)\Pi(x) \qquad (6.5.6)
$$

appearing in the last integrand. Suppose, for instance, that the coefficient of x^2 in the Taylor expansion of (6.5.6) is $c_2 \neq 0$. Then the contribution of this term to the integral in (6.5.6) will be

$$c_2 \int_0^1 f''(r) r^2 \, dr = c_2 f'(1) - 2c_2 \int_0^1 f'(r) r \, dr = c_2 f'(1) - 2c_2 f(1) + 2c_2 \int_0^1 f(r) \, dr. \tag{6.5.7}$$

For (6.5.5) to hold for all f, the coefficient of $\int_0^1 f(r)\,dr$ must be zero since the coefficient is zero on the left-hand side. We therefore necessarily have $c_2 = 0$. Similarly, the coefficients of all the terms x^n with $n > 2$ must also be zero and we have that the expression in (6.5.6) must be of the form $c_0 + c_1 x$. Thus, to find Π and therefore P, we must solve the differential equation

$$\frac{1}{2\theta} I(\tilde{Q}^2)\Pi''''(x) - 4\theta I(\tilde{Q}Q')\Pi''(x) + 8\theta^3 I(Q'^2)\Pi(x) = c_0 + c_1 x, \tag{6.5.8}$$

for some c_0, c_1.

In the case $k = 0$, we have $Q(x) = 1$ and $Q'(x) = 0$. Consequently, the differential equation becomes

$$\frac{1}{6\theta}\Pi''''(x) = c_0 + c_1 x. \tag{6.5.9}$$

By integrating four times, we see that the solution to this is $\Pi(x)$ given by a degree 5 polynomial, say

$$\Pi(x) = a_0 + a_1 x + a_4 x^4 + a_5 x^5. \tag{6.5.10}$$

We may then substitute this $\Pi(x)$ and $Q(x) = 1$ back into (6.5.5) to get

$$\begin{aligned} 2\theta f(1) + f'(1) &= \frac{1}{6\theta}\int_0^1 f''(r)\big(24a_4 + 120a_5 r\big)\,dr \\ &= \frac{4a_4}{\theta} f'(1) + \frac{20a_5}{\theta} f'(1) - \frac{20a_5}{\theta} f(1). \end{aligned} \tag{6.5.11}$$

From this we read off the solution $a_5 = -\theta^2/10$ and $a_4 = \theta(2\theta+1)/4$. Differentiating $\Pi(x)$ twice to obtain $P(x)$, we find that the optimal polynomial to take in the case $k = 0$ is, up to scaling by a constant,

$$P_0(x) = 3\theta(2\theta+1)x^2 - 2\theta^2 x^3. \tag{6.5.12}$$

With $\theta = 1/2$, the optimal choice is then $P_0(x) = 3x^2 - x^3/2$ and we compute that

$$\mathcal{R}(P_0(x), 1) = \frac{1}{7}. \tag{6.5.13}$$

This gives us the lower bound on the proportion p_0 of $7/8 = 0.875$.

Returning to the general case of $k \geq 1$, we solve the differential equation (6.5.8) by considering its characteristic polynomial (see, for instance Chapter 4 of [TP85])

$$\frac{1}{2\theta} I(\tilde{Q}^2) X^4 - 4\theta I(\tilde{Q}Q') X^2 + 8\theta^3 I(Q'^2). \tag{6.5.14}$$

Using Mathematica, we may factor this polynomial and find that its roots are $\pm\alpha \pm i\beta$, where

$$\begin{aligned} \alpha &= \frac{2\theta}{\sqrt{2I(\tilde{Q}^2)}} \sqrt{\sqrt{I(\tilde{Q}^2) I(Q'^2)} + I(\tilde{Q}Q')} \\ &= 2\theta \sqrt{k(2k+1)\left(\sqrt{\frac{4k+3}{4k-1}} + \frac{4k+3}{4k+1}\right)}, \end{aligned} \tag{6.5.15}$$

and

$$\begin{aligned} \beta &= \frac{2\theta}{\sqrt{2I(\tilde{Q}^2)}} \sqrt{\sqrt{I(\tilde{Q}^2) I(Q'^2)} - I(\tilde{Q}Q')} \\ &= 2\theta \sqrt{k(2k+1)\left(\sqrt{\frac{4k+3}{4k-1}} - \frac{4k+3}{4k+1}\right)}. \end{aligned} \tag{6.5.16}$$

Thus, the solution to the homogeneous differential equation

$$\frac{1}{2\theta} I(\tilde{Q}^2) \Pi''''(x) - 4\theta I(\tilde{Q}Q') \Pi''(x) + 8\theta^3 I(Q'^2) \Pi(x) = 0 \tag{6.5.17}$$

is $\Pi(x)$ equal to a linear combination of $e^{(\pm\alpha \pm i\beta)x}$. The particular solution to (6.5.8) is

$$\Pi(x) = a_0 + a_1 x, \tag{6.5.18}$$

where $a_j = \left(8\theta^3 I(Q'^2)\right)^{-1} c_j$. Adding these two solutions together gives us the general form of a solution $\Pi(x)$ to (6.5.8). Differentiating $\Pi(x)$ twice to obtain $P(x)$ then eliminates the $a_0 + a_1 x$ terms of $\Pi(x)$ and we conclude that $P(x)$ must be given by a linear combination of the terms $e^{(\pm\alpha \pm i\beta)x}$, i.e.

$$P(x) = v_1 e^{(\alpha+i\beta)x} + v_2 e^{(\alpha-i\beta)x} + v_3 e^{(-\alpha+i\beta)x} + v_4 e^{(-\alpha-i\beta)x} \tag{6.5.19}$$

for some $v_1, \dots, v_4 \in \mathbb{C}$. Since we have $P(0) = P'(0) = 0$, we know that

$$v_1 + v_2 + v_3 + v_4 = 0, \tag{6.5.20}$$

and

$$v_1(\alpha + i\beta) + v_2(\alpha - i\beta) + v_3(-\alpha + i\beta) + v_4(-\alpha - i\beta) = 0. \tag{6.5.21}$$

This allows us to eliminate the variables v_3 and v_4 and with the help of Mathematica, we get that $P(x)$ satisfies

$$\begin{aligned}\beta P(x) = {} & 2\sinh(\alpha x)\big(\beta(v_1 + v_2)\cos(\beta x) + i\beta(v_1 - v_2)\sin(\beta x)\big) \\ & + 2\alpha(v_1 + v_2)\sin(\beta x)\big(\sinh(\alpha x) - \cosh(\alpha x)\big).\end{aligned} \tag{6.5.22}$$

We simplify this using $2(\sinh(\alpha x) - \cosh(\alpha x)) = -2e^{-\alpha x}$ and since we now have two variables and only one condition for P to satisfy, namely (6.5.5), we may scale the variables and assume that $v_1 + v_2 = 1$. Then we have that, up to scaling, P is of the form

$$P(x) = \sinh(\alpha x)\big(\cos(\beta x) - Y\sin(\beta x)\big) - \frac{\alpha}{\beta} e^{-\alpha x}\sin(\beta x), \tag{6.5.23}$$

where we set $Y = -i(v_1 - v_2)$. We can then solve for Y by substituting this P back into (6.5.5).

At this point we show that the $P(x)$ we have found above is in fact the unique choice which minimises $\mathcal{R}(P, Q)$. Suppose for a contradiction that there is some function F such that $\mathcal{R}(F, Q) < \mathcal{R}(P, Q)$. Recall that our initial condition on P was that the derivative of $\mathcal{R}(P + \varepsilon f, Q)$ must be zero at $\varepsilon = 0$ for any allowable f. In this case, this implies that the derivative of

$$\mathcal{R}(\varepsilon) := \mathcal{R}(P + \varepsilon(F - P), Q) \tag{6.5.24}$$

with respect to ε is zero at $\varepsilon = 0$ but that $\mathcal{R}(1) < \mathcal{R}(0)$. As a function of ε, we have that by the definition of $\mathcal{R}(P, Q)$,

$$\mathcal{R}(\varepsilon) = \frac{a_0 + a_1\varepsilon + a_2\varepsilon^2}{(b_0 + b_1\varepsilon)^2} \tag{6.5.25}$$

is a ratio of non-negative quadratics for certain constants a_j and b_j. Now, if $b_0 + b_1\varepsilon$ does not divide the numerator, then this rational function has a unique minimum lying below its horizontal asymptote at a_2/b_1^2. The location of this minimum is given by $\mathcal{R}'(\varepsilon) = 0$ but we have that $\mathcal{R}'(0) = 0$. It follows that $\varepsilon = 0$ gives the unique minimum of $\mathcal{R}(\varepsilon)$ and thus we cannot have $\mathcal{R}(1) < \mathcal{R}(0)$. Alternatively, if $b_0 + b_1\varepsilon$ does divide the numerator, then $\mathcal{R}(\varepsilon)$ is given by a ratio of linear polynomials and the fact that $\mathcal{R}'(0) = 0$ implies that $\mathcal{R}(\varepsilon)$ must in fact be a constant. This then implies that there are infinitely many distinct functions P which minimise $\mathcal{R}(P, Q)$. However, we have shown that there is only one function P which satisfies the differential equation (6.5.8) and so this cannot be the case. Consequently, we have that $\mathcal{R}(P, Q)$ is indeed minimised by the $P(x)$ found above.

We now return to solving for the variable Y to find the precise choice of P we are after. First, we observe that since $\Pi(x)$ satisfies the differential equation (6.5.8), the integral in (6.5.5) becomes

$$\begin{aligned}\int_0^1 f''(r)\big(c_0 + c_1 r\big)\, dr &= c_0 f'(1) + c_1 f'(1) - c_1 f(1) \\ &= 8\theta^3 I(Q'^2)\left((a_0 + a_1)f'(1) - a_1 f(1)\right). \end{aligned} \tag{6.5.26}$$

On the other hand, the contribution of the $a_0 + a_1 x$ terms in $\Pi(x)$ to the terms $8\theta^3 I(Q'^2)f(1)\Pi'(1)$ and $-8\theta^3 I(Q'^2)f'(1)\Pi(1)$ on the right of (6.5.5) will be

$$8a_1\theta^3 I(Q'^2)f(1) - 8(a_0 + a_1)\theta^3 I(Q'^2)f'(1). \tag{6.5.27}$$

Therefore, the total contribution of the $a_0 + a_1 x$ terms in $\Pi(x)$ to (6.5.5) is zero. Thus, when substituting $P(x)$ back into (6.5.5), we may disregard the $a_0 + a_1 x$ terms of $\Pi(x)$. In particular, we may obtain $\Pi(x)$ and $\Pi'(x)$ by taking indefinite integrals of $P(x)$ and ignoring any constants of integration.

For $Q(x) = x^{2k}$, the left side of (6.5.5) is

$$2\theta f(1) + \frac{1}{2k+1} f'(1). \tag{6.5.28}$$

So, we may substitute our $P(x)$ above back into (6.5.5) and set the ratio of the $f(1)$ and $f'(1)$ coefficients on the right-hand side to be $2\theta(2k+1)$ as well. This gives us a single linear equation in Y and therefore a unique solution. While we do obtain an explicit formula for Y for any $k \geq 1$ and $\theta > 0$ using Mathematica, the formula is very messy and so we won't reproduce it here. Numerically for $\theta = 1/2$, we compute the values of Y for $k = 1, 2, \ldots, 4$ as -0.8827, -0.7078, -0.6537 and -0.6268.

Evaluating $\mathcal{R}(P, x^{2k})$ using these values leads to the non-vanishing proportions given in Theorem 6.2.9.

As mentioned above, the explicit formula for Y is not concise. However, we do obtain a very simple formula for the asymptotic behaviour of Y as $k \to \infty$. Using Mathematica, we have that as $k \to \infty$, $\alpha = 2\theta(2k+1) + O(k^{-1})$ and $\beta = \theta + O(k^{-1})$, and we find that Y is given asymptotically for large k by

$$\begin{aligned} Y &= \frac{2\alpha\beta\,(2\alpha^2+\beta^2)\cos\beta - (4\alpha^4+3\alpha^2\beta^2+\beta^4)\sin\beta}{2\alpha\beta\,(2\alpha^2+\beta^2)\sin\beta + (4\alpha^4+3\alpha^2\beta^2+\beta^4)\cos\beta} + O(k^{-1}) \\ &= -\tan\beta + O(k^{-1}) \\ &= -\tan\theta + O(k^{-1}). \end{aligned} \tag{6.5.29}$$

With the optimal choice for $P(x)$ now determined, we are ready to analyse the rate at which $\mathcal{R}(P, x^{2k})$ tends to zero.

Proposition 6.5.1. *For $P(x) = P_{2k}(x)$ the optimal choice of polynomial defined above and for any $0 < \theta \leq 1/2$, we have that*

$$\mathcal{R}(P, x^{2k}) = \frac{1}{64k^2} + O(k^{-3}), \tag{6.5.30}$$

as $k \to \infty$.

