Copy Edited by: P.A.
J. London Math. Soc. Page 1 of 19
doi:10.1112/jlms/jdr021

On the syntomic regulator for products of elliptic curves

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Abstract

We consider the syntomic regulator on the integral motivic cohomology of a smooth proper surface over a p-adic field and apply a recent formula of Besser that uses p-adic integration theory, in particular his theory of triple indices on Coleman integrals, to the case of a self-product of an elliptic curve. The method is suitable to separate decomposable from indecomposable elements in the (integral) motivic cohomology. As an interesting example, we construct an element that, though not given in decomposable form, becomes decomposable after taking p-adic completion.

Introduction

The purpose of this paper is to apply a new method of Besser how to compute the syntomic regulator of the (integral) motivic cohomology of a smooth proper surface X over a p-adic field L with good reduction to the case of a product of elliptic curves.

Let $V = H^2_{\text{et}}(\bar{X}, \mathbb{Q}_p(2))$ be the second étale cohomology considered as a $G_L = \text{Gal}(\bar{L}/L)$-representation. We know that V is a crystalline representation. Let $H^1_f(L, V)$ be the Bloch–Kato group in $H^1(L, V) = \text{Ext}^1_{G_L}(L, V)$ classifying extensions

$$0 \longrightarrow V \longrightarrow W \longrightarrow L \longrightarrow 0$$

of G_L-representations that are crystalline.

For a regular scheme Z, we denote by $H^i(Z, \mathcal{K}_2)$ the Zariski cohomology of the algebraic K-sheaf $\mathcal{K}_2$ and let $H^i(\widehat{Z, \mathcal{K}_2}) := \varprojlim_n H^i(Z, \mathcal{K}_2)/p^n$ be its p-adic completion. If $\mathcal{X}$ is a proper smooth model over the ring of integers $\mathcal{O}_L$ in L, then the syntomic regulator r_{syn} is a map from $H^1_{\text{zar}}(\mathcal{X}, \mathcal{K}_2)$ to the syntomic cohomology $H^3_{\text{syn}}(\mathcal{X}, S_{\mathbb{Q}_p}(2))$ which is isomorphic to $H^1_f(L, V)$ by the p-adic points conjecture (compare [**16**]). Note that if X is a surface, an element in $H^1(X, \mathcal{K}_2)$ is represented by a finite formal sum $\theta = \sum_i (Z_i, f_i)$, where Z_i are curves on X and the f_i are rational functions on the Z_i's satisfying the condition Q2

$$\sum_i \text{div}(f_i) = 0$$

on X. For a scheme $\mathcal{X}$ which is smooth over $\mathcal{O}_L$, one has a similar description for $H^1(\mathcal{X}, \mathcal{K}_2)$ with Z_i being irreducible subschemes of codimension 1 on $\mathcal{X}$.

Besser's technique reduces the computation of $r_{\text{syn}}(z)$ to p-adic integration theory on curves, in particular his theory of triple indices for Coleman integrals plays a crucial role. It is relatively easy to see that r_{syn} induces an injection

$$r_{\text{syn}} : H^1_{\text{zar}}(\widehat{\mathcal{X}, \mathcal{K}_2}) \otimes \mathbb{Q}_p \hookrightarrow H^3_{\text{syn}}(\mathcal{X}, S_{\mathbb{Q}_p}(2))$$

(we recall the argument in paragraph 3) and one might expect that this regulator map is, at least for a large class of varieties, an isomorphism.

For example, if the geometric genus of X is zero, then $H^3_{\text{syn}}(\mathcal{X}, S_{\mathbb{Q}_p}(2)) \cong H^1_f(L, \text{NS}(\bar{X}) \otimes \mathbb{Q}_p(1))$ and if $\text{NS}(\bar{X}) = \text{NS}(X)$, the Néron–Severi group of X, one can show that the image of

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Received 7 December 2009; revised 27 October 2010.
2010 *Mathematics Subject Classification* 14F30, 19E15, 19F27. Q3, Q4

the decomposable part $\mathrm{Pic}(\mathcal{X}) \otimes \mathcal{O}_L^*$ under r_{syn} generates $H_f^1(L, \mathrm{NS}(\bar{X}) \otimes \mathbb{Q}_p(1))$ (for a detailed argument see paragraph 3). We conjecture that r_{syn} is an isomorphism if $X = E \times E$ is the self-product of an elliptic curve. It follows from recent work of Saito and Sato that this would imply the finiteness of the torsion subgroup in the Chow group of zero-cycles $\mathrm{Ch}_0(E \times E_{\mathbb{Q}_p})$ (compare [19]).

From now on let $\mathcal{X} = \mathcal{E} \times \mathcal{E}_{\mathbb{Z}_p}$ where $\mathcal{E}$ is a smooth proper model over $\mathbb{Z}_p$ of an elliptic curve E defined over $\mathbb{Q}$ with complex multiplication by the ring of integers in an imaginary quadratic field, with ordinary good reduction at p.

It then follows from diagram (3.6) below that the image of the decomposable part $\mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Z}_p^*$ in $H^1(\mathcal{X}, \mathcal{K}_2)$ generates a 4-dimensional subspace in $H^3_{\mathrm{syn}}(\mathcal{X}, S_{\mathbb{Q}_p}(2))$, the latter being a 5-dimensional $\mathbb{Q}_p$-vector space. Let $\{f, g\}$ be a Steinberg symbol in $K_2(k(E))$ (where f, g are rational functions on E supported at torsion points) that appears in the p-adic Deligne–Beilinson formula of Coleman and de Shalit, and relates the Coleman–de Shalit regulator (= p-adic integral) of $\{f, g\}$ to the value of the p-adic L-function of E at $s = 0$. If $\mathcal{U} \subset \mathcal{E}$ is the complement of finitely many $\mathbb{Z}_p$-sections of $\mathcal{E}$ including all points in $\mathrm{supp}((f)) \cup \mathrm{supp}((g))$, then one can, via the diagonal embedding $\mathcal{U} \hookrightarrow \mathcal{U} \times \mathcal{U}$, consider $(\mathcal{U}, f)$ as an element in $H^1(\mathcal{U} \times \mathcal{U}, \mathcal{K}_2)$.

Throughout the paper, we shall consider two liftings of $(\mathcal{U}, f^C)$ for some $C > 0$.

(i) If $\mathcal{U} \to \mathcal{U} \times \mathcal{E}$ denotes the diagonal embedding, then we may consider $z'' = (\mathcal{U}, f)$ as an element in $H^1(\mathcal{U} \times \mathcal{E}, \mathcal{K}_2)$.

(ii) We show that $(\mathcal{U}, f^M)$ lifts globally to an element $z \in H^1(\mathcal{E} \times \mathcal{E}, \mathcal{K}_2)$. Here M is chosen such that the torsion points occurring in $\mathrm{supp}(\mathrm{div}(f))$ have order dividing M. (See Proposition 1.4 and Definition 1.8.)

Then we prove the following two results about these liftings.

(i) By applying a projection formula of Besser on finite polynomial cohomology, we relate $r_{\mathrm{syn}}(z'') \in H^3_{\mathrm{syn}}(\mathcal{U} \times \mathcal{E}, S_{\mathbb{Q}_p}(2))$ to the Coleman–de Shalit regulator $r_p(\{f, g\})$, and hence to the p-adic L-function $L_p(E, s)$ (see Proposition 2.33).