Proof. Recall that $\mathcal{R}(P, Q)$ is given by

$$\mathcal{R}(P,Q) = \frac{\int_0^1 \left((2\theta)^{-1} I(\tilde{Q}^2)P''(x)^2 - 4\theta I(\tilde{Q}Q')P(x)P''(x) + 8\theta^3 I(Q'^2)P(x)^2\right)dx}{\left(2\theta P(1)Q(1) + P'(1)\tilde{Q}(1)\right)^2}. \tag{6.5.31}$$

With $Q(x) = x^{2k}$, the denominator of $\mathcal{R}(P, x^{2k})$ is the square of

$$2\theta P(1) + \frac{1}{2k+1}P'(1). \tag{6.5.32}$$

Using the expression for $P(x)$ in (6.5.23), we have that

$$P(1) = \sinh(\alpha)\big(\cos\beta - Y\sin\beta\big) - \frac{\alpha}{\beta}e^{-\alpha}\sin\beta = \frac{e^\alpha}{2}\big(\cos\beta - Y\sin\beta\big) + O(\alpha e^{-\alpha}). \tag{6.5.33}$$

Similarly, we have that

$$P'(1) = \frac{\alpha e^{\alpha}}{2}\big(\cos\beta - Y\sin\beta\big) - \frac{e^{\alpha}}{2}\big(\beta\sin\beta + \beta\cos\beta\big) = \frac{\alpha e^{\alpha}}{2}\big(\cos\beta - Y\sin\beta\big) + O(e^{\alpha}). \tag{6.5.34}$$

Therefore we have

$$\begin{aligned} 2\theta P(1) + \frac{1}{2k+1}P'(1) &= \frac{e^{\alpha}}{2}\left(\left(2\theta + \frac{\alpha}{(2k+1)}\right)(\cos\beta - Y\sin\beta)\right) \\ &\quad + O(\alpha e^{-\alpha}) + O(k^{-1}e^{\alpha}) \\ &= \frac{2e^{\alpha}\theta}{\cos\beta} + O(k^{-1}e^{\alpha}), \end{aligned} \tag{6.5.35}$$

where we have used the estimates $\alpha = 2\theta(2k+1) + O(k^{-1})$ and $Y = -\tan\beta + O(k^{-1})$ so that

$$\cos\beta - Y\sin\beta = \frac{1}{\cos\beta} + O(k^{-1}). \tag{6.5.36}$$

This shows us that the denominator of $\mathcal{R}(P, x^{2k})$ is asymptotically of size $e^{2\alpha}$.

Now we consider the numerator of $\mathcal{R}(P, x^{2k})$. As $P(x)$ is given by a linear combination of the terms $e^{(\alpha+i\beta)x}$, the main term of the numerator will come from the $e^{+\alpha x}$ terms of $P(x)$. In particular, the leading terms will be those with a factor of $e^{2\alpha}$ in them. Thus we may disregard the $e^{-\alpha x}$ terms of $P(x)$ when analysing the numerator of $\mathcal{R}(P, x^{2k})$ since these cannot give a factor of $e^{2\alpha}$. We therefore focus only on the contribution of

$$\begin{aligned} P_+(x) &:= \frac{e^{\alpha x}}{2}\big(\cos\beta - Y\sin\beta\big) \\ &= \frac{1}{4}\big((1+iY)e^{(\alpha+i\beta)x} + (1-iY)e^{(\alpha-i\beta)x}\big) \end{aligned} \tag{6.5.37}$$

to the numerator. Specifically, we are left to evaluate

$$\int_0^1 \big((2\theta)^{-1}I(\tilde{Q}^2)P_+''(x)^2 - 4\theta I(\tilde{Q}Q')P_+(x)P_+''(x) + 8\theta^3 I(Q'^2)P_+(x)^2\big)\,dx. \tag{6.5.38}$$

By definition, $\pm\alpha \pm i\beta$ are the roots of the polynomial

$$\frac{1}{2\theta}I(\tilde{Q}^2)X^4 - 4\theta I(\tilde{Q}Q')X^2 + 8\theta^3 I(Q'^2), \tag{6.5.39}$$

and hence $(\alpha \pm i\beta)^2$ are the roots of

$$\frac{1}{2\theta} I(\tilde{Q}^2) X^2 - 4\theta I(\tilde{Q}Q') X + 8\theta^3 I(Q'^2). \tag{6.5.40}$$

Consequently, if we plug one of the individual terms of $P_+(x)$ into (6.5.38) we get zero. Hence the terms which give a non-zero contribution to (6.5.38) are those which involve a product of the two terms of $P_+(x)$. The contribution of these terms is

$$\frac{1+Y^2}{16}\Big(2(2\theta)^{-1} I(\tilde{Q}^2)(\alpha^2+\beta^2)^2 - 4\theta I(\tilde{Q}Q')\big((\alpha+i\beta)^2 + (\alpha-i\beta)^2\big) \\ + 16\theta^3 I(Q'^2)\Big) \int_0^1 e^{2\alpha x}\, dx. \tag{6.5.41}$$

As $(\alpha \pm i\beta)^2$ are the roots of (6.5.40), we have that

$$4\theta I(\tilde{Q}Q')(\alpha \pm i\beta)^2 = \frac{1}{2\theta} I(\tilde{Q}^2)(\alpha \pm i\beta)^4 + 8\theta^3 I(Q'^2). \tag{6.5.42}$$

Substituting this into (6.5.41) gives us

$$\frac{1+Y^2}{16}\frac{I(\tilde{Q}^2)}{2\theta}\left(2(\alpha^2+\beta^2)^2 - (\alpha+i\beta)^4 - (\alpha-i\beta)^4\right)\int_0^1 e^{2\alpha x}\, dx. \tag{6.5.43}$$

We write the integral as

$$\int_0^1 e^{2\alpha x}\, dx = \frac{e^{2\alpha}-1}{2\alpha} = \frac{e^{2\alpha}}{2\alpha}\left(1 + O(e^{-2\alpha})\right), \tag{6.5.44}$$

and then simplifying our previous expression gives us

$$\frac{e^{2\alpha}}{2\alpha}\frac{1+Y^2}{2\theta} I(\tilde{Q}^2)(\alpha\beta)^2\left(1 + O(e^{-2\alpha})\right). \tag{6.5.45}$$

Lastly, using $\alpha = 2\theta(2k+1) + O(k^{-1})$, $\beta = \theta + O(k^{-1})$, $1 + Y^2 = (\cos\beta)^{-2} + O(k^{-1})$ and plugging in

$$I(\tilde{Q}^2) = \frac{1}{(4k+3)(2k+1)^2}, \tag{6.5.46}$$

gives us the final expression

$$\frac{e^{2\alpha}\theta^2}{2\cos^2\beta}\frac{1}{(4k+3)(2k+1)}\left(1 + O(k^{-1})\right) = \frac{e^{2\alpha}\theta^2}{16k^2\cos^2\beta}\left(1 + O(k^{-1})\right) \tag{6.5.47}$$

for the numerator of $\mathcal{R}(P, x^{2k})$. Recalling that the denominator of $\mathcal{R}(P, x^{2k})$ is given by the square of (6.5.35), we therefore have that

$$\mathcal{R}(P, x^{2k}) = \frac{1}{64k^2}\left(1 + O(k^{-1})\right), \tag{6.5.48}$$

as required. □

The proof of Theorem 6.2.9 is completed by recalling the statement of Theorem 6.2.7 with $Q(x) = x^{2k}$ and applying the result of Proposition 6.5.1.

Chapter 7

The mixed second moment of prime Dirichlet L-functions over function fields

In this chapter we study moments of derivatives of Dirichlet L-functions with prime conductor in the function field setting. Derivatives of L-functions, and hence their moments, are of interest for numerous reasons. For example, Speiser's theorem [Spe35] states that the Riemann hypothesis is equivalent to $\zeta'(s)$ having no zeros in the region $0 < \mathrm{Re}(s) < 1/2$. For L-functions attached to arithmetic objects, the derivatives of the L-function at the central point control the order of vanishing of the L-function, which in turn is conjectured to contain deep arithmetic information. For instance, consider the L-function $L_E(s)$ associated to an elliptic curve E defined over $\mathbb{Q}$. By the Mordell-Weil Theorem, the group of $\mathbb{Q}$-rational points $E(\mathbb{Q})$ on E is finitely generated. That is,

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\mathrm{tor}} \times \mathbb{Z}^r,$$

where $E(\mathbb{Q})_{\mathrm{tor}}$ is the finite torsion subgroup of $E(\mathbb{Q})$ and $r \geq 0$ is the rank of the curve. Define the analytic rank of the L-function $L_E(s)$ to be the order of vanishing of the L-function at the central point $s = 1/2$. Then the Birch and Swinnerton-Dyer conjecture famously asserts that the analytic rank of $L_E(s)$ is equal to the rank of the curve E.

7.1 Moments of derivatives over function fields

In the function field setting, the study of moments of derivatives of L-functions has already seen a significant amount of interest. Andrade and Yiasemides [AY21] have obtained asymptotic formulae for the first, second and mixed fourth moment of derivatives of Dirichlet L-functions, where the average is over all non-trivial characters modulo a monic, irreducible $Q \in \mathbb{F}_q[t]$. Similarly to the Riemann zeta-function, this family of L-functions has unitary symmetry type and as such, one sees a very clear analogy between the results of [AY21] and those of Conrey [Con88] on the zeta function.

Andrade and Rajagopal [AR16] and Andrade and Jung [AJ21] studied the mean values of the derivatives $L^{(n)}(\frac{1}{2}, \chi_D)$ of the quadratic Dirichlet L-functions over the hyperelliptic ensemble $\mathcal{H}_{2g+1}$. A general formula for the first moment is given in [AJ21, Theorem 3.1] which implies that for any integer $n \geq 1$, as $g \to \infty$,

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} L^{(n)}(\tfrac{1}{2}, \chi_D) \sim \frac{(-1)^n}{2(n+1)} \cdot \mathcal{A}(1) \cdot (2g+1)^{n+1}. \tag{7.1.1}$$

The factor $\mathcal{A}(1)$ is an arithmetic term given in the form of an Euler product which also appears in the asymptotic formula for the first moment of $L(\frac{1}{2}, \chi_D)$ of Andrade and Keating [AK12]. In [BJ19], Bae and Jung used the approach of Florea [Flo17c] to obtain lower order terms in the asymptotic formula for the first moment of $L''(\frac{1}{2}, \chi_D)$ obtained in [AR16]. It is also shown in [BJ19] that for these quadratic L-functions,

$$\frac{L'(\frac{1}{2}, \chi_D)}{-\log q} = gL(\tfrac{1}{2}, \chi_D). \tag{7.1.2}$$

For a unitary symplectic matrix $X \in Sp(2N)$, we have that the characteristic polynomial satisfies $\Lambda'_X(1) = N\Lambda_X(1)$ and so given that the family $L(s, \chi_D)$ has symplectic symmetry, the relation (7.1.2) is not surprising. As a consequence, the moments of $L'(\frac{1}{2}, \chi_D)$ may be obtained easily from the moments of $L(\frac{1}{2}, \chi_D)$. In particular, one has all the moments of $L'(\frac{1}{2}, \chi_D)$ up to the fourth using the results of Florea [Flo17c, Flo17b, Flo17a].

Djanković and Đokić [DĐ21] considered the mixed second moment of $L(s, \chi_D)$ and its second derivative and obtained the asymptotic formula [1]

[1] In [DĐ21], the main result is given in terms of the completed L-function rather than $L(s, \chi_D)$ but, as discussed in [DĐ21, Remark 1.2], the formula given in (7.1.3) follows from [DĐ21, Theorem 1.1] and Florea's formula for the second moment of $L(\frac{1}{2}, \chi_D)$ in [Flo17b].

$$\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\frac{L(\frac{1}{2},\chi_D)L''(\frac{1}{2},\chi_D)}{\log^2 q}\sim\frac{1}{80}\cdot\frac{\mathcal{B}(1)}{\zeta_q(2)}\cdot(2g+1)^5. \tag{7.1.3}$$

Similarly to (7.1.1), $\mathcal{B}(1)$ is an arithmetic factor, given as an Euler product, also appearing in the main term of the second moment of $L(\frac{1}{2},\chi_D)$ obtained by Florea [Flo17b].

The mean values of derivatives of quadratic Dirichlet L-functions over monic and irreducible polynomials were first studied by Andrade [And19] who obtained an asymptotic formula for the first moment of $L'(\frac{1}{2},\chi_P)$ and $L''(\frac{1}{2},\chi_P)$. In [Jun22], Jung extended the results of [And19] to give an asymptotic formula for the first moment of $L^{(n)}(\frac{1}{2},\chi_P)$ over $\mathcal{P}_{2g+1}$ for all integers $n\geq 1$. Jung's result implies that at as $g\to\infty$,

$$\frac{1}{|\mathcal{P}_{2g+1}|}\sum_{P\in\mathcal{P}_{2g+1}}L^{(n)}(\tfrac{1}{2},\chi_P)\sim\frac{(-1)^n}{2(n+1)}\cdot(2g+1)^{n+1}. \tag{7.1.4}$$

We note the similarity between (7.1.1) and (7.1.4) as both families of L-functions have symplectic symmetry.