(ii) We show that $r_{\mathrm{syn}}(z)$ lies in the decomposable part of $H^3_{\mathrm{syn}}(\mathcal{X}, S_{\mathbb{Q}_p}(2))$, that is, in $H^1_f(\mathbb{Q}_p, \mathrm{NS}(\bar{X}) \otimes \mathbb{Q}_p(1))$ (see Theorem 3.1).

In the language of Asakura–Sato [1] our element z turns out to be regulator-decomposable. Obviously, z is not given in decomposable form (like elements in $\mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Z}_p^*$), but it becomes decomposable after p-adic completion. The proof is a nice application of Besser's theory of triple indices on Coleman integrals.

In his recent Durham PhD Thesis [22], Zigmond studied similar elements, which he calls triangle configurations, with respect to the Deligne–Beilinson regulator. Take for some positive integer M two M-torsion points P and Q on E and let h be a rational function on E such that $\mathrm{div}(h) = M[P] - M[Q]$. Then the element

$$T_{P,Q} = (E \times \{P\}, h^{-1}) + (\Delta, h) + (\{Q\} \times E, h^{-1})$$

is a triangle configuration. It is easy to see that $T_{P,Q} \in H^1(E \times E_F, \mathcal{K}_2)$ for some extension field F. Zigmond shows that the image of triangle configurations under the Deligne–Beilinson regulator in real Deligne cohomology (associated to the variety $E \times E$ over F)

$$r_{\mathrm{DB}} : K_1^{(2)}(E \times E_F) = H^1(E \times E_F, \mathcal{K}_2) \otimes \mathbb{Q} \longrightarrow H^3_{\mathcal{D}}(E \times E_{\mathbb{R}}, \mathbb{R}(2))$$

is contained in the image of decomposables (that is, elements coming from $\mathrm{Pic}(E \times E_F) \otimes F^*$).

According to Beilinson's Conjecture, the regulator map r_{DB}, when restricted to the integral motivic cohomology $K_1^{(2)}(E \times E_F)_{\mathbb{Z}}$, is injective.

Zigmond's main result [22, Theorem 4.7] shows that if α is a sum of triangle configurations that already lies in $K_1^{(2)}(E \times E_F)_{\mathbb{Z}}$, then $r_{\mathrm{DB}}(\alpha)$ is contained in the image of decomposables with coefficients in $\mathcal{O}_F^*$.

As we are in the CM-case, let us assume that F is chosen such that F contains the CM-field, E/F has good reduction everywhere and the M-torsion points P and Q are defined over F. Let $\mathcal{E}$ be a smooth proper model of E over $\mathcal{O}_F$. Then

$$K_1^{(2)}(E \times E_F)_{\mathbb{Z}} = \mathrm{Image}(H^1(\mathcal{E} \times \mathcal{E}, \mathcal{K}_2) \longrightarrow H^1(E \times E_F, \mathcal{K}_2)) \otimes \mathbb{Q}.$$

It is easy to prove that there always exists a decomposable element $\sigma_{P,Q} \in \mathrm{Pic}(E \times E_F) \otimes F^*$ (supported on $E_F \times \{P\}$, Δ and $\{Q\} \times E_F$) and an integer k such that $kT_{P,Q} - \sigma_{P,Q}$ is integral, that is, in $K_1^{(2)}(E \times E_F)_{\mathbb{Z}}$.

Beilinson's Conjectures in this case predict that all elements in $K_1^{(2)}(E \times E_F)_{\mathbb{Z}}$ are decomposable [**22**, Conjecture 2.4], hence the above element $kT_{P,Q} - \sigma_{P,Q}$ is expected to be decomposable, which implies that $T_{P,Q}$ is decomposable in $H^1(E \times E_F, \mathcal{K}_2) \otimes \mathbb{Q}$.

Our element z' that we define in (1.8) can be seen to be a sum of triangle configurations. Hence, Zigmond's results and the Beilinson-Conjectures imply that z' is decomposable and hence also z, which is obtained as a push-forward from z' under some norm map. It seems to be out of reach to prove directly that z is decomposable. Our main result gives further evidence, by using p-adic regulators instead of the Deligne–Beilinson regulator, for this expectation.

We hope that the same method will also lead to a proof that in another well-known family of elements in $H^1(\mathcal{X}, \mathcal{K}_2)$, namely those defined by Mildenhall and Flach [**17**], which are integral at p, we find a regulator indecomposable one, which would imply that r_{syn} is an isomorphism. Q6

1. *p-adic regulators on* CM-*elliptic curves*

Let E be an elliptic curve defined over $\mathbb{Q}$ with complex multiplication by the ring of integers $\mathcal{O}_K$ in an imaginary quadratic field K. For a good reduction prime p for E, Coleman and de Shalit [**13**] constructed a p-adic regulator map as a homomorphism from K_2 of the function field of E to its tangent space,

$$r_{p,E} : K_2(\overline{\mathbb{Q}}_p(E)) \longrightarrow \mathrm{Hom}(H^0(E_{\mathbb{Q}_p}, \Omega^1_E), \overline{\mathbb{Q}}_p),$$

whose value at the Steinberg symbol $\{f, g\}$ is the linear functional

$$r_{p,E}(\{f, g\})(\omega) = \int_{(f)} \log(g) \cdot \omega \tag{1.1}$$

($\omega \in H^0(E, \Omega^1_E)$). Here *log* is a fixed branch of the p-adic logarithm and the integral is defined via Coleman's p-adic integration theory. Coleman defines a function $F_{\log(g)\cdot\omega} : E(\overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$, unique up to a constant such that if $(f) = \sum n_i(x_i)$ is the divisor of f, then

$$\int_{(f)} \log(g) \cdot \omega = \sum n_i F_{\log(g)\cdot\omega}(x_i).$$

Here $F_{\log(g)\cdot\omega}$ is a Coleman integral and primitive for the differential in the sense that $dF_{\log(g)\cdot\omega} = \log(g) \cdot \omega$.

The main result of Coleman and de Shalit [**13**] is a p-adic analogue of the Deligne–Beilinson conjecture, which relates the p-adic regulator to a value of the p-adic L-function of E as follows.

THEOREM 1.2. *Let p be a good ordinary reduction prime for E (hence p splits in $\mathcal{O}_K$). For rational functions $f, g \in K(E)$ with divisors $D = \mathrm{div}(f)$ and $D' = \mathrm{div}(g)$ supported at torsion points of E (subject to some mild restrictions), we have*

$$r_{p,E}(\{f, g\})(\omega) = c_{f,g} \cdot \Omega_p \cdot L_p(E, 0),$$

where Ω_p is a p-adic period extended by a Euler factor at p and $c_{f,g} \in \mathbb{Q}$.

There exists a pair (f, g) for which $c_{f,g} \in \mathbb{Q}^*$. We fix such a pair throughout this paper. Following Coleman and de Shalit, we may also assume that the divisor D looks as follows. There is a non-trivial ideal $\mathfrak{a} \subset \mathcal{O}_K$ with $(\mathfrak{a}, 2pN) = 1$, where N is the conductor of E, such that

$$D = (N(\mathfrak{a}) - 1)(0) - \sum_{\substack{P \in E[\mathfrak{a}] \\ P \neq 0}} P. \tag{1.3}$$

Here $E[\mathfrak{a}]$ denotes the group of $\mathfrak{a}$-torsion points in E and $N(\mathfrak{a})$ the norm of the ideal $\mathfrak{a}$ in $\mathbb{Z}$; see [**13**, Paragraph 5; **18**].