7.2 Statement of results

Our main result in this chapter concerns the mixed second moment of derivatives of quadratic Dirichlet L-functions over monic, irreducible polynomials.

Theorem 7.2.1. *Let $\mu,\nu\geq 0$ be integers. Then, as $g\to\infty$,*

$$\frac{1}{|\mathcal{P}_{2g+1}|}\sum_{P\in\mathcal{P}_{2g+1}}\frac{L^{(\mu)}(\frac{1}{2},\chi_P)L^{(\nu)}(\frac{1}{2},\chi_P)}{(\log q)^{\mu+\nu}}=\frac{c(\mu,\nu)}{\zeta_q(2)}\cdot(2g+1)^{\mu+\nu+3}+O(g^{\mu+\nu+2}), \tag{7.2.1}$$

where $\zeta_q(s)$ is the zeta-function of $\mathbb{F}_q[t]$,

$$c(\mu,\nu)=\frac{1}{2^{\mu+\nu+3}}\left((-1)^{\mu+\nu}A(\mu,\nu)+\sum_{m=0}^{\mu}\sum_{n=0}^{\nu}\binom{\mu}{m}\binom{\nu}{n}(-2)^{\mu+\nu-m-n}A(m,n)\right), \tag{7.2.2}$$

and

$$A(m,n):=\frac{1}{2(m+n+3)}\int_0^1\left(x^{m+1}(2-x)^n+x^{n+1}(2-x)^m\right)dx. \tag{7.2.3}$$

Remark 7.2.2. *It's possible that following the approach of Bui and Florea in [BF20] on the second moment of $L(\frac{1}{2}, \chi_P)$, one could improve the error in Theorem 7.2.1 and obtain a secondary main term.*

As an application of the asymptotic formula for the mixed second moment, we are able to obtain the following simultaneous non-vanishing result on the derivatives of the L-functions at $s = 1/2$.

Corollary 7.2.1. *For $\mu, \nu \geq 0$ integers, we have*

$$\frac{1}{|\mathcal{P}_{2g+1}|} |\{P \in \mathcal{P}_{2g+1} : L^{(\mu)}(\tfrac{1}{2}, \chi_P) L^{(\nu)}(\tfrac{1}{2}, \chi_P) \neq 0\}| \gg \frac{1}{g^4}, \tag{7.2.4}$$

as $g \to \infty$.

We are not able to obtain a positive proportion of non-vanishing in Corollary 7.2.1 as we cannot compute the mollified mixed second moment in Theorem 7.2.1. As in Chapter 6, computing the mollified moments requires a formula for the twisted moments with a power saving error term. However, evaluating the second moment of $L(\frac{1}{2}, \chi_P)$ (and therefore the twisted moment) with a power saving is beyond current techniques.

In Chapter 4, we studied the joint moments of derivatives of characteristic polynomials of random matrices. In the case of the unitary symplectic group $Sp(2N)$, for non-negative integers k_1, k_2 and n_1, n_2, we proved that

$$\int_{Sp(2N)} \left(\Lambda_X^{(n_1)}(1)\right)^{k_1} \left(\Lambda_X^{(n_2)}(1)\right)^{k_2} dX \sim b_{k_1,k_2}^{Sp}(n_1, n_2) \cdot (2N)^{k(k+1)/2 + k_1 n_2 + k_2 n_2}, \tag{7.2.5}$$

where $k = k_1 + k_2$ and the leading order coefficient $b_{k_1,k_2}^{Sp}(n_1, n_2)$ can be written explicitly in the form of a combinatorial sum over partitions. Using our asymptotic formula (7.2.5), we made conjectures for the corresponding mixed moments of L-functions with symplectic symmetry. In particular, for the family $L(s, \chi_P)$, the conjecture is that

$$\begin{aligned} \frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P \in \mathcal{P}_{2g+1}} & \frac{L^{(n_1)}(\frac{1}{2}, \chi_P)^{k_1} L^{(n_2)}(\frac{1}{2}, \chi_P)^{k_2}}{(\log q)^{n_1+n_2}} \\ & \sim \eta_k b_{k_1,k_2}^{Sp}(n_1, n_2) \cdot (2g+1)^{k(k+1)/2 + k_1 n_1 + k_2 n_2}, \end{aligned} \tag{7.2.6}$$

where η_k is a certain arithmetic factor in the form of an Euler product. More specifically, η_k is the same arithmetic factor present in the conjectural asymptotic formula for the k-th moment of $L(\frac{1}{2}, \chi_P)$ due to Andrade, Jung and Shamesaldeen [AJ21]. See Conjecture 2.2 and Theorem 4.1 in [AJ21] for further details on their conjecture and the arithmetic term. A similar formula is also conjectured to hold for the family $L(s, \chi_D)$ over $\mathcal{H}_{2g+1}$ with a corresponding arithmetic factor. The results of (7.1.1), (7.1.3) and (7.1.4) all agree with the prediction of the conjecture based on random matrix theory since we have from Chapter 4 that

$$b_{0,1}^{Sp}(0,n) = \frac{(-1)^n}{2(n+1)} \text{ and } b_{1,1}^{Sp}(0,2) = \frac{1}{80}. \tag{7.2.7}$$

In the case of the mixed second moment considered in Theorem 7.2.1, we see that the main term is of the correct size as predicted by the conjecture. The conjecture also states that the factor $c(\mu, \nu)$ in the leading order coefficient should satisfy

$$c(n_1, n_2) = b_{1,1}^{Sp}(n_1, n_2), \tag{7.2.8}$$

since $\eta_2 = \zeta_q(2)^{-1}$ is the relevant arithmetic factor for the second moment. By Theorem 4.2.2, the random matrix theory coefficient $b_{1,1}^{Sp}(n_1, n_2)$ is given by

$$\begin{aligned} b_{1,1}^{Sp}(n_1, n_2) = \frac{(-1)^{n_1+n_2}}{2^{n_1+n_2+3}}(n_1!)(n_2!) \sum_{2l_1+2l_2 \le n_1} \sum_{2m_1+2m_2 \le n_2} \frac{1}{(n_1 - 2l_1 - 2l_2)!} \\ \times \frac{1}{(n_2 - 2m_1 - 2m_2)!} \frac{(2l_2 + 2m_2 - 2l_1 - 2m_1 - 2)}{(2l_1 + 2m_1 + 3)!(2l_2 + 2m_2 + 1)!}, \end{aligned} \tag{7.2.9}$$

where the sum is over non-negative integers l_1, l_2 and m_1, m_2. We do not currently have a proof of (7.2.8) for all n_1, n_2 here but we have checked using Mathematica that it does indeed hold for $n_1, n_2 \le 20$.

Remark 7.2.3. *By a change of variables $x \mapsto 2x$, we have that*

$$A(m,n) = \frac{2^{m+n+1}}{m+n+3}\left(B(\tfrac{1}{2}; m+2, n+1) + B(\tfrac{1}{2}; n+2, m+1)\right) \tag{7.2.10}$$

where

$$B(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt \tag{7.2.11}$$

is the incomplete beta function. Recall that the beta function is given by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{7.2.12}$$

and the regularised beta function is defined as

$$I(x;a,b) = \frac{B(x;a,b)}{B(a,b)} \tag{7.2.13}$$

For m, n positive integers and $0 \le x < 1$, the regularised beta function satisfies [AS72, Eq. 6.6.4]

$$I(x;m,n-m+1) = \sum_{j=m}^{n} \binom{n}{j} x^j (1-x)^{n-j} = \sum_{j=0}^{n-m} \binom{n}{j+m} x^{j+m}(1-x)^{n-j-m}. \tag{7.2.14}$$

Hence, taking $x = 1/2$, we have that

$$B(\tfrac{1}{2}; m+2, n+1) = \frac{1}{2^{m+n+2}} \frac{(m+1)!n!}{(m+n+2)!} \sum_{j=0}^{n} \binom{m+n+2}{j+m+2}, \tag{7.2.15}$$

and we have a similar expression for $B(\frac{1}{2}; n+2, m+1)$. Therefore, we may write $A(m,n)$ as

$$A(m,n) = \frac{m!n!}{2(m+n+3)!} \left((m+1) \sum_{j=0}^{n} \binom{m+n+2}{j+m+2} + (n+1) \sum_{j=0}^{m} \binom{m+n+2}{j+n+2} \right). \tag{7.2.16}$$

Thus, the claim (7.2.8) may be viewed as a certain identity of combinatorial sums.

We also have the following result on the twisted first moment of $L^{(k)}(\frac{1}{2}, \chi_P)$ which generalises Jung's result [Jun22, Theorem 2.1]. Before stating the result, we need some additional notation. For any integers $k, n \ge 0$, we let

$$J_k(n) := \sum_{m=1}^{n} m^k. \tag{7.2.17}$$

Faulhaber's formula (see, for instance [CG96, p. 107]), states that

$$J_k(n) = \frac{1}{k+1} \sum_{m=0}^{k} \binom{k+1}{m} B_m^+ n^{k+1-m}, \tag{7.2.18}$$

where B_n^+ are the second Bernoulli numbers. In particular, $J_k(n)$ is a polynomial in n of degree $k+1$ with zero constant term.

Theorem 7.2.4. *Let $k \ge 0$ be an integer. Also, let $l \in \mathcal{M}$ and write $l = l_1 l_2^2$ with*

$l_1, l_2 \in \mathcal{M}$ and l_1 square-free. Then, as $g \to \infty$,

$$\begin{aligned}
&\frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P \in \mathcal{P}_{2g+1}} \frac{L^{(k)}(\frac{1}{2}, \chi_P)\chi_P(l)}{(\log q)^k} \\
&= \frac{(-1)^k}{|l_1|^{1/2}} \sum_{m=0}^{k} \binom{k}{m} 2^m d(l_1)^{k-m} J_m([\tfrac{g-d(l_1)}{2}]) \\
&\quad + \frac{1}{|l_1|^{1/2}} \sum_{m=0}^{k} \binom{k}{m} (-2g)^{k-m} \sum_{i=0}^{m} \binom{m}{i} 2^i d(l_1)^{m-i} J_i([\tfrac{g-1-d(l_1)}{2}]) \\
&\quad + O\left(q^{-g/2} g^{k+1} d(l)\right), \qquad (7.2.19)
\end{aligned}$$

where $d(f)$ denotes the degree of the polynomial f.

7.2.1 Preliminaries for the proofs

We denote the divisor function on $\mathbb{F}_q[t]$ by $\tau(f)$ which satisfies

$$\sum_{f \in \mathcal{M}_n} \tau(f) = (n+1)q^n. \qquad (7.2.20)$$

For $\alpha, \beta \in \mathbb{C}$, we let

$$\tau_{\alpha,\beta}(f) = \sum_{f = f_1 f_2} \frac{1}{|f_1|^\alpha |f_2|^\beta}. \qquad (7.2.21)$$

Also, for integers $m, n \geq 0$ we will denote

$$\tau^{(m,n)}(f) := \frac{\partial^{m+n}}{\partial \alpha^m \partial \beta^n} \tau_{\alpha,\beta}(f)|_{\alpha=\beta=0} = (-\log q)^{m+n} \sum_{f = f_1 f_2} d(f_1)^m d(f_2)^n, \qquad (7.2.22)$$

and note that we have the bound

$$|\tau^{(m,n)}(f)| \ll \sum_{f = f_1 f_2} d(f)^{m+n} \ll \tau(f) d(f)^{m+n}. \qquad (7.2.23)$$

Lastly, we have the following Weil bound for character sums over monic, irreducible polynomials.

Lemma 7.2.5. *For $f \in \mathcal{M}$ not a square, as $g \to \infty$,*

$$\frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P \in \mathcal{P}_{2g+1}} \chi_P(f) \ll q^{-g} d(f).$$

Proof. This follows from equation (2.5) in [Rud10] and the Prime Polynomial Theorem. $\square$

7.3 The twisted moment of the k-th derivative

Here we will prove Theorem 7.2.4 following the approach used in [Jun22]. Let $l \in \mathcal{M}$ and write $l = l_1 l_2^2$ with $l_1, l_2 \in \mathcal{M}$ and l_1 square-free. For $h \in \{g, g-1\}$, we define the sum

$$\begin{aligned} S_h(m;l) &:= \sum_{f \in \mathcal{M}_{\leq h}} \frac{d(f)^m}{|f|^{1/2}} \frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P \in \mathcal{P}_{2g+1}} \chi_P(fl) \\ &= \sum_{n=0}^{h} n^m q^{-n/2} \sum_{f \in \mathcal{M}_n} \frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P \in \mathcal{P}_{2g+1}} \chi_P(fl), \end{aligned} \tag{7.3.1}$$

and compute an asymptotic formula for $S_h(m;l)$ in the following lemma.