Let $M = N(\mathfrak{a})$. Then all points occurring in D are M-torsion points. Let $U = E - \operatorname{supp}(D)$. Then $(U, (f)) \in H^1(U \times U, \mathcal{K}_2)$ where U is considered to be embedded diagonally, so $\Delta : U \to U \times U$.

Then we have the following proposition.

Proposition 1.4. *There exists a $C \in \mathbb{N}$ such that*

$$(U, (f^C)) \in \operatorname{Image}(H^1(E \times E, \mathcal{K}_2) \longrightarrow H^1(U \times U, \mathcal{K}_2)).$$

Proof. Localization in algebraic K-theory yields an exact sequence

$$H^1(E \times E, \mathcal{K}_2) \longrightarrow H^1(U \times U, \mathcal{K}_2) \xrightarrow{\partial} \operatorname{Ch}_0(E \times E \backslash U \times U), \tag{1.5}$$

where $E \times E \backslash U \times U$ is a normal crossing divisor on $E \times E$, whose irreducible components consist of curves $\{x\} \times E$, $x \in \operatorname{supp}(D)$ or $E \times \{x\}$, $x \in \operatorname{supp}(D)$ intersecting transversally in points (x, y), $x, y \in \operatorname{supp}(D)$.

We write

$$D = \operatorname{div}(f) = \sum_{i=1}^{M-1} -[x_i] + (M-1)[0],$$

so

$$-D = \operatorname{div}(f^{-1}) = \sum_{i=1}^{M-1} [x_i] - (M-1)[0].$$

Let $F/\mathbb{Q}$ be such that all points x_i are defined over F. Then

$$\partial(U, (f^{-1})) = \sum_{i=1}^{M-1} [(x_i, x_i)] - (M-1)[(0,0)] \in \operatorname{Ch}_0(E \times E \backslash U \times U). \tag{1.6}$$

Now consider the points

$$[(x_i, x_i)] \in \operatorname{Pic}(\{x_i\} \times E) = \operatorname{Pic}(E).$$

We have

$$Mx_1 = 0, \quad Mx_2 = 0 \quad \text{on } E,$$

hence

$$M[x_1] - M[x_2] = \operatorname{div}(h_1) \quad \text{for } h_1 \in F(E),$$

which implies that

$$M[(x_1, x_1)] - M[(x_1, x_2)] = 0 \quad \text{in } \operatorname{Ch}_0(E \times E_F \backslash U \times U_F).$$

Next

$$-M[(x_2, x_2)] + M[(x_1, x_2)] \in \operatorname{Div}(E \times \{x_2\}),$$

which is principal, and hence 0 in $\operatorname{Pic}(E_F \times \{x_2\})$.

Then

$$2M[(x_2, x_2)] - 2M[(x_2, x_3)]$$

is 0 in $\mathrm{Pic}(\{x_2\} \times E_F)$, and hence 0 in $\mathrm{Ch}_0(E \times E_F \setminus U \times U_F)$.

Next

$$-2M[(x_3, x_3)] + 2M[(x_2, x_3)]$$

is 0 in $\mathrm{Ch}_0(E \times E_F \setminus U \times U_F)$. By induction one shows, for all j, that

$$jM[(x_j, x_j)] - jM[(x_j, x_{j+1})] = 0$$

and

$$-jM[(x_{j+1}, x_{j+1})] + jM[(x_j, x_{j+1})] = 0$$

in $\mathrm{Ch}_0(E \times E_F \setminus U \times U_F)$.

By using (1.6) we get

$$\begin{aligned} \partial((U, (f^{-M}))) = {} & M[(x_1, x_1)] - M[(x_1, x_2)] \\ & - M[(x_2, x_2)] + M[(x_1, x_2)] \\ & + 2M[(x_2, x_2)] - 2M[(x_2, x_3)] \\ & \pm \ldots \\ & + M(M-1)[(x_{M-1}, x_{M-1})] - (M-1)M[(x_{M-1}, 0)] \\ & - M(M-1)[(0,0)] + (M-1)M[(x_{M-1}, 0)], \end{aligned}$$

which is a sum of principal divisors in $(E \times E \setminus U \times U)_F$ and hence 0 in $\mathrm{Ch}_0(E \times E_F \setminus U \times U_F)$.

Then $\partial((U, (f^M))) = 0$ in $\mathrm{Ch}_0(E \times E_F \setminus U \times U_F)$ as well, where we consider $(U, (f^M))$ as an element in $H^1(U \times U_F, \mathcal{K}_2)$. Hence, there exists $z' \in H^1(E \times E_F, \mathcal{K}_2)$ with image $(U, (f^M))$ in $H^1(U \times U_F, \mathcal{K}_2)$.

Now consider the following commutative diagram with respect to the push-forward $\pi_* : ?/F \to ?/\mathbb{Q}$: Q7

$$\begin{array}{ccccc} H^1(E \times E_F, \mathcal{K}_2) & \longrightarrow & H^1(U \times U_F, \mathcal{K}_2) & \xrightarrow{\partial} & \mathrm{Ch}_0(E \times E_F \setminus U \times U_F) \\ \downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* \\ H^1(E \times E, \mathcal{K}_2) & \longrightarrow & H^1(U \times U, \mathcal{K}_2) & \xrightarrow{\partial} & \mathrm{Ch}_0(E \times E \setminus U \times U). \end{array} \tag{1.7}$$

We have $\pi_*(U, (f^M)) = (U, (f^C))$ where C is a multiple of M. Then $z := \pi_*(z')$ has image $(U, (f^C))$ in $H^1(U \times U, \mathcal{K}_2)$. This completes the proof of Proposition 1.4. □

Since all $\mathfrak{a}$-torsion points occurring above extend uniquely to $\mathfrak{a}$-torsion points on a smooth proper model $\mathcal{E}_{/\mathbb{Z}_p}$ of E and all principal divisors supported in $\mathfrak{a}$-torsion points extend to principal divisors in $\mathcal{E}_{/\mathbb{Z}_p}$ (resp. $\mathcal{E} \times \mathcal{E}_{\mathbb{Z}_p} \setminus \mathcal{U} \times \mathcal{U}_{\mathbb{Z}_p}$), the whole construction in Proposition 1.4 can be performed on the model $\mathcal{E} \times \mathcal{E}_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ as well, so we may assume that Q8

$$(\mathcal{U}, (f)) \in H^1(\mathcal{U} \times \mathcal{U}_{\mathbb{Z}_p}, \mathcal{K}_2),$$

and that $(\mathcal{U}, (f^C))$ lies in the image of

$$H^1(\mathcal{E} \times \mathcal{E}_{\mathbb{Z}_p}, \mathcal{K}_2) \longrightarrow H^1(\mathcal{U} \times \mathcal{U}_{\mathbb{Z}_p}, \mathcal{K}_2).$$

Let L denote the completion of F at a prime lying above p with ring of integers $\mathcal{O}_L$. While in Proposition 1.4 we have shown the existence of an element mapping to $(\mathcal{U}, f^M)$ under the above restriction map, we now work with an explicit element $z' \in H^1(\mathcal{X}_{\mathcal{O}_L}, \mathcal{K}_2)$, which has (Δ, f^M) as a summand (where $\Delta : E \to E \times E$ is the diagonal), and whose construction is already suggested by the proof of Proposition 1.4. It is given as follows.

Let $h_i \in L(E)^*$ be rational functions such that $\operatorname{div}(h_i) = M[x_i] - M[0]$. Let

$$z' = \sum_{i=1}^{M-1} ((E_L \times \{0\}, h_i) + (\{x_i\} \times E_L, h_i)) + (\Delta, f^M). \tag{1.8}$$

Then

$$\operatorname{div}(h_i | E_L \times \{0\}) = M[x_i, 0] - M[0, 0]$$

and

$$\operatorname{div}(h_i | \{x_i\} \times E_L) = M[x_i, x_i] - M[x_i, 0].$$

As $\operatorname{div} f^M|_\Delta = M(M-1)[0,0] - \sum_i M[x_i, x_i]$, we see that $z' \in H^1(X_L, \mathcal{K}_2)$ and we can achieve $z' \in H^1(\mathcal{X}_{\mathcal{O}_L}, \mathcal{K}_2)$ by possibly multiplying the functions h_i by an appropriate p-power. Evidently, z' is a sum of triangle configurations.

The group $\operatorname{Gal}(L/\mathbb{Q}_p)$ acts on the set $\{x_i : i = 1, \ldots, M-1\}$; hence, any $\sigma \in \operatorname{Gal}(L/\mathbb{Q}_p)$ defines a permutation of the summands $\{x_i \times E_L, h_i\}$ resp. $\{E_L \times \{0\}, h_i\}$. We conclude that σ leaves z' invariant, so $z' \in H^1(E \times E_L, \mathcal{K}_2)^{\operatorname{Gal}(L/\mathbb{Q}_p)}$; hence, the image of z' in $H^1(E \times E_{\overline{\mathbb{Q}}_p}, \mathcal{K}_2)$ lies in $H^1(E \times E_{\overline{\mathbb{Q}}_p}, \mathcal{K}_2)^{\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$. We apply a result from Galois descent theory [**14**, Proposition 4.6] and conclude that

$$H^1(\widehat{E \times E_{\mathbb{Q}_p}}, \mathcal{K}_2) \otimes \mathbb{Q}_p = H^1(\widehat{E \times E_{\overline{\mathbb{Q}}_p}}, \mathcal{K}_2)^{\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \otimes \mathbb{Q}_p;$$

hence, we can regard z' as an element in $H^1(\widehat{\mathcal{X}_{\mathbb{Z}_p}}, \mathcal{K}_2) \otimes \mathbb{Q}_p$. As before, let $z = N_{L/\mathbb{Q}_p *}(z') \in H^1(\mathcal{X}_{\mathbb{Z}_p}, \mathcal{K}_2)$. We have a commutative diagram of norm maps

$$\begin{array}{ccc} H^1(E \times E_L, \mathcal{K}_2) & \longrightarrow & H^1(\widehat{E \times E_L}, \mathcal{K}_2) \otimes \mathbb{Q}_p \\ \downarrow N_{L/\mathbb{Q}_p *} & & \downarrow N_{L/\mathbb{Q}_p *} \\ H^1(E \times E_{\mathbb{Q}_p}, \mathcal{K}_2) & \longrightarrow & H^1(\widehat{E \times E_{\mathbb{Q}_p}}, \mathcal{K}_2) \otimes \mathbb{Q}_p. \end{array}$$

We have seen that the image $\tilde{z}'$ of z' in $H^1(\widehat{E \times E_L}, \mathcal{K}_2) \otimes \mathbb{Q}_p$ already lies in $H^1(\widehat{E \times E_{\mathbb{Q}_p}}, \mathcal{K}_2) \otimes \mathbb{Q}_p$. Hence, the image $\tilde{z}$ of $z = N_{L/\mathbb{Q}_p}(z')$ in $H^1(\widehat{E \times E_{\mathbb{Q}_p}}, \mathcal{K}_2) \otimes \mathbb{Q}_p$ satisfies $\tilde{z} = [L : \mathbb{Q}_p]\tilde{z}'$ (because the composition map

$$H^1(\widehat{E \times E_{\mathbb{Q}_p}}, \mathcal{K}_2) \otimes \mathbb{Q}_p \longrightarrow H^1(\widehat{E \times E_L}, \mathcal{K}_2) \otimes \mathbb{Q}_p \xrightarrow{N_{L/\mathbb{Q}_p *}} H^1(\widehat{E \times E_{\mathbb{Q}_p}}, \mathcal{K}_2) \otimes \mathbb{Q}_p$$

is multiplication by the degree $[L : \mathbb{Q}_p]$).

As the syntomic regulator $r_{\mathrm{syn}}(z)$ or $r_{\mathrm{syn}}(z')$ only depends, respectively, on the class of z or z' in $H^1(\widehat{\mathcal{X}_{\mathbb{Z}_p}}, \mathcal{K}_2) \otimes \mathbb{Q}_p \hookrightarrow H^1(\widehat{E \times E_{\mathbb{Q}_p}}, \mathcal{K}_2) \otimes \mathbb{Q}_p$, we conclude that $r_{\mathrm{syn}}(z) = [L : \mathbb{Q}_p] r_{\mathrm{syn}}(z')$. Hence, we later assume without loss of generality that $z = z'$.

Now we consider again both elements $f, g \in K(E)$ as in Theorem 1.2. After possibly replacing C by a multiple of C, we may assume that all zeroes and poles of f and g are torsion points of order C. Let

$$t = \coprod_{x \in E} t_x : K_2(K(E)) \longrightarrow \coprod_{x \in E} k(x)^*$$

be the tame symbol map. Let $\tilde{L}$ be a finite extension of $\mathbb{Q}_p$ containing L such that all points appearing in $\operatorname{supp}(f), \operatorname{supp}(g)$ are defined over $\tilde{L}$. Then according to a lemma of Bloch [**7**, Lecture 8], there exist functions $f_i \in \tilde{L}(E)$ with divisors D_{f_i} such that $\operatorname{supp}(D_{f_i}) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$ and $c_i \in \tilde{L}^*$ such that

$$\{f, g\}^C \prod_i \{f_i, c_i\} \in \Gamma(E_{\tilde{L}}, \mathcal{K}_2) = \ker t. \tag{1.9}$$

We have the following equality of tame symbols

$$t(\{f,g\}^C) = t(\{f^C, g\}),$$

hence we have

$$\{f^C, g\} + \sum_i \pi_{\tilde{L}/\mathbb{Q}_p^*}\{f_i, c_i\} \otimes \frac{1}{m} \in \Gamma(E_{\mathbb{Q}_p}, \mathcal{K}_2) \otimes \mathbb{Q}, \tag{1.10}$$

for some $m \in \mathbb{N}$ (compare [**15**, Paragraph 5]).