Lemma 7.3.1. *For an integer $m \geq 0$ and $h \in \{g, g-1\}$, we have that as $g \to \infty$,*

$$S_h(m;l) = \frac{1}{|l_1|^{1/2}} \sum_{i=0}^{m} \binom{m}{i} 2^i d(l_1)^{m-i} J_i([\tfrac{h-d(l_1)}{2}]) + O\left(q^{-g/2} g^{m+1} d(l)\right). \tag{7.3.2}$$

Proof. We split the sum $S_h(m;l) = S_h(m;l)_\square + S_h(m;l)_{\neq\square}$ according to whether $fl = \square$ or $fl \neq \square$. For the contribution of non-squares, we use Lemma 7.2.5 to obtain

$$\begin{aligned} |S_h(m;l)_{\neq\square}| &\ll \sum_{n=0}^{g} n^m q^{-n/2} \sum_{\substack{f \in \mathcal{M}_n \\ fl \neq \square}} \left| \frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P \in \mathcal{P}_{2g+1}} \chi_P(fl) \right| \\ &\ll q^{-g} \sum_{n=0}^{g} n^m q^{-n/2} \sum_{f \in \mathcal{M}_n} d(fl) \\ &\ll q^{-g}(g + d(l)) \sum_{n=0}^{g} n^m q^{n/2} \\ &\ll q^{-g/2} g^m (g + d(l)). \end{aligned} \tag{7.3.3}$$

For the contribution of the squares in $S_h(m;l)_\square$, we use the facts that $\chi_P(f^2) = \chi_P(f)^2$ and since $d(f) \leq g$, we have that $P \nmid f$ for all $P \in \mathcal{P}_{2g+1}$. Thus, for $fl = \square$, we have

$$\sum_{P\in\mathcal{P}_{2g+1}} \chi_P(fl) = \sum_{P\in\mathcal{P}_{2g+1}} 1 - \sum_{\substack{P\in\mathcal{P}_{2g+1}\\ P|l}} 1 = |\mathcal{P}_{2g+1}| + O(d(l)). \tag{7.3.4}$$

Now, as $fl = \square$, we write $f = l_1 f_1^2$ with f_2 monic. Then, since $d(f) = d(l_1)+2d(f_1) \leq h$, we can rewrite the sum over f_1 with $d(f_1) \leq (h - d(l_1))/2$ which gives us that

$$\begin{aligned}
S_h(m;l)_\square &= \sum_{n=0}^{h} n^m q^{-n/2} \sum_{\substack{f\in\mathcal{M}_n\\ fl=\square}} \frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P\in\mathcal{P}_{2g+1}} \chi_P(fl)\\
&= \sum_{n=0}^{h} n^m q^{-n/2} \sum_{\substack{f\in\mathcal{M}_n\\ fl=\square}} 1 + O\left(\frac{d(l)}{|\mathcal{P}_{2g+1}|} \sum_{n=0}^{h} n^m q^{-n/2} \sum_{\substack{f\in\mathcal{M}_n\\ fl=\square}} 1 \right)\\
&= q^{-d(l_1)/2} \sum_{n=0}^{(h-d(l_1))/2} (d(l_1)+2n)^m q^{-n} \sum_{f_1\in\mathcal{M}_n} 1 + O\left(q^{-2g} g\, d(l) \sum_{n=0}^{h} n^m q^{n/2} \right)\\
&= \frac{1}{|l_1|^{1/2}} \sum_{n=0}^{(h-d(l_1))/2} (d(l_1)+2n)^m + O\left(q^{-3g/2} g^{m+1} d(l) \right),
\end{aligned} \tag{7.3.5}$$

where we have used the Prime Polynomial Theorem in bounding the error. For the main term, we use a binomial expansion and the function $J_i(n) = \sum_{m=0}^{n} m^i$ to write it as

$$\begin{aligned}
\frac{1}{|l_1|^{1/2}} \sum_{n=0}^{(h-d(l_1))/2} (2n+d(l_1))^m &= \frac{1}{|l_1|^{1/2}} \sum_{n=0}^{(h-d(l_1))/2} \sum_{i=0}^{m} \binom{m}{i} (2n)^i d(l_1)^{m-i}\\
&= \frac{1}{|l_1|^{1/2}} \sum_{i=0}^{m} \binom{m}{i} 2^i d(l_1)^{m-i} \sum_{n=0}^{(h-d(l_1))/2} n^i\\
&= \frac{1}{|l_1|^{1/2}} \sum_{i=0}^{m} \binom{m}{i} 2^i d(l_1)^{m-i} J_i([\tfrac{h-d(l_1)}{2}]).
\end{aligned} \tag{7.3.6}$$

Combining this with the bounds for the error terms completes the proof. $\square$

Proof of Theorem 7.2.4. Using the expression for $L^{(k)}(\frac{1}{2}, \chi_P)$ given in [AJ21, Lemma 5.1] and multiplying by $\chi_P(l)$, we have that

$$\frac{L^{(k)}(\frac{1}{2},\chi_P)\chi_P(l)}{(\log q)^k} = (-1)^k \sum_{n=0}^{g} n^k q^{-n/2} \sum_{f\in\mathcal{M}_n} \chi_P(fl) + \sum_{m=0}^{k} \binom{k}{m} (-2g)^{k-m} \sum_{n=0}^{g-1} n^m q^{-n/2} \sum_{f\in\mathcal{M}_n} \chi_P(fl). \tag{7.3.7}$$

Taking the average over $\mathcal{P}_{2g+1}$ and recalling the definition of $S_h(m;l)$ in (7.3.1), we can then write

$$\frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P\in\mathcal{P}_{2g+1}} \frac{L^{(k)}(\frac{1}{2},\chi_P)\chi_P(l)}{(\log q)^k} = (-1)^k S_g(k;l) + \sum_{m=0}^{k} \binom{k}{m} (-2g)^{k-m} S_{g-1}(m;l). \tag{7.3.8}$$

Using this expression for the twisted moment, we now apply the formula for $S_h(m;l)$ in Lemma 7.3.1 which gives us

$$\begin{aligned}\frac{L^{(k)}(\frac{1}{2},\chi_P)\chi_P(l)}{(\log q)^k} &= (-1)^k \frac{1}{|l_1|^{1/2}} \sum_{i=0}^{k} \binom{k}{i} 2^i d(l_1)^{k-i} J_i([\tfrac{g-d(l_1)}{2}]) \\ &+ \frac{1}{|l_1|^{1/2}} \sum_{m=0}^{k} \binom{k}{m} (-2g)^{k-m} \sum_{i=0}^{m} \binom{m}{i} 2^i d(l_1)^{m-i} J_i([\tfrac{g-1-d(l_1)}{2}]) \\ &+ O\left(q^{-g/2} g^{k+1} d(l)\right) + O\left(q^{-g/2} d(l) \sum_{m=0}^{k} \binom{k}{m} (-2g)^{k-m} g^{m+1}\right).\end{aligned} \tag{7.3.9}$$

The last error term above is easily seen to be $O(q^{-g/2} g^{k+1} d(l))$ which completes the proof.

□

7.4 The mixed second moment

In this section we will prove Theorem 7.2.1. Our starting point will be the approximate functional equation for the product of two shifted L-functions given in [BFK23, Lemma 2.1]. Namely, we have that for $P \in \mathcal{P}_{2g+1}$,

$$L(\tfrac{1}{2}+\alpha,\chi_P)L(\tfrac{1}{2}+\beta,\chi_P) = \sum_{f\in\mathcal{M}_{\leq 2g}} \frac{\tau_{\alpha,\beta}(f)\chi_P(f)}{|f|^{1/2}} \\ + q^{-2g(\alpha+\beta)} \sum_{f\in\mathcal{M}_{\leq 2g-1}} \frac{\tau_{-\alpha,-\beta}(f)\chi_P(f)}{|f|^{1/2}}. \tag{7.4.1}$$

Using the approximate functional equation, we write

$$\frac{1}{|\mathcal{P}_{2g+1}|}\sum_{P\in\mathcal{P}_{2g+1}} L(\tfrac{1}{2}+\alpha,\chi_P)L(\tfrac{1}{2}+\beta,\chi_P) = F_{2g}(\alpha,\beta) + q^{-2g(\alpha+\beta)}F_{2g-1}(-\alpha,-\beta), \tag{7.4.2}$$

where

$$F_{2g}(\alpha,\beta) = \sum_{f\in\mathcal{M}_{\leq 2g}} \frac{\tau_{\alpha,\beta}(f)}{|f|^{1/2}} \frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P\in\mathcal{P}_{2g+1}} \chi_P(f), \tag{7.4.3}$$

and $F_{2g-1}(\alpha,\beta)$ is given by a similar expression with $\mathcal{M}_{\leq 2g}$ replaced by $\mathcal{M}_{\leq 2g-1}$.

From here, one could attempt to obtain the mixed second moment by first computing a formula for the shifted second moment and then differentiating with respect to the shifts. This would require asymptotically evaluating the right-hand side of (7.4.2) for small shifts α,β. The CFKRS recipe [CFK+05] was applied to this family of L-functions in [AJS21] and so we can write down the formula one would expect to obtain. Explicitly, the shifted second moment should be given by

$$\frac{1}{|\mathcal{P}_{2g+1}|}\sum_{P\in\mathcal{P}_{2g+1}} L(\tfrac{1}{2}+\alpha_1,\chi_P)L(\tfrac{1}{2}+\alpha_2,\chi_P) \\ = \sum_{\substack{\epsilon_i=\pm 1\\ i=1,2}} A(\tfrac{1}{2};\epsilon_1\alpha_1,\epsilon_2\alpha_2) \prod_{1\leq i\leq j\leq 2} \zeta_q(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j)\,(1+o(1))\,, \tag{7.4.4}$$

where

$$A(\tfrac{1}{2};\alpha_1,\alpha_2) = \prod_{P\in\mathcal{P}}\prod_{1\leq i\leq j\leq 2}\left(1-\frac{1}{|P|^{1+\alpha_i+\alpha_j}}\right) \\ \times \frac{1}{2}\left(\prod_{i=1}^{2}\left(1-\frac{1}{|P|^{1/2+\alpha_i}}\right)^{-1} + \prod_{i=1}^{2}\left(1+\frac{1}{|P|^{1/2+\alpha_i}}\right)^{-1}\right). \tag{7.4.5}$$

Since we can compute an asymptotic formula for the second moment of $L(\frac{1}{2}, \chi_P)$, we should in principle be able to compute a formula for the second moment with small shifts using the same techniques. For instance, formulae for the first, second and third shifted moments of the quadratic Dirichlet L-functions $L(s, \chi_D)$ over $\mathcal{H}_{2g+1}$ were computed in [BFK23] using the same approach as Florea [Flo17c, Flo17b, Flo17a] for the moments without the shifts.

However, even if one obtains a formula in the form of (7.4.4), it is not straightforward to take derivatives to yield the mixed moments. Especially when the order of the derivatives is high, computing the main term will be very messy. This difficulty was previously noted in [DĐ21, Remark 1.3], where the authors consequently use an alternate method. For this reason, the approach we take is to differentiate (7.4.2) with respect to the shifts first and then determine the asymptotic behaviour. With this in mind, for $m, n \geq 0$ and $h \in \{2g, 2g-1\}$, we define

$$T_h(m,n) := \frac{\partial^{m+n}}{\partial \alpha^m \partial \beta^n} F_h(\alpha, \beta)|_{\alpha=\beta=0} = \sum_{f \in \mathcal{M}_{\leq h}} \frac{\tau^{(m,n)}(f)}{|f|^{1/2}} \frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P \in \mathcal{P}_{2g+1}} \chi_P(f). \tag{7.4.6}$$

Then, differentiating (7.4.2) gives us

$$\begin{aligned} &\frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P \in \mathcal{P}_{2g+1}} L^{(\mu)}(\tfrac{1}{2}, \chi_P) L^{(\nu)}(\tfrac{1}{2}, \chi_P) \\ &\qquad = T_{2g}(\mu, \nu) + (-1)^{\mu+\nu} \sum_{m=0}^{\mu} \sum_{n=0}^{\nu} \binom{\mu}{m} \binom{\nu}{n} (2g \log q)^{\mu+\nu-m-n} \, T_{2g-1}(m,n). \end{aligned} \tag{7.4.7}$$

We are then left to analyse the sums $T_h(m,n)$ and we give an asymptotic formula in the next proposition.

Proposition 7.4.1. *For integers $m, n \geq 0$ and $h \in \{2g, 2g-1\}$, we have that as $g \to \infty$,*

$$T_h(m,n) = \frac{(-\log q)^{m+n} A(m,n)}{\zeta_q(2)} g^{m+n+3} + O(g^{m+n+2}), \tag{7.4.8}$$

with $A(m,n)$ as defined in (7.2.3).

We complete the proof of Theorem 7.2.1 by applying the result of Proposition 7.4.1 to (7.4.7) and isolating the coefficient of $(2g+1)^{\mu+\nu+3}$.

7.4.1 Proof of Proposition 7.4.1

We will prove Proposition 7.4.1 by analysing separately the contribution of the square and the non-square polynomials to the sum $T_h(m,n)$. So, we write

$$T_h(m,n) = T_h(m,n)_{\square} + T_h(m,n)_{\neq\square}, \tag{7.4.9}$$

where $T_h(m,n)_{\square}$ and $T_h(m,n)_{\neq\square}$ denote the sums over f a perfect square and f not a square in (7.4.6), respectively. We bound $T_h(m,n)_{\neq\square}$ using Lemma 7.2.5 to obtain

$$\begin{aligned}
|T_h(m,n)_{\neq\square}| &\ll \sum_{\substack{f\in\mathcal{M}_{\leq 2g}\\ f\neq\square}} \frac{\tau^{(m,n)}(f)}{|f|^{1/2}} \left| \frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P\in\mathcal{P}_{2g+1}} \chi_P(f) \right| \\
&\ll q^{-g} \sum_{f\in\mathcal{M}_{\leq 2g}} \frac{\tau^{(m,n)}(f)}{|f|^{1/2}} d(f) \\
&\ll g q^{-g} \sum_{j=0}^{2g} q^{-j/2} \sum_{f\in\mathcal{M}_j} \tau(f) d(f)^{m+n} \\
&\ll g q^{-g} \sum_{j=0}^{2g} j^{m+n+1} q^{j/2} \\
&\ll g^{m+n+2},
\end{aligned} \tag{7.4.10}$$

where we have used the fact that $\sum_{f\in\mathcal{M}_j} \tau(f) \ll jq^j$.

Next, for the terms in $T_h(m,n)_{\square}$ with $f=\square$, we have that $d(f)\leq 2g$ and so $P\nmid f$ for all $P\in\mathcal{P}_{2g+1}$. Therefore, as $f=l^2$ say, we have that $\chi_P(f)=\chi_P(l)^2=1$ and consequently

$$\frac{1}{|\mathcal{P}_{2g+1}|} \sum_{P\in\mathcal{P}_{2g+1}} \chi_P(f) = 1. \tag{7.4.11}$$

Thus, we have that

$$T_h(m,n)_{\square} = \sum_{\substack{f\in\mathcal{M}_{\leq h}\\ f=\square}} \frac{\tau^{(m,n)}(f)}{|f|^{1/2}} = \sum_{f\in\mathcal{M}_{\leq h/2}} \frac{\tau^{(m,n)}(f^2)}{|f|}. \tag{7.4.12}$$

Remark 7.4.2. *At this point, the standard method to compute a partial sum such as that in (7.4.12) would be to consider the generating series and apply Perron's formula. In this case however, this approach is not directly applicable since $\tau^{(m,n)}(f^2)$ is not a multiplicative function. So, to apply this method, we would have to consider* *the generating series*

$$G(\alpha,\beta;u)=\sum_{f\in\mathcal{M}}\frac{\tau_{\alpha,\beta}(f^2)}{|f|}u^{d(f)} \tag{7.4.13}$$

and take the derivatives after applying Perron's formula. By the multiplicativity of $\tau_{\alpha,\beta}(f)$, one can show that

$$G(\alpha,\beta;u)=\frac{\mathcal{Z}(q^{-1-2\alpha}u)\mathcal{Z}(q^{-1-2\beta}u)\mathcal{Z}(q^{-1-\alpha-\beta}u)}{\mathcal{Z}(q^{-2-2\alpha-2\beta}u^2)}, \tag{7.4.14}$$

where $\mathcal{Z}(u)=(1-qu)^{-1}$ is the zeta function of $\mathbb{F}_q[t]$. The point is that following this method would essentially equate to computing the second shifted moment and then taking the derivatives. But, as discussed earlier, we choose to avoid this approach as taking arbitrary order derivatives of (7.4.14) leads to very complicated expressions.

Using the definition of $\tau^{(m,n)}(f)$, we write the main term $T_h(m,n)_\square$ more explicitly as

$$\sum_{f\in\mathcal{M}_{\leq h/2}}\frac{\tau^{(m,n)}(f^2)}{|f|}=(-\log q)^{m+n}\sum_{j=0}^{h/2}q^{-j}\sum_{f\in\mathcal{M}_j}\sum_{f^2=f_1f_2}d(f_1)^m d(f_2)^n. \tag{7.4.15}$$

Then, for a given j, we use the hyperbola method and write

$$\begin{aligned}\sum_{f\in\mathcal{M}_j}\sum_{f^2=f_1f_2}d(f_1)^m d(f_2)^n=&\sum_{f\in\mathcal{M}_j}\sum_{\substack{f^2=f_1f_2\\ d(f_1)=j}}j^{m+n}+\sum_{f\in\mathcal{M}_j}\sum_{\substack{f^2=f_1f_2\\ d(f_1)<j}}d(f_1)^m d(f_2)^n\\ &+\sum_{f\in\mathcal{M}_j}\sum_{\substack{f^2=f_1f_2\\ d(f_2)<j}}d(f_1)^m d(f_2)^n.\end{aligned} \tag{7.4.16}$$

The second and third sums here are similar so we need only focus on the first two. In particular, it suffices to evaluate the sum

$$\sum_{f\in\mathcal{M}_j}\sum_{\substack{f^2=f_1f_2\\ d(f_1)=k}}d(f_1)^m d(f_2)^n=k^m(2j-k)^n\sum_{f\in\mathcal{M}_j}\sum_{\substack{f^2=f_1f_2\\ d(f_1)=k}}1, \tag{7.4.17}$$

for $k\leq j$. To compute the above sum, we reformulate the counting problem

$$\sum_{f\in\mathcal{M}_j}\sum_{\substack{f^2=f_1f_2\\ d(f_1)=k}}1=\sum_{d(f_1)=k}\sum_{\substack{d(f_2)=2j-k\\ f_1f_2=\square}}1 \tag{7.4.18}$$

as follows. We observe that $f_1f_2 = \square$ if and only if $f_1 = l_1l_2^2$ and $f_2 = l_1l_3^2$ with $l_1, l_2, l_3 \in \mathcal{M}$ and l_1 square-free. Also, since we require that $d(f_1) = k$, we must have $d(l_1) \leq k$ and $d(l_2) = (k - d(l_1))/2$. Then, to ensure that $d(f_2) = 2j - k$, we have $d(l_3) = (2j - k - d(l_1))/2$. Note that $d(l_3) \geq 0$ since $d(l_1) \leq k \leq j$. So, by summing over l_1, l_2, l_3, we may write

$$\begin{aligned}
\sum_{f \in \mathcal{M}_j} \sum_{\substack{f^2 = f_1f_2 \\ d(f_1)=k}} 1 &= \sum_{\substack{l_1 \in \mathcal{H} \\ d(l_1) \leq k \\ k-d(l_1) \text{ even}}} \sum_{\substack{l_2 \in \mathcal{M} \\ d(l_2)=(k-d(l_1))/2}} \sum_{\substack{l_3 \in \mathcal{M} \\ d(l_3)=j-k/2-d(l_1)/2}} 1 \\
&= q^{j-k/2} \sum_{\substack{l_1 \in \mathcal{H} \\ d(l_1) \leq k \\ k-d(l_1) \text{ even}}} q^{-d(l_1)/2} \sum_{\substack{l_2 \in \mathcal{M} \\ d(l_2)=(k-d(l_1))/2}} 1 \\
&= q^j \sum_{\substack{l_1 \in \mathcal{H} \\ d(l_1) \leq k \\ k-d(l_1) \text{ even}}} q^{-d(l_1)}. \qquad (7.4.19)
\end{aligned}$$

For the final sum over l_1, if k is even, then

$$\sum_{\substack{l_1 \in \mathcal{H} \\ d(l_1) \leq k \\ k-d(l_1) \text{ even}}} q^{-d(l_1)} = \sum_{\substack{i=0 \\ i \text{ even}}}^{k} \sum_{l_1 \in \mathcal{H}_i} q^{-i} = 1 + \sum_{i=1}^{k/2} \sum_{l_1 \in \mathcal{H}_{2i}} q^{-2i} = 1 + (1 - q^{-1})\frac{k}{2}, \quad (7.4.20)$$

where we have used the fact that for $n \geq 1$,

$$|\mathcal{H}_n| = \frac{q^n}{\zeta_q(2)} = q^n(1 - q^{-1}). \quad (7.4.21)$$

Similarly, if k is odd, we have

$$\sum_{\substack{l_1 \in \mathcal{H} \\ d(l_1) \leq k \\ k-d(l_1) \text{ even}}} q^{-d(l_1)} = 1 + (1 - q^{-1})\frac{k-1}{2}. \quad (7.4.22)$$

Thus, we have that

$$\sum_{f \in \mathcal{M}_j} \sum_{\substack{f^2 = f_1f_2 \\ d(f_1)=k}} 1 = q^j \left(1 + (1 - q^{-1}) \left[\tfrac{k}{2}\right]\right), \quad (7.4.23)$$

and incorporating this into (7.4.16) using (7.4.17) gives us

$$\sum_{f\in\mathcal{M}_j}\sum_{f^2=f_1f_2} d(f_1)^m d(f_2)^n = q^j\Bigg(j^{m+n}\left(1+(1-q^{-1})\left[\tfrac{j}{2}\right]\right)$$
$$+\sum_{k=0}^{j-1}\left(k^m(2j-k)^n+k^n(2j-k)^m\right)\left(1+(1-q^{-1})\left[\tfrac{k}{2}\right]\right)\Bigg). \tag{7.4.24}$$

We now analyse the behaviour of the expression in (7.4.24) as $j\to\infty$. As $[x]=x+O(1)$, the first term inside the brackets in (7.4.24) is

$$(1-q^{-1})\frac{j^{m+n+1}}{2}+O(j^{m+n}). \tag{7.4.25}$$

We also have that

$$\sum_{k=0}^{j-1} k^m(2j-k)^n\left(1+(1-q^{-1})[\tfrac{k}{2}]\right) = \frac{(1-q^{-1})}{2}\sum_{k=0}^{j-1} k^{m+1}(2j-k)^n$$
$$+O\left(\sum_{k=0}^{j-1} k^m(2j-k)^n\right), \tag{7.4.26}$$

so we now focus on the sum $\sum_{k=0}^{j-1} k^m(2j-k)^n$. First, by factoring out a power of j, we write

$$\sum_{k=0}^{j-1} k^m(2j-k)^n = j^{m+n+1}\cdot\frac{1}{j}\sum_{k=0}^{j-1}\left(\frac{k}{j}\right)^m\left(2-\frac{k}{j}\right)^n. \tag{7.4.27}$$

Then we see that as $j\to\infty$,

$$\frac{1}{j}\sum_{k=0}^{j-1}\left(\frac{k}{j}\right)^m\left(2-\frac{k}{j}\right)^n \to \int_0^1 x^m(2-x)^n dx \tag{7.4.28}$$

since the sum is a left Riemann sum for the integral. Therefore, we have that

$$\sum_{k=0}^{j-1} k^m(2j-k)^n = j^{m+n+1}\cdot\frac{1}{j}\sum_{k=0}^{j-1}\left(\frac{k}{j}\right)^m\left(2-\frac{k}{j}\right)^n \sim j^{m+n+1}\int_0^1 x^m(2-x)^n dx, \tag{7.4.29}$$

as $j\to\infty$. On the other hand, by Faulhaber's formula, we know that the sum is a polynomial in j so we have

$$\sum_{k=0}^{j-1} k^m(2j-k)^n = j^{m+n+1}\int_0^1 x^m(2-x)^n dx + O(j^{m+n}). \tag{7.4.30}$$

Using (7.4.26) and (7.4.30) in (7.4.24) yields

$$\begin{aligned}\sum_{f\in\mathcal{M}_j}\sum_{f^2=f_1f_2} d(f_1)^m d(f_2)^n = \frac{q^j j^{m+n+2}}{2\zeta_q(2)}\int_0^1 \left(x^{m+1}(2-x)^n + x^{n+1}(2-x)^m\right) dx\\ + O(q^j j^{m+n+1}).\end{aligned} \tag{7.4.31}$$

Therefore, by Faulhaber's formula, we have

$$\begin{aligned}T_h(m,n)_\square &= \sum_{f\in\mathcal{M}_{\le h/2}} \frac{\tau^{(m,n)}(f^2)}{|f|} \qquad\qquad (7.4.32)\\ &= (-\log q)^{m+n}\sum_{j=0}^{h/2} q^{-j}\sum_{f\in\mathcal{M}_j}\sum_{f^2=f_1f_2} d(f_1)^m d(f_2)^n\\ &= \frac{(-\log q)^{m+n}}{2\zeta_q(2)}\int_0^1 \left(x^{m+1}(2-x)^n + x^{n+1}(2-x)^m\right) dx \sum_{j=0}^{h/2} j^{m+n+2}\\ &\quad + O\left(\sum_{j=0}^{h/2} j^{m+n+1}\right)\\ &= \frac{(-\log q)^{m+n}h^{m+n+3}}{2^{m+n+4}(m+n+3)\zeta_q(2)}\int_0^1 \left(x^{m+1}(2-x)^n + x^{n+1}(2-x)^m\right) dx\\ &\quad + O(h^{m+n+2}).\end{aligned} \tag{7.4.33}$$

Recalling the definition of $A(m,n)$ given in (7.2.3) and choosing $h\in\{2g,2g-1\}$ completes the proof.