Now consider the composite map

$$\Gamma(E_{\tilde{L}}, \mathcal{K}_2) \longrightarrow K_2(\bar{\mathbb{Q}}_p(E)) \xrightarrow{r_{p,E}} \mathrm{Hom}(H^0(E, \Omega^1_E), \bar{\mathbb{Q}}_p),$$

again denoted by $r_{p,E}$. By the main result of Besser [**3**] on syntomic regulators for K_2 of curves, we have

$$r_{p,E}\left(\{f^C, g\} + \sum_i \{f_i, c_i\}\right) = r_{\mathrm{syn}}\left(\{f^C, g\} + \sum_i \{f_i, c_i\}\right), \tag{1.11}$$

where

$$\begin{aligned} r_{\mathrm{syn}} : K_2(E_{\tilde{L}}) \otimes \mathbb{Q} &\longrightarrow H^1_{dR}(E_{\tilde{L}}) \cong H^2_{\mathrm{syn}}(\mathcal{E}_{\mathcal{O}_{\tilde{L}}}, S_{\mathbb{Q}_p}(2)) \\ &\longrightarrow \mathrm{Hom}(H^0(E_{\bar{\mathbb{Q}}_p}, \Omega^1_{E_{\bar{\mathbb{Q}}_p}}), \bar{\mathbb{Q}}_p) \end{aligned}$$

denotes the syntomic regulator for K_2 of curves.

For functions $a, b \in \mathbb{Q}_p(E)$, we have that the p-adic integral $\int_{(b)} \log(a)\omega$ vanishes if a or b is constant. Hence, we have

$$r_{p,E}\left(\{f^C, g\} + \sum_i \{f_i, c_i\}\right)(\omega) = r_{p,E}(\{f^C, g\})(\omega) = \int_{(f^C)} \log g\ \omega = -\int_{(g)} \log f^C \omega, \tag{1.12}$$

where we have used the properties of the p-adic regulator pairing (compare [**13**, Theorem 3.5]).

Consider again the model $\mathcal{U} \subset \mathcal{E}_{\mathbb{Z}_p}$ of U over $\mathbb{Z}_p$, where U is now the affine in $E_{\mathbb{Q}_p}$ with complement $E_{\mathbb{Q}_p} \backslash U = \mathrm{supp}(f) \cup \mathrm{supp}(g)$, and we assume without loss of generality that $f, g \in \mathcal{O}^*_{\mathcal{U}}$. To $\mathcal{U}$ one can associate a basic wide open Y in the sense of Coleman or, equivalently, an affinoid Dagger space in the sense of Grosse–Kloenne. It has an underlying affinoid variety Y' which is obtained as tube of the reduction $U_{\mathbb{F}_p}$ of $\mathcal{U}$ under the specialization map

$$\mathrm{sp} : \hat{\mathcal{E}}_{\mathbb{Q}_p} \longrightarrow \hat{\mathcal{E}}_{\mathbb{Z}_p},$$

where $\hat{\mathcal{E}}_{\mathbb{Z}_p}$ is the formal completion of $\mathcal{E}$ along its closed fibre and $\hat{\mathcal{E}}_{\mathbb{Q}_p}$ the generic fibre of $\hat{\mathcal{E}}$ considered as rigid analytic variety. Note that Y is equipped with an overconvergent structure sheaf. For each point e in $\mathrm{supp}(\bar{f}) \cup \mathrm{supp}(\bar{g})$, let Y_e be the annuli end of Y at e. The collection of all annuli ends of Y is denoted by $\mathrm{End}(Y)$.

For the convenience of the reader, we recall Besser's theory of double resp. triple indices [**3**, **5**, **6**], which are defined on a certain class of Coleman integrals. Coleman integration theory defines for wide open Y as above the $\mathbb{C}_p$-algebra $A_{\mathrm{Col}}(Y)$ of Coleman integrals, which forms a subclass of locally analytic functions, and the $A_{\mathrm{Col}}(Y)$-modules $\Omega^1_{\mathrm{Col}}(Y)$ fitting into an exact sequence

$$0 \longrightarrow \mathbb{C}_p \longrightarrow A_{\mathrm{Col}}(Y) \xrightarrow{d} \Omega^1_{\mathrm{Col}}(Y) \longrightarrow 0.$$

$\Omega^1_{\mathrm{Col}}(Y)$ contains the space $\Omega^1(Y)$ of overconvergent forms on Y and $A_{\mathrm{Col}}(Y)$ contains the space $A(Y)$ of overconvergent functions. Fix a branch of the p-adic logarithm $\log : \mathbb{C}_p^* \to \mathbb{C}_p$. Q9
For $x \in E_{\mathbb{F}_p}$, $x \notin S$, a Coleman function is analytic on the open residue disc $\mathcal{U}_x$ and it is an element of the polynomial algebra $A(Y_e)[\log z_e]$ where z_e is a local parameter of the residue disc of e if $x = e \in S$.

Define $A_{\mathrm{Col},1}(Y)$ as the inverse image of $\Omega^1(Y)$ under d and $\Omega^1_{\mathrm{Col},1}(Y) = A_{\mathrm{Col},1}(Y) \cdot \Omega^1(Y)$. Besser defines double and triple indices on the Coleman integrals in $A_{\mathrm{Col},1}(Y)$. First, we recall the local definitions.

Let K be a complete subfield of $\mathbb{C}_p$. We define $A_{\log} := K((z))[\log z]$ of polynomials over the formal variable $\log z$ over the field of Laurent polynomials over z. It admits a differential d into the module of differentials $K((z))[\log z]dz$ by $d \log z = dz/z$. Let $A_{\log,1} := d^{-1}(K((z))\, dz) = K((z)) + K \log z$. For $F \in A_{\log,1}$ we define the residue of its differential Res $dF := a_{-1}$ if $dF = \sum_{n>-\infty}^{\infty} a_n z^n\, dz$. Then we have the following proposition.

Proposition [3]. *There is a unique antisymmetric function*

$$\langle\ ,\ \rangle : A_{\log,1} \times A_{\log,1} \longrightarrow K$$

Q10 *with* $\langle F, G\rangle = \mathrm{Res} F\, dG$ *if* $F \in K((z))$. *This is the local double index.*

For the triple index one starts with triples $F, G, H \in A_{\log,1}$ together with choices of integrals in $A_{\log}$ of all pairs $R\, dS$ with different R, S satisfying $\int R\, dS + \int S\, dR = SR$ (called auxiliary data). The triple index

$$\begin{aligned} \langle\ ,\ \rangle : A_{\log,1} \times A_{\log,1} \times A_{\log,1} &\longrightarrow K \\ (F, G, H) &\longmapsto \langle F, G, H\rangle \end{aligned}$$

is a function that

(i) is trilinear and symmetric in its first two variables;

(ii) satisfies a triple identity

$$\langle F, G, H\rangle + \langle F, H, G\rangle + \langle H, G, F\rangle = 0;$$

(iii) reduces to the double index

$$\langle F, G, H\rangle = \langle F, \textstyle\int G dH\rangle,$$

for $G \in K((z))$.

According to [6, Proposition 6.3], $\langle\ ,\ ,\ \rangle$ exists and is unique. It is called the local triple index.

The theory of global indices can be viewed as a generalization of the residue theorem. Suppose that we are in the previous situation, that is, given a wide open Y on a rigid analytic curve X with annuli ends Y_e, we have given Coleman integrals $F, G, H \in A_{\mathrm{Col},1}(Y)$. At each annuli end the restrictions of the functions are in $A_{\log,1}$ and hence local indices are defined at all annuli ends. For the local triple indices, one first chooses auxiliary data globally as Coleman integrals and then restrict to Y_e.