7.4.2 Proof of Corollary 7.2.1

Here we will prove the simultaneous non-vanishing result for the derivatives of the L-functions $L(s,\chi_P)$ at $s=1/2$. Fix integers $\mu,\nu\ge 0$ and let $\mathcal{P}^*_{2g+1}$ denote the subset of $P\in\mathcal{P}_{2g+1}$ for which $L^{(\mu)}(\frac{1}{2},\chi_P)L^{(\nu)}(\frac{1}{2},\chi_P)\neq 0$. First, we need upper bounds for moments of derivatives of these L-functions. By Cauchy's integral formula, we may write

$$L^{(\mu)}(\tfrac{1}{2}, \chi_P) = \frac{\mu!}{2\pi i} \oint_{|z|=r} \frac{L(\frac{1}{2}+z, \chi_P)}{z^{\mu+1}} dz. \tag{7.4.34}$$

We choose $r = 1/g$ and then by applying Hölder's inequality to the integral, we have that

$$\begin{aligned} |L^{(\mu)}(\tfrac{1}{2}, \chi_P)|^k &\ll \left(\oint_{|z|=1/g} |L(\tfrac{1}{2}+z, \chi_P)|^k \, dz \right) \left(\oint_{|z|=1/g} \frac{dz}{z^{k(\mu+1)/(k-1)}} \right)^{k-1} \\ &\ll g^{k\mu+1} \oint_{|z|=1/g} |L(\tfrac{1}{2}+z, \chi_P)|^k \, dz, \end{aligned} \tag{7.4.35}$$

where we have bounded the second integral using the estimation lemma. We now introduce the sum over $\mathcal{P}_{2g+1}$ to give

$$\begin{aligned} \sum_{P \in \mathcal{P}_{2g+1}} |L^{(\mu)}(\tfrac{1}{2}, \chi_P)|^k &\ll g^{k\mu+1} \left(\oint_{|z|=1/g} \sum_{P \in \mathcal{P}_{2g+1}} |L(\tfrac{1}{2}+z, \chi_P)|^k \, dz \right) \\ &\ll g^{k\mu} \max_{|z|=1/g} \left(\sum_{P \in \mathcal{P}_{2g+1}} |L(\tfrac{1}{2}+z, \chi_P)|^k \right). \end{aligned} \tag{7.4.36}$$

For small shifts $z \asymp 1/g$, the shifted moments

$$\sum_{P \in \mathcal{P}_{2g+1}} |L(\tfrac{1}{2}+z, \chi_P)|^k \tag{7.4.37}$$

can be bounded in the same way as the k-th moment at the central point. Specifically, by Theorem 1.1 in [GZ23], we have the bound

$$\sum_{P \in \mathcal{P}_{2g+1}} |L(\tfrac{1}{2}+z, \chi_P)|^k \ll q^{2g} g^{k(k+1)/2-1}. \tag{7.4.38}$$

Combining this with the bound in (7.4.36) then gives us the upper bound

$$\sum_{P \in \mathcal{P}_{2g+1}} |L^{(\mu)}(\tfrac{1}{2}, \chi_P)|^k \ll q^{2g} g^{k(k+1)/2+k\mu-1} \tag{7.4.39}$$

for the moments of the derivatives. Recall that by the Prime Polynomial Theorem, we have that

$$|\mathcal{P}_{2g+1}| \sim \frac{q^{2g}}{g}, \tag{7.4.40}$$

as $g \to \infty$ and so the above bound can be written as

$$\frac{1}{|\mathcal{P}_{2g+1}|}\sum_{P\in\mathcal{P}_{2g+1}}|L^{(\mu)}(\tfrac{1}{2},\chi_P)|^k \ll g^{k(k+1)/2+k\mu}. \tag{7.4.41}$$

This bound is sharp and of the order predicted by the random matrix theory results of Chapter 4.

Now, to obtain the simultaneous non-vanishing result, we again apply Hölder's inequality, this time to yield

$$\begin{aligned}\Big(\sum_{P\in\mathcal{P}_{2g+1}} & L^{(\mu)}(\tfrac{1}{2},\chi_P)L^{(\nu)}(\tfrac{1}{2},\chi_P)\Big)^4 \\ &\le \Big(\sum_{P\in\mathcal{P}^*_{2g+1}} 1\Big)^2\Big(\sum_{P\in\mathcal{P}_{2g+1}} L^{(\mu)}(\tfrac{1}{2},\chi_P)^4\Big)\Big(\sum_{P\in\mathcal{P}_{2g+1}} L^{(\nu)}(\tfrac{1}{2},\chi_P)^4\Big).\end{aligned} \tag{7.4.42}$$

Note that the derivatives of $L(s,\chi_P)$ at $s = 1/2$ are all real so by raising the L-functions to the power 4, we may drop the absolute values present in Hölder's inequality. By Theorem 7.2.1 and the Prime Polynomial Theorem, we know that the left-hand side of (7.4.42) is asymptotically of size $(q^{2g}g^{\mu+\nu+2})^4$. For the last two terms on the right-hand side, we take $k = 4$ in (7.4.39) to get the bound

$$\sum_{P\in\mathcal{P}_{2g+1}} L^{(\mu)}(\tfrac{1}{2},\chi_P)^4 \ll q^{2g}g^{9+4\mu}. \tag{7.4.43}$$

Therefore, by rearranging the inequality in (7.4.42), we find that

$$\Big(\sum_{P\in\mathcal{P}^*_{2g+1}} 1\Big)^2 \gg \frac{(q^{2g})^2}{g^{10}}. \tag{7.4.44}$$

By taking the square-root and recalling the definition of $\mathcal{P}^*_{2g+1}$ we therefore obtain

$$|\{P\in\mathcal{P}_{2g+1} : L^{(\mu)}(\tfrac{1}{2},\chi_P)L^{(\nu)}(\tfrac{1}{2},\chi_P)\neq 0\}| \gg \frac{q^{2g}}{g^5}, \tag{7.4.45}$$

Applying the Prime Polynomial Theorem once again then yields the desired result.

Bibliography

[AB22] J. C. Andrade and C. G. Best. Random matrix theory and moments of moments of L-functions. *Random Matrices: Theory and Appl.*, 12(3):2350002, 2022.

[AB24] J. C. Andrade and C. G. Best. Joint moments of derivatives of characteristic polynomials of random symplectic and orthogonal matrices. *J. Phys. A: Math. Theor.*, 57(20):205205, 2024.

[ABB17] L.-P. Arguin, D. Belius, and P. Bourgade. Maximum of the characteristic polynomial of random unitary matrices. *Commun. Math. Phys.*, 349(2):703–751, 2017.

[ABB$^+$19] L.-P. Arguin, D. Belius, P. Bourgade, M. Radziwiłł, and K. Soundararajan. Maximum of the Riemann zeta function on a short interval of the critical line. *Commun. Pure Appl. Math.*, 72(3):500–535, 2019.

[ABGS21] T. Assiotis, B. Bedert, M. A. Gunes, and A. Soor. On a distinguished family of random variables and Painlevé equations. *Probab. Math. Phys.*, 2(3):613–642, 2021.

[ABK22] T. Assiotis, E. C. Bailey, and J. P. Keating. On the moments of the moments of the characteristic polynomials of Haar distributed symplectic and orthogonal matrices. *Ann. Inst. Henri Poincaré D*, 9(3):567–604, 2022.

[ABP$^+$14] S. A. Altuğ, S. Bettin, I. Petrow, Rishikesh, and I. Whitehead. A recursion formula for moments of derivatives of random matrix polynomials. *Q. J. Math.*, 65(4):1111–1125, 2014.

[ABR20] L.-P. Arguin, P. Bourgade, and M. Radziwiłł. The Fyodorov-Hiary-Keating conjecture. I. *arXiv preprint arXiv:2007.00988*, 2020.

[ABR23] L.-P. Arguin, P. Bourgade, and M. Radziwiłł. The Fyodorov-Hiary-Keating conjecture. II. *arXiv preprint arXiv:2307.00982*, 2023.

[ACRS24] E. Alvarez, B. Conrey, M. O. Rubinstein, and N. C. Snaith. Moments of the derivative of the characteristic polynomial of unitary matrices. *arXiv preprint arXiv:2407.13124*, 2024.

[AGKW24] T. Assiotis, M. A. Gunes, J. P. Keating, and F. Wei. Exchangeable arrays and integrable systems for characteristic polynomials of random matrices. *arXiv preprint arXiv:2407.19233*, 2024.

[AGS22] T. Assiotis, M. A. Gunes, and A. Soor. Convergence and an explicit formula for the joint moments of the circular Jacobi β-ensemble characteristic polynomial. *Math. Phys. Anal. Geom.*, 25(2):15, 2022.

[AJ21] J. Andrade and H. Jung. Mean values of derivatives of L-functions in function fields: IV. *J. Korean Math. Soc.*, 58(6):1529–1547, 2021.

[AJS21] J. C. Andrade, H. Jung, and A. Shamesaldeen. The integral moments and ratios of quadratic Dirichlet L-functions over monic irreducible polynomials in $\mathbb{F}_q[T]$. *Ramanujan J.*, 56(1):23–66, 2021.

[AK12] J. C. Andrade and J. P. Keating. The mean value of $L(\frac{1}{2}, \chi)$ in the hyperelliptic ensemble. *J. Number Theory*, 132(12):2793–2816, 2012.

[AK13] J. C. Andrade and J. P. Keating. Mean value theorems for L-functions over prime polynomials for the rational function field. *Acta Arith.*, 161(4):371–385, 2013.

[AK14] J. C. Andrade and J. P. Keating. Conjectures for the integral moments and ratios of L-functions over function fields. *J. Number Theory*, 142:102–148, 2014.

[AK21] T. Assiotis and J. P. Keating. Moments of moments of characteristic polynomials of random unitary matrices and lattice point counts. *Random Matrices: Theory and Appl.*, 10(02):2150019, 2021.

[AKW22] T. Assiotis, J. P. Keating, and J. Warren. On the joint moments of the characteristic polynomials of random unitary matrices. *Int. Math. Res. Not.*, 2022(18):14564–14603, 2022.

[Alb24] B. Alberts. Explicit analytic continuation of Euler products. *arXiv preprint arXiv:2406.18190*, 2024.

[Alv22] E. Alvarez. *Moments of characteristic polynomials and their derivatives in the classical compact ensembles.* PhD Thesis, University of Bristol, 2022.

[And16] J. Andrade. Rudnick and Soundararajan's theorem for function fields. *Finite Fields Appl.*, 37:311–327, 2016.

[And19] J. Andrade. Mean values of derivatives of L-functions in function fields: III. *Proc. Roy. Soc. Edinburgh Sect. A*, 149(4):905–913, 2019.

[AR12] M. W. Alderson and M. O. Rubinstein. Conjectures and experiments concerning the moments of $L(1/2, \chi_d)$. *Exp. Math.*, 21(3):307–328, 2012.

[AR16] J. Andrade and S. Rajagopal. Mean values of derivatives of L-functions in function fields: I. *J. Math. Anal. Appl.*, 443(1):526–541, 2016.

[Art24] E. Artin. Quadratische Körper im Gebiete der höheren Kongruenzen. i, ii. *Math. Z.*, 19(1):153–206, 1924.

[AS72] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions (tenth printing).* Dover, 1972.

[Ass22] T. Assiotis. On the moments of the partition function of the $C\beta E$ field. *J. Stat. Phys.*, 187(2):14, 2022.

[AY21] J. C. Andrade and M. Yiasemides. The fourth moment of derivatives of Dirichlet L-functions in function fields. *Math. Z.*, 299(1–2):671–697, 2021.

[BA20] Y. Barhoumi-Andréani. A new approach to the characteristic polynomial of a random unitary matrix. *arXiv preprint arXiv:2011.02465*, 2020.

[Bar00] E. W. Barnes. The theory of the G-function. *Q. J. Math.*, 31:264–314, 1900.

[BBB$^+$19a] E. C. Bailey, S. Bettin, G. Blower, J. B. Conrey, A. Prokhorov, M. O. Rubinstein, and N. C. Snaith. Mixed moments of characteristic polynomials of random unitary matrices. *J. Math. Phys.*, 60(8):083509, 2019.

[BBB+19b] E. Basor, P. Bleher, R. Buckingham, T. Grava, A. Its, E. Its, and J. P. Keating. A representation of joint moments of CUE characteristic polynomials in terms of Painlevé functions. *Nonlinearity*, 32(10):4033–4078, 2019.

[BCD+17] A. Bucur, E. Costa, C. David, J. Guerreiro, and D. Lowry-Duda. Traces, high powers and one level density for families of curves over finite fields. *Math. Proc. Cambridge Philos. Soc.*, 165(2):225–248, 2017.

[BCDT01] C. Breuil, B. Conrad, F. Diamond, and R. Taylor. On the modularity of elliptic curves over $\mathbf{Q}$. *J. Amer. Math. Soc.*, 14:843–939, 2001.

[BCY11] H. M. Bui, B. Conrey, and M. P. Young. More than 41% of the zeros of the zeta function are on the critical line. *Acta Arith.*, 150(1):35–64, 2011.

[BDPW24] J. Bergström, A. Diaconu, D. Petersen, and C. Westerland. Hyperelliptic curves, the scanning map, and moments of families of quadratic L-functions. *arXiv preprint arXiv:2302.07664v2*, 2024.

[Bes24] C. G. Best. The mixed second moment of prime quadratic Dirichlet L-functions over function fields. *Submitted*, 2024.

[BF18] H. M. Bui and A. Florea. Zeros of quadratic Dirichlet L-functions in the hyperelliptic ensemble. *Trans. Amer. Math. Soc.*, 370(11):8013–8045, 2018.

[BF20] H. M. Bui and A. Florea. Moments of Dirichlet L–functions with prime conductors over function fields. *Finite Fields Appl.*, 64:101659, 2020.

[BFH90] D. Bump, S. Friedberg, and J. Hoffstein. Nonvanishing theorems for L-functions of modular forms and their derivatives. *Invent. Math.*, 102(1):543–618, 1990.

[BFK23] H. M. Bui, A. Florea, and J. P. Keating. The Ratios Conjecture and upper bounds for negative moments of L-functions over function fields. *Trans. Amer. Math. Soc.*, 376(6):4453–4510, 2023.

[BG17] S. Bettin and S. M. Gonek. The $\theta = \infty$ conjecture implies the Riemann hypothesis. *Mathematika*, 63:29–33, 2017.

[BGR18] E. Basor, F. Ge, and M. O. Rubinstein. Some multidimensional integrals in number theory and connections with the Painlevé V equation. *J. Math. Phys.*, 59(9):091404, 2018.

[BJ19] S. Bae and H. Jung. Note on the mean values of derivatives of quadratic Dirichlet L-functions in function fields. *Finite Fields Appl.*, 57:249–267, 2019.

[BK95] E. B. Bogomolny and J. P. Keating. Random matrix theory and the Riemann zeros. I. three- and four-point correlations. *Nonlinearity*, 8(6):1115, 1995.

[BK96] E. B. Bogomolny and J. P. Keating. Random matrix theory and the Riemann zeros II: n-point correlations. *Nonlinearity*, 9(4):911, 1996.

[BK19] E. C. Bailey and J. P. Keating. On the moments of the moments of the characteristic polynomials of random unitary matrices. *Commun. Math. Phys.*, 371(2):689–726, 2019.

[BK21] E. C. Bailey and J. P. Keating. On the moments of the moments of $\zeta(1/2+it)$. *J. Number Theory*, 223:79–100, 2021.

[BK22] E. C. Bailey and J. P. Keating. Maxima of log-correlated fields: some recent developments. *J. Phys. A: Math. Theor.*, 55(5):053001, 2022.

[BS06] A. Borodin and E. Strahov. Averages of characteristic polynomials in random matrix theory. *Comm. Pure Appl. Math.*, 59(2):161–253, 2006.

[CF00] J. B. Conrey and D. W. Farmer. Mean values of L-functions and symmetry. *Int. Math. Res. Notices*, 2000(17):883–908, 2000.

[CFK$^+$03] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith. Autocorrelation of random matrix polynomials. *Commun. Math. Phys.*, 237(3):365–395, 2003.

[CFK$^+$05] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith. Integral moments of L-functions. *Proc. London Math. Soc. (3)*, 91(01):33–104, 2005.

[CFS05] J. B. Conrey, P. J. Forrester, and N. C. Snaith. Averages of ratios of characteristic polynomials for the compact classical groups. *Int. Math. Res. Not.*, 2005(7):397–431, 2005.

[CFZ08] B. Conrey, D. W. Farmer, and M. R. Zirnbauer. Autocorrelation of ratios of l-functions. *Commun. Number Theory Phys.*, 2(3):593–636, 2008.

[CG84] J. B. Conrey and A. Ghosh. On mean values of the zeta-function. *Mathematika*, 31(1):159–161, 1984.

[CG92] J. B. Conrey and A. Ghosh. Mean values of the zeta-function, III. *Proc. Amalfi Conf. Analytic Number Theory*, pages 35–59, 1992.

[CG96] J. H. Conway and R. K. Guy. *The book of numbers.* Copernicus New York, NY, 1996.

[CG01] J. B. Conrey and S. M. Gonek. High moments of the Riemann zeta-function. *Duke Math. J.*, 107(3):577–604, 2001.

[Cho65] S. Chowla. The Riemann hypothesis and Hilbert's tenth problem. *Norske Vid. Selsk. Forh. (Trondheim)*, 38:62–64, 1965.

[CK15a] T. Claeys and I. Krasovsky. Toeplitz determinants with merging singularities. *Duke Math. J.*, 164(15):2897–2987, 2015.

[CK15b] B. Conrey and J. P. Keating. Moments of zeta and correlations of divisor-sums: I. *Phil. Trans. R. Soc. A*, 373(2040):20140313, 2015.

[CK15c] B. Conrey and J. P. Keating. Moments of zeta and correlations of divisor-sums: II. *Advances in the Theory of Numbers (New York: Springer)*, pages pp 75–85, 2015.

[CK15d] B. Conrey and J. P. Keating. Moments of zeta and correlations of divisor-sums: III. *Indagat. Math.*, 26(5):736–747, 2015.

[CK16] B. Conrey and J. P. Keating. Moments of zeta and correlations of divisor-sums: IV. *Res. Number Theory*, 2(1):24, 2016.

[CK18] B. Conrey and J. P. Keating. Moments of zeta and correlations of divisor-sums: V. *Proc. London Math. Soc. (3)*, 118(4):729–752, 2018.

[CKRS06] J. B. Conrey, J. P. Keating, M. O. Rubinstein, and N. C. Snaith. Random matrix theory and the Fourier coefficients of half-integral-weight forms. *Exp. Math.*, 15(1):67–82, 2006.

[CMN18] R. Chhaibi, T. Madaule, and J. Najnudel. On the maximum of the CβE field. *Duke Math. J.*, 167(12):2243–2345, 2018.

[Con83] B. Conrey. Zeros of derivatives of Riemann's ξ-function on the critical line. *J. Number Theory*, 16(1):49–74, 1983.

[Con88] J. B. Conrey. The fourth moment of derivatives of the Riemann zeta-function. *Q. J. Math.*, 39(153):21–36, 1988.

[Con89] J. B. Conrey. More than two fifths of the zeros of the Riemann zeta function are on the critical line. *J. Reine Angew. Math.*, 399:1–26, 1989.

[CRS06] J. B. Conrey, M. O. Rubinstein, and N. C. Snaith. Moments of the derivative of characteristic polynomials with an application to the Riemann zeta function. *Commun. Math. Phys.*, 267(3):611–629, 2006.

[CS07] J. B. Conrey and N. C. Snaith. Applications of the L-functions ratios conjectures. *Proc. London Math. Soc. (3)*, 94(3):594–646, 2007.

[Cur23] M. J. Curran. Freezing transition and moments of moments of the Riemann zeta function. *arXiv preprint arXiv:2301.10634*, 2023.

[DĐ21] G. Djanković and D. Đokić. The mixed second moment of quadratic Dirichlet L-functions over function fields. *Rocky Mountain J. Math.*, 51(6):2003–2017, 2021.

[Deh08] P.-O. Dehaye. Joint moments of derivatives of characteristic polynomials. *Algebra Number Theory*, 2(1):31–68, 2008.

[DFF$^+$10] E. Dueñez, D. W. Farmer, S. Froehlich, C. P. Hughes, F. Mezzadri, and T. Phan. Roots of the derivative of the Riemann-zeta function and of characteristic polynomials. *Nonlinearity*, 23(10):2599–2621, 2010.

[DGH03] A. Diaconu, D. Goldfeld, and J. Hoffstein. Multiple dirichlet series and moments of zeta and L-functions. *Compos. Math.*, 139(3):297–360, 2003.

[Dia19] A. Diaconu. On the third moment of $L(\frac{1}{2}, \chi_d)$ I: The rational function field case. *J. Number Theory*, 198:1–42, 2019.

[dlVP96] C. J. de le Vallée Poussin. Recherches analytiques la théorie des nombrespremiers. *Ann. Soc. Sci. Bruxelles*, 20(2):183–256, 281–297, 1896.

[DW21] A. Diaconu and I. Whitehead. On the third moment of $L(\frac{1}{2}, \chi_d)$ II: the number field case. *J. Euro. Math. Soc.*, 23(6):2051–2070, 2021.

[Dys62] F. J. Dyson. Statistical theory of the energy levels of complex systems. i. *J. Math. Phys.*, 3(1):140–156, 1962.

[ELS20] J. S. Ellenberg, W. Li, and M. Shusterman. Nonvanishing of hyperelliptic zeta functions over finite fields. *Algebra Number Theory*, 14(7):1895–1909, 2020.

[Eul44] L. Euler. Variae observationes circa series infinitas. *Comm. Acd. Sci. Petropolitanae*, 9:160–188, 1744.

[Fah21] B. Fahs. Uniform asymptotics of Toeplitz determinants with Fisher–Hartwig singularities. *Commun. Math. Phys.*, 383(2):685–730, 2021.

[Far93] D. W. Farmer. Long mollifiers of the Riemann zeta-function. *Mathematika*, 40(1):71–87, 1993.

[FHK12] Y. V. Fyodorov, G. A. Hiary, and J. P. Keating. Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta function. *Phys. Rev. Lett.*, 108(17):170601, 2012.

[FK14] Y. V. Fyodorov and J. P. Keating. Freezing transitions and extreme values: random matrix theory, and disordered landscapes. *Philos. Trans. R. Soc. Math. Phys. Eng. Sci.*, 372(2007):20120503, 2014.

[Flo17a] A. Florea. The fourth moment of quadratic Dirichlet L-functions over function fields. *Geom. Funct. Anal.*, 27(3):541–595, 2017.

[Flo17b] A. Florea. The second and third moment of $L(1/2, \chi)$ in the hyperelliptic ensemble. *Forum Math.*, 29(4):873–892, 2017.

[Flo17c] A. M. Florea. Improving the error term in the mean value of $L(\frac{1}{2}, \chi)$ in the hyperelliptic ensemble. *Int. Math. Res. Not.*, 2017(20):6119–6148, 2017.

[FW06] P. J. Forrester and N. S. Witte. Boundary conditions associated with the Painlevé III$'$ and V evaluations of some random matrix averages. *J. Phys. A: Math. Gen.*, 39(28):8983–8995, 2006.

[Gao21] P. Gao. Sharp upper bounds for moments of quadratic Dirichlet L-functions. *arXiv preprint arXiv:2101.08483*, 2021.

[GH85] D. Goldfeld and J. Hoffstein. Eisenstein series of 1/2-integral weight and the mean value of real Dirichlet L-series. *Invent. Math.*, 80(2):185–208, 1985.

[GHK07] S. M. Gonek, C. P. Hughes, and J. P. Keating. A hybrid Euler-Hadamard product for the Riemann zeta function. *Duke Math. J.*, 136(3):507–549, 2007.

[Gon89] S. M. Gonek. On negative moments of the Riemann zeta-function. *Mathematika*, 36(1):71–88, 1989.

[Gun24] M. A. Gunes. Characteristic polynomials of orthogonal and symplectic random matrices, Jacobi ensembles & L-functions. *Random Matrices: Theory and Appl.*, 13(2):2450006, 2024.

[GW23] R. Gharakhloo and N. S. Witte. Modulated bi-orthogonal polynomials on the unit circle: the $2j-k$ and $j-2k$ systems. *Constr. Approx.*, 58(1):1–74, 2023.

[GZ23] P. Gao and L. Zhao. Moments of quadratic Dirichlet L-functions over function fields. *Finite Fields Appl.*, 85:102113, 2023.

[Had96] J. Hadamard. Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques. *Bull. Soc. Math. France*, 24:199–220, 1896.