Then Besser defines global indices

$$\langle F, G\rangle_{\mathrm{gl}} := \sum_e \langle F, G\rangle_e,$$

$$\langle F, G, H\rangle_{\mathrm{gl}} := \sum_e \langle F, G, H\rangle_e,$$

as the sum of all local indices at all annuli ends.

As is shown in [3, Proposition 4.10], the global double index only depends on the cohomology classes of dF and dG in $H^1_{dR}(Y)$.

The global triple index does not depend on the auxiliary choices. Later on, when we need them, we shall recall further properties of the triple index as shown in [5, 6].

Now we return to the situation at the beginning of this section, that is, we have the basic wide open Y in $\hat{\mathcal{E}}_{\mathbb{Q}_p}$ with annuli ends Y_e at all points $e \in S = \mathrm{supp}((\bar{f})) \cup \mathrm{supp}((\bar{g}))$.

Let ω be a holomorphic 1-form on E, that is, $\omega \in H^0(E_{\mathbb{Q}_p}, \Omega^1_{E/\mathbb{Q}_p})$, and F_ω be a Coleman integral. Then we have the following version of a proposition of Besser [3].

Proposition 1.13.

$$\sum_{e\in \mathrm{End}(Y)} \langle \log g, F_\omega, \log f^C\rangle_e =: \langle \log g, F_\omega, \log f^C\rangle_{\mathrm{gl}} = \sum_{x\in E} \log t_x(g, f^C)F_\omega(x) + \int_{(g)} \log(f^C)\cdot\omega.$$

Here $\langle\ ,\ ,\ \rangle_e$ denotes Besser's triple index for three Coleman functions at the annuli end Y_e and $\langle\ ,\ ,\ \rangle_{\mathrm{gl}}$ the global triple index.

Indeed, this is a combination of [3, Propositions 3.4 and 5.3]. We have reformulated it by replacing the double index $\mathrm{ind}_e(\log g, \int F_\omega d\log(f^C))$ at the annuli end Y_e by the triple index $\langle \log g, F_\omega, \log(f^C)\rangle_e$. This follows from the definition of the triple index as $\mathrm{Res}_e\omega = 0$ for all annuli ends Y_e (note that ω is a global holomorphic form on E).

For the pair $\{c_i, f_i\}$ we have

$$\begin{aligned}\sum_{e\in \mathrm{End}(Y)} \langle \log c_i, F_\omega, \log f_i\rangle_e &= \langle \log c_i, F_\omega, \log f_i\rangle_{\mathrm{gl}} \\ &\overset{(1)}{=} -\langle \log c_i, \log f_i, F_\omega\rangle_{\mathrm{gl}} - \langle F_\omega, \log f_i, \log c_i\rangle_{\mathrm{gl}} \\ &\overset{(2)}{=} 0.\end{aligned}$$

Here the equality (1) follows from the triple identity and (2) follows from [6, Proposition 7.4 and Lemma 7.3].

Now we replace the pair of functions (g, f^C) in Proposition 1.13 by the pair (c_i, f_i) for any i. As $\int_{(c_i)} \log(f_i)\omega$ vanishes, we get

$$\sum_{x\in E} \log t_x(c_i, f_i)F_\omega(x) = \langle \log c_i, F_\omega, \log f_i\rangle_{\mathrm{gl}} = 0. \tag{1.14}$$

Recall that the tame symbol t_x of $\{g, f^C\}\prod_i\{c_i, f_i\}$ vanishes at all x. This implies that

$$\sum_{x\in E} \log t_x(g, f^C)F_\omega(x) = 0 \tag{1.15}$$

as well.

For the element $\{g, f^C\} + \sum_i\{c_i, f_i\} \in K_2(E_{\tilde{L}})$, we get the following useful formula for its syntomic regulator.

Lemma 1.16.

$$r_{\mathrm{syn}}(\{g, f^C\} + \sum_i \{c_i, f_i\})(\omega) = \int_{(g)} \log(f^C)\omega = \langle \log g, F_\omega, \log(f^C)\rangle_{\mathrm{gl}}.$$

In the next section, we relate this result to the syntomic regulator $r_{\mathrm{syn}}(z'')$ of the element $z'' \in H^1(\mathcal{U}\times \mathcal{E}_{\mathbb{Z}_p}, \mathcal{K}_2)$, given by $z'' = (\mathcal{U}, f)$.

2. *Cup product in finite polynomial cohomology using Besser's triple index*

Now we recall modified syntomic resp. finite polynomial cohomology, as defined by Besser [2, 4, 5]. Let $X \to \mathrm{Spec}R$ be a smooth scheme over a discrete valuation ring R with generic fibre X_K and closed fibre X_k, where K and k denotes, respectively, the fraction field and residue field of R.

One then has complexes

$$R\Gamma_{\mathrm{rig}}(X_k, K), R\Gamma_{dR}(X_K, K)$$

with $F^n R\Gamma_{dR}$ the Hodge filtration, and a canonical map

$$\tau : R\Gamma_{dR}(X_K, K) \longrightarrow R\Gamma_{\mathrm{rig}}(X_k, K). \tag{2.1}$$

We fix a Frobenius endomorphism $\varphi : X_k \to X_k$ with $\deg \varphi = q$. Let $\mathcal{P}$ be the multiplication monoid of all polynomials $P(t) \in \mathbb{Q}[t]$ with constant coefficient 1. Let $P(t) = \prod_1^{\deg P}(1 - \alpha_i t)$ and $\mathcal{P}_m \subset \mathcal{P}$ be the submonoid of polynomials that are pure of weight m, that is, $1/\alpha_i$ has complex absolute value $q^{m/2}$. Let $P \in \mathcal{P}$. We define the syntomic P-complex $R\Gamma_{f,P}(X, n)$ as

$$R\Gamma_{f,P}(X, n) := \mathrm{MF}(F^n R\Gamma_{dR}(X_K/K)) \overset{P(\varphi^*)}{\longrightarrow} R\Gamma_{\mathrm{rig}}(X_k/K) \tag{2.2}$$

with cohomology group $H^i_{f,P}(X, n)$.

One has commutative diagrams for $P, Q \in \mathcal{P}$

$$\begin{array}{ccc} F^n R\Gamma_{dR}(X_K/K) & \overset{P(\varphi^*)}{\longrightarrow} & R\Gamma_{\mathrm{rig}}(X_k/K) \\ \downarrow = & & \downarrow Q(\varphi^*) \\ F^n R\Gamma_{dR}(X_K/K) & \overset{PQ(\varphi^*)}{\longrightarrow} & R\Gamma_{\mathrm{rig}}(X_k/K). \end{array}$$

One gets an induced map (compare [4, Definition 2.3])

$$R\Gamma_{f,P}(X, n) \longrightarrow R\Gamma_{f,PQ}(X, n). \tag{2.3}$$

We consider the special polynomials $P_i(t) = 1 - t^i/q^{ni}$. For $i < j$ we have the relation $P_i \mid P_j$, hence the $P_i(t)$ form a directed subset of $\mathbb{Q}_p[t]$, ordered by division. Using the maps (2.3) as transition maps one, defines

$$R\Gamma_{\mathrm{ms}}(X, n) := \varinjlim_i R\Gamma_{f,P_i}(X, n), \tag{2.4}$$

the modified syntomic complex of X. By a result of Besser, $R\Gamma_{\mathrm{ms}}(X, n)$ is independent of the choice of Frobenius.

The finite polynomial complex, twisted n times, of weight m is defined as

$$R\Gamma_{\mathrm{fp}}(X, n, m) := \varinjlim_{P \in \mathcal{P}_m} R\Gamma_{f,P}(X, n), \tag{2.5}$$

where the monoid $\mathcal{P}_m$ is ordered by division.

Its cohomology is denoted by $H^j_{\mathrm{fp}}(X, n, m)$ and called finite polynomial cohomology.

By Besser [5, (2.6)] one has canonical maps

$$H^i_{\mathrm{ms}}(X, n) \longrightarrow H^i_{\mathrm{fp}}(X, n, 2n) \tag{2.6}$$

from modified syntomic to finite polynomial cohomology since all polynomials P_i have weight $2n$.

Remark. Note that we have an isomorphism

$$R\Gamma_{\mathrm{syn}}(X, n) \overset{\sim}{\longrightarrow} R\Gamma_{\mathrm{ms}}(X, n) \tag{2.7}$$

between the syntomic and the modified syntomic complex [2].

Besser also defines finite polynomial cohomology with compact support, denoted by $R\Gamma_{\mathrm{fp,c}}(X, n, m)$ as the homotopy limit of the diagrams:

$$\begin{array}{ccccccc} F^n R\Gamma_{dR,c}(X_K/K) & & & R\Gamma_{\mathrm{rig},c}(X_s/K) & & & \mathrm{MF}(P(\varphi^*)) \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & & R\Gamma_{dR,c}(X_K/K) & & R\Gamma_{\mathrm{rig},c}(X_s/K) & & \end{array} \tag{2.8}$$

where $\mathrm{MF}(P(\varphi^*))$ is the mapping fibre of $P(\varphi^*)$ acting on $R\Gamma_{\mathrm{rig},c}(X_s/K)$, for $P \in \mathcal{P}_m$. Note that on the level of cohomology with compact support, one has canonical maps

$$R\Gamma_{\mathrm{rig},c}(X_s/K) \longrightarrow R\Gamma_{dR,c}(X_K/K) \tag{2.9}$$

called cospecialization maps.

One has by definition canonical maps [5, Proposition 4.4]

$$\pi : H^j_{\mathrm{fp},c}(X,n,m) \longrightarrow H^j_{\mathrm{rig},c}(X_s/K). \tag{2.10}$$

Also, one has cup products

$$R\Gamma_{\mathrm{fp}}(X,n_1,m_1) * R\Gamma_{\mathrm{fp},c}(X,n_2,m_2) \longrightarrow R\Gamma_{\mathrm{fp},c}(X,n_1+n_2,m_1+m_2), \tag{2.11}$$

and short exact sequences

$$H^{i-1}_{\mathrm{rig}}(X_s/K) \xrightarrow{\iota} H^i_{\mathrm{fp}}(X,n,m) \longrightarrow F^n H^i_{dR}(X_K/K). \tag{2.12}$$

For $x \in H^i_{\mathrm{rig}}(X_s/K)$ and $y \in H^j_{\mathrm{fp},c}(X,n,m)$ one has the formula

$$\pi(\iota(x) \cup y) = x \cup \pi(y), \tag{2.13}$$

where the cup product on the right-hand side is induced from products

$$R\Gamma_{\mathrm{rig}}(X_s/K) \times R\Gamma_{\mathrm{rig},c}(X_s/K) \longrightarrow R\Gamma_{\mathrm{rig},c}(X_s/K). \tag{2.14}$$

We make essential use of Besser's projection formula [5, (4.4)]: for finite polynomial cohomology, namely, for $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ a proper morphism between smooth $\mathrm{Spec}R$-schemes, we have

$$f_*(\alpha \cup f^*\beta) = f_*\alpha \cup \beta, \tag{2.15}$$

for $\alpha \in H^i_{\mathrm{fp}}(\mathcal{Y}_1,n_1,n_2)$ and $\beta \in H^j_{\mathrm{fp},c}(\mathcal{Y}_2,m_1,m_2)$. Note that the push-forward in syntomic resp. finite polynomial cohomology is induced from corresponding Gysin maps $R\Gamma_{\mathrm{syn}}(\mathcal{Y}_1,n) \to R\Gamma_{\mathrm{syn}}(\mathcal{Y}_2,n+d)[2d]$ with $\dim \mathcal{Y}_1 + d = \dim \mathcal{Y}_2$ (see [5, 20]). It is shown in [9] that the push-forward in syntomic cohomology commutes with the push-forward in motivic cohomology. Assume that $\mathcal{Y}_1$ is a smooth $\mathrm{Spec}R$-curve and one has a finite morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ to a smooth surface $\mathcal{Y}_2$ over $\mathrm{Spec}R$. Then f induces push-forward maps

$$f_* : H^1_{\mathrm{ms}}(\mathcal{Y}_1,1) \longrightarrow H^3_{\mathrm{ms}}(\mathcal{Y}_2,2) \quad \text{and} \quad f_* : H^1_{\mathrm{fp}}(\mathcal{Y}_1,1,2) \longrightarrow H^3_{\mathrm{fp}}(\mathcal{Y}_2,2,4). \tag{2.16}$$

Denote by γ the image of the syntomic regulator of $(\mathcal{U}, f^C)$ under the map $H^3_{\mathrm{ms}}(\mathcal{U} \times \mathcal{E}_{\mathbb{Z}_p},2) \to H^3_{\mathrm{fp}}(\mathcal{U} \times \mathcal{E}_{\mathbb{Z}_p},2,4)$. Let $\omega \in H^1_{\mathrm{ms}}(\mathcal{U},1) \subset H^1_{\mathrm{fp}}(\mathcal{U},1,2)$ and $\eta \in H^0(E,\Omega^1) \subset H^1_{dR}(E) \cong H^1_{\mathrm{fp},c}(\mathcal{U},0,1)$ where the last isomorphism follows from [5, Lemma 5.2].

Let $\pi_1 : \mathcal{U} \times \mathcal{E} \to \mathcal{U}$, $E \times E \to E$ be the projection maps to the first components. Then the induced maps

$$\begin{aligned} \Delta^*\pi_1^* &: H^0(E,\Omega^1) \longrightarrow H^0(E,\Omega^1) \\ &: H^1_{\mathrm{fp}}(\mathcal{U},1,2) \longrightarrow H^1_{\mathrm{fp}}(\mathcal{U},1,2) \\ &: H^1_{\mathrm{fp},c}(\mathcal{U},0,1) \longrightarrow H^1_{\mathrm{fp},c}(\mathcal{U},0,1) \end{aligned}$$

are the identity maps. Note that $\pi_1 : \mathcal{U} \times \mathcal{E} \to \mathcal{U}$ is proper, so $\pi_1^*\eta$ is well defined.