[Hal99] R. R. Hall. The behaviour of the Riemann zeta-function on the critical line. *Mathematika*, 46(2):281–313, 1999.

[Hal04] R. R. Hall. On the stationary points of Hardy's function $Z(t)$. *Acta Arith.*, 111(2):125–140, 2004.

[Har13] A. J. Harper. Sharp conditional bounds for moments of the Riemann zeta function. *arXiv preprint arXiv:1305.4618*, 2013.

[Har19] A. J. Harper. On the partition function of the Riemann zeta function, and the Fyodorov–Hiary–Keating conjecture. *arXiv preprint arXiv:1906.05783*, 2019.

[Hay66] D. Hayes. The expression of a polynomial as a sum of three irreducibles. *Acta Arith.*, 11(4):461–488, 1966.

[HB79] D. R. Heath-Brown. The fourth power moment of the Riemann zeta function. *Proc. London Math. Soc. (3)*, 38(3):385–422, 1979.

[HB81] D. R. Heath-Brown. Fractional moments of the Riemann zeta-function. *J. Lond. Math. Soc. (2)*, 24(1):65–78, 1981.

[HL16] G. H. Hardy and J. E. Littlewood. Contributions to the theory of the riemann zeta-function and the theory of the distribution of primes. *Acta Math.*, 41(1):119–196, 1916.

[HPZ16] A. Huckleberry, A. Püttmann, and M. R. Zirnbauer. Haar expectations of ratios of random characteristic polynomials. *Complex Anal. Synerg.*, 2(1):Paper No. 1, 73 pp, 2016.

[HRS19] W. Heap, M. Radziwiłł, and K. Soundararajan. Sharp upper bounds for fractional moments of the Riemann zeta function. *Q. J. Math.*, 70(4):1387–1396, 2019.

[HS22] W. Heap and K. Soundararajan. Lower bounds for moments of zeta and L-functions revisited. *Mathematika*, 68(1):1–14, 2022.

[Hug01] C. P. Hughes. *On the characteristic polynomial of a random unitary matrix and the Riemann zeta function.* PhD Thesis, University of Bristol, 2001.

[Ing27] A. E. Ingham. Mean-value theorems in the theory of the Riemann zeta-function. *Proc. London Math. Soc. (2)*, 27(4):273–300, 1927.

[Isi71] A. Isihara. *Statistical Physics.* Academic Press, New York, 1971.

[Iwa90] H. Iwaniec. On the order of vanishing of modular L-functions at the critical point. *Sém. Theor. Nombres Bordeaux (2)*, 2(2):365–376, 1990.

[Jun22] H. Jung. Mean values of derivatives of quadratic prime Dirichlet L-functions in function fields. *Commun. Korean Math. Soc.*, 37(3):635–648, 2022.

[Jut81] M. Jutila. On the mean value of $L(\frac{1}{2}, \chi)$ for real characters. *Analysis*, 1(2):149–161, 1981.

[KMV00] E. Kowalski, P. Michel, and J. VanderKam. Non-vanishing of high derivatives of automorphic L-functions at the center of the critical strip. *J. reine angew. Math.*, 526:1–34, 2000.

[KN04] R. Killip and I. Nenciu. Matrix models for circular ensembles. *Int. Math. Res. Not.*, 2004(50):2665–2701, 2004.

[KO08] J. P. Keating and B. E. Odgers. Symmetry transitions in random matrix theory & L-functions. *Commun. Math. Phys.*, 281(2):499–528, 2008.

[KRRGR18] J. P. Keating, B. Rodgers, E. Roditty-Gershon, and Z. Rudnick. Sums of divisor functions in $\mathbb{F}_q[t]$ and matrix integrals. *Math. Z.*, 288(1–2):167–198, 2018.

[KS99a] N. Katz and P. Sarnak. Zeroes of zeta functions and symmetry. *Bull. Amer. Math. Soc.*, 36(1):1–26, 1999.

[KS99b] N. M. Katz and P. Sarnak. *Random matrices, Frobenius eigenvalues, and monodromy*, volume 45 of *Amer. Math. Soc. Colloq. Publ.* American Mathematical Society, Providence, RI, 1999.

[KS00a] J. P. Keating and N. C. Snaith. Random matrix theory and L-functions at $s = 1/2$. *Commun. Math. Phys.*, 214(1):91–110, 2000.

[KS00b] J. P. Keating and N. C. Snaith. Random matrix theory and $\zeta(1/2+it)$. *Commun. Math. Phys.*, 214(1):57–89, 2000.

[KS03] J. P. Keating and N. C. Snaith. Random matrices and L-functions. *J. Phys. A: Math. Gen.*, 36(12):2859–2881, 2003.

[KW22] J. P. Keating and M. D. Wong. On the critical–subcritical moments of moments of random characteristic polynomials: a GMC perspective. *Commun. Math. Phys.*, 394(3):1247–1301, 2022.

[KW24a] J. P. Keating and F. Wei. Joint moments of higher order derivatives of CUE characteristic polynomials I: asymptotic formulae. *Int. Math. Res. Not.*, 2024(12):9607–9632, 2024.

[KW24b] J. P. Keating and F. Wei. Joint moments of higher order derivatives of CUE characteristic polynomials II: structures, recursive relations, and applications. *Nonlinearity*, 37(8):085009, 2024.

[Lev74] N. Levinson. More than one third of zeros of Riemann's zeta-function are on $\sigma = 1/2$. *Adv. Math.*, 13(4):383–436, 1974.

[Li18] W. Li. Vanishing of hyperelliptic L-functions at the central point. *J. Number Theory*, 191:85–103, 2018.

[Li24] X. Li. Moments of quadratic twists of modular L-functions. *Invent. math.*, 237(8):697–733, 2024.

[LMF24] The LMFDB Collaboration. The L-functions and modular forms database. `https://www.lmfdb.org`, 2024. [Online; accessed 30 August 2024].

[Mec19] E. S. Meckes. *The random matrix theory of the classical compact groups*, volume 218 of *Cambridge tracts in mathematics*. Cambridge University Press, 2019.

[Mez03] F. Mezzadri. Random matrix theory and the zeros of $\zeta'(s)$. *J. Phys. A: Math. Gen.*, 36(12):2945–2962, 2003.

[MM91] M. R. Murty and V. K. Murty. Mean values of derivatives of modular L-series. *Ann. of Math.*, 133(3):447–475, 1991.

[Mon72] H. L. Montgomery. The pair correlation of zeros of the zeta function. *Proc. Sympos. Pure Math.*, 24:181–193, 1972.

[MPPRW24] J. Miller, P. Patzt, D. Petersen, and O. Randal-Williams. Uniform twisted homological stability. *arXiv preprint arXiv:2402.00354*, 2024.

[MV00] P. Michel and J. VanderKam. Non-vanishing of high derivatives of Dirichlet L-functions at the central point. *J. Number Theory*, 81(1):130–148, 2000.

[Naj18] J. Najnudel. On the extreme values of the Riemann zeta function on random intervals of the critical line. *Probab. Theory Relat. Fields*, 172(1–2):387–452, 2018.

[Nor04] J.-M. Normand. Calculation of some determinants using the s-shifted factorial. *J. Phys. A: Math. Gen.*, 37(22):5737–5762, 2004.

[ÖC99] A. E. Özlük and Snyder. C. On the distribution of the nontrivial zeros of quadratic L-functions close to the real axis. *Acta Arith.*, 91(3):209–228, 1999.

[Odl89] A. Odlyzko. The 10^{20}-th zero of the Riemann zeta function and 70 million of its neighbors. *https://www-users.cse.umn.edu/ odlyzko/unpublished/index.html, unpublished manuscript*, 1989.

[Pet12] I. Petrow. Moments of $L'(\frac{1}{2})$ in the family of quadratic twists. *Int. Math. Res. Not. IMRN*, 2014(6):1576–1612, 2012.

[PRZZ20] K. Pratt, N. Robles, A. Zaharescu, and D. Zeindler. More than five-twelfths of the zeros of ζ are on the critical line. *Res. Math. Sci.*, 7:2, December 2020.

[PZ17] E. Paquette and O. Zeitouni. The maximum of the CUE field. *Int. Math. Res. Not.*, 2018(16):5028–5119, 2017.

[Rad12] M. Radziwill. Limitations to mollifying $\zeta(s)$. *arXiv preprint arXiv:1207.6583*, 2012.

[Ram80] K. Ramachandra. Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series-II. *Hardy-Ramanujan J.*, 3:1–24, 1980.

[Rie59] B. Riemann. Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse. *Monatsber. Akad. Berlin*, pages 671–680, 1859.

[Ros02] M. Rosen. *Number theory in function fields*, volume 210 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[RS96] Z. Rudnick and P. Sarnak. Zeros of principal L-functions and random matrix theory. *Duke Math. J.*, 81(2):269–322, 1996.

[RS05] Z. Rudnick and K. Soundararajan. Lower bounds for moments of L -functions. *Proc. Natl. Acad. Sci.*, 102(19):6837–6838, 2005.

[RS06] Z. Rudnick and K. Soundararajan. Lower bounds for moments of L-functions: symplectic and orthogonal examples. Multiple Dirichlet series, automorphic forms and analytic number theory. *Proc. Sympos. Pure Math.*, 75:293–303, 2006.

[RS13] M. Radziwiłł and K. Soundararajan. Continuous lower bounds for moments of zeta and L-functions. *Mathematika*, 59(1):119–128, 2013.

[RS15] M. Radziwiłł and K. Soundararajan. Moments and distribution of central L -values of quadratic twists of elliptic curves. *Invent. math.*, 202(3):1029–1068, 2015.

[Rud10] Z. Rudnick. Traces of high powers of the Frobenius class in the hyperelliptic ensemble. *Acta Arith.*, 143(1):81–99, 2010.

[RW15] M. O. Rubinstein and K. Wu. Moments of zeta functions associated to hyperelliptic curves over finite fields. *Philos. Trans. R. Soc. A*, 373(2040):20140307, 2015.

[Sel42] A. Selberg. On the zeros of Riemann's zeta-function. *Skr. Norske Vid. Akad. Oslo I.*, 1942(10):1–59, 1942.

[Sel46] A. Selberg. Contributions to the theory of the Riemann zeta-function. *Arch. Math. Naturvid.*, 48(5):89–155, 1946.

[She21] Q. Shen. The fourth moment of quadratic Dirichlet L-functions. *Math. Z.*, 298(2):713–745, 2021.

[Son20] K. Sono. The second moment of quadratic Dirichlet L-functions. *J. Number Theory*, 206:194–230, 2020.

[Sou98] K. Soundararajan. The horizontal distribution of zeros of $\zeta'(s)$. *Duke Math. J.*, 91(1):33–59, 1998.

[Sou00] K. Soundararajan. Nonvanishing of quadratic Dirichlet L-functions at $s = \frac{1}{2}$. *Ann. of Math. (2)*, 152(2):447–488, 2000.

[Sou09] K. Soundararajan. Moments of the Riemann zeta function. *Ann. Math.*, 170(2):981–993, 2009.

[Spe35] A. Speiser. Geometrisches zur Riemannschen Zetafunktion. *Math. Ann.*, 110(1):514–521, 1935.

[SS24] Q. Shen and J. Stucky. The fourth moment of quadratic Dirichlet L-functions II. *arXiv preprint arXiv:2402.01497*, 2024.

[SY10] K. Soundararajan and M. P. Young. The second moment of quadratic twists of modular L-functions. *J. Eur. Math. Soc.*, 12(5):1097–1116, 2010.

[Tit86] E. C. Titchmarsh. *The Theory of the Riemann Zeta-Function (2nd edition)*. Oxford University Press, 1986.

[TP85] M. Tenenbaum and H. Pollard. *Ordinary differential equations (2nd edition)*. Dover Publications, Inc., New York, 1985.

[TW95] R. Taylor and A. Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572, 1995.

[Wei48] A. Weil. Sur les courbes algébriques et les variétés qui s'en déduisent. *Publ. Inst. Math. Univ. Strasbourg*, 7 (1945), 1948.

[Wey66] H. Weyl. *The Classical Groups: Their Invariants and Representations*. Princeton University Press, 1966.

[Wil95] A. Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

[Win12] B. Winn. Derivative moments for characteristic polynomials from the CUE. *Commun. Math. Phys.*, 315(2):531–562, 2012.

[You09] M. P. Young. The first moment of quadratic Dirichlet L-functions. *Acta Arith.*, 138(1):73–99, 2009.

[You10] M. P. Young. A short proof of Levinson's theorem. *Arch. Math.*, 95(6):539–548, 2010.

[You12] M. P. Young. The third moment of quadratic Dirichlet L-functions. *Sel. Math. New Ser.*, 19(2):509–543, 2012.

[Zha01] Y. Zhang. On the zeros of $\zeta'(s)$ near the critical line. *Duke Math. J.*, 110(3):555–572, 2001.