We want to compute the cup product

$$\gamma \cup \pi_1^*\omega \cup \pi_1^*\eta,$$

under the product pairing:

$$\begin{aligned} &H^3_{\mathrm{fp}}(\mathcal{U} \times \mathcal{E}_{\mathbb{Z}_p},2,4) \times H^1_{\mathrm{fp}}(\mathcal{U} \times \mathcal{E}_{\mathbb{Z}_p},1,2) \times H^1_{\mathrm{fp},c}(\mathcal{U} \times \mathcal{E}_{\mathbb{Z}_p},0,1) \\ &\longrightarrow H^5_{\mathrm{fp},c}(\mathcal{U} \times \mathcal{E}_{\mathbb{Z}_p},3,7) \cong H^4_{dR,c}(U \times E_{/\mathbb{Q}_p}) \xrightarrow{\mathrm{Tr}} \mathbb{Q}_p. \end{aligned} \tag{2.17}$$

We apply Besser's projection formula for finite polynomial cohomology to the closed immersion

$$\Delta : \mathcal{U} \longrightarrow \mathcal{U} \times \mathcal{E}.$$

Note that $\gamma = \Delta_*(\mathrm{reg}(f^C))$, where $\mathrm{reg}(f^C)$ is the syntomic regulator of $f^C \in \mathcal{O}^*(\mathcal{U})$ in $H^1_{\mathrm{ms}}(\mathcal{U}, 1) \subset H^1_{\mathrm{fp}}(\mathcal{U}, 1, 2)$. We obtain

$$\gamma \cup \pi_1^*\omega \cup \pi_1^*\eta = \mathrm{reg}(f^C) \cup \omega \cup \eta, \tag{2.18}$$

where the cup product on the right-hand side is now computed under the pairing

$$\begin{aligned} H^1_{\mathrm{ms}}(\mathcal{U},1) \times H^1_{\mathrm{ms}}(\mathcal{U},1) \times H^1_{\mathrm{fp},c}(\mathcal{U},0,1) &\overset{\text{'}\cup\times\mathrm{id}\text{'}}{\longrightarrow} H^2_{\mathrm{ms}}(\mathcal{U},2) \times H^1_{\mathrm{fp},c}(\mathcal{U},0,1) \\ &\overset{\mathrm{id}\times\pi}{\longrightarrow} H^2_{\mathrm{ms}}(\mathcal{U},2) \times H^1_{\mathrm{rig},c}(\mathcal{U}_s/\mathbb{Q}_p) \\ &\overset{\sim}{\longrightarrow} H^1_{\mathrm{rig}}(\mathcal{U}_s/\mathbb{Q}_p) \times H^1_{\mathrm{rig},c}(\mathcal{U}_s/\mathbb{Q}_p) \\ &\longrightarrow H^2_{\mathrm{rig},c}(\mathcal{U}_s/\mathbb{Q}_p) \overset{\mathrm{Tr}}{\longrightarrow} \mathbb{Q}_p, \end{aligned} \tag{2.19}$$

where we have used the isomorphism in [**3**, Proposition 3.2].

Let Ψ be the image of $\mathrm{reg}(f^C) \cup \omega$ under the pairing

$$H^1_{\mathrm{ms}}(\mathcal{U},1) \times H^1_{\mathrm{ms}}(\mathcal{U},1) \longrightarrow H^2_{\mathrm{ms}}(\mathcal{U},2) \overset{\sim}{\longrightarrow} H^1_{\mathrm{rig}}(\mathcal{U}_s/\mathbb{Q}_p),$$

and η' be the image of η under the map

$$H^0(E,\Omega^1) \hookrightarrow H^1_{\mathrm{fp},c}(\mathcal{U},0,1) \cong H^1_{dR}(E) \overset{\pi}{\longrightarrow} H^1_{\mathrm{rig},c}(\mathcal{U}_s/\mathbb{Q}_p),$$

where π is defined as in (2.10).

Then we have

$$\mathrm{reg}(f^C) \cup \omega \cup \eta = \Psi \cup \eta' \in \mathrm{Image}(H^2_{\mathrm{rig},c}(\mathcal{U}_s/\mathbb{Q}_p) \overset{\mathrm{Tr}}{\longrightarrow} \mathbb{Q}_p). \tag{2.20}$$

This follows from (2.12).

We apply this fact in the particular situation when $\omega = \mathrm{reg}(g)$ for $g \in \mathcal{O}(\mathcal{U})^*$.

We have the following proposition, which is a modified version of [**5**, Proposition 5.3].

Proposition 2.21. *Let $\omega = \mathrm{reg}(g) \in H^1_{\mathrm{ms}}(\mathcal{U},1)$, $g \in \mathcal{O}^*(\mathcal{U})$, $\eta \in H^0(E,\Omega^1)$. Then we have*

$$\mathrm{reg}(f^C) \cup \omega \cup \eta = \langle F_\eta, \log f^C, \log g\rangle_{\mathrm{gl}},$$

where the right-hand side is the global triple index associated to the Coleman functions $F_\eta, \log f^C, \log g$ on the basic wide open Y associated to $\mathcal{U}$ with annuli ends e_i at the C-torsion points on $E_{\mathbb{F}_p}$ occurring in $\mathrm{supp}(\bar{f}) \cup \mathrm{supp}(\bar{g})$.

Corollary 2.22. *We have the following identities:*

$$\mathrm{reg}(f^C) \cup \mathrm{reg}(g) \cup \eta = -\langle \log g, F_\eta, \log f^C\rangle_{\mathrm{gl}} r_{p,E}(\{f^C, g\})(\eta),$$

which is the p-adic regulator of the Steinberg symbol $\{f^C, g\}$, evaluated at the homomorphic 1-form η (see § 1).

Q11

Proof of the corollary. Only the first equality requires a proof: by the triple identity for the global triple index we obtain

$$\langle F_\eta, \log f^C, \log g\rangle_{\mathrm{gl}} = -\langle \log g, F_\eta, \log f^C\rangle_{\mathrm{gl}} - \langle \log g, \log f^C, F_\eta\rangle_{\mathrm{gl}},$$

and the second term vanishes because of [**6**, Lemma 7.4]. □

Remark. Corollary 2.22 follows directly from [**3**, Propositions 3.4 and 5.3]; however, we prefer to prove here the Proposition 2.21 as it sheds some light on Besser's triple index formula and its beautiful proof.

Proof of the proposition. We first recall the computation of the cup product

$$\mathrm{reg}(f^C) \cup \mathrm{reg}(g) \in H^2_{\mathrm{ms}}(\mathcal{U}, 2) \cong H^1_{\mathrm{rig}}(\mathcal{U}_s),$$

following Besser [**3**].

Note that by definition

$$H^1_{\mathrm{ms}}(\mathcal{U}, 1) = \varinjlim_k \left\{ \omega \in \Omega^1(U)_{\log}, h \in A^\dagger, dh = \left(1 - \frac{\varphi*}{p}\right)^k \omega \right\},$$

where $A^\dagger$ is the weak completion of $\mathcal{O}_U$ and $\Omega^1(U)_{\log}$ are the algebraic differential forms on U with logarithmic singularities along $E \backslash U$ and $\varphi : Y \to Y$ is a lifting of the Frobenius on the basic wide open Y.

The first Chern class of f^C, resp. $g \in \mathcal{O}^*_{\mathcal{U}}$ is given by

$$c_1^1(f^C) = \left(d \log f^C, \frac{1}{p} \log f_0^C \right),$$

resp.

$$c_1^1(g) = \left(d \log g, \frac{1}{p} \log g_0 \right),$$

where $f_0 = f^p / \varphi^* f \equiv 1$ modulo p, hence $\log f_0$ is well defined.

Then $\mathrm{reg}(f^C) \cup \mathrm{reg}(g) \in H^2_{\mathrm{ms}}(\mathcal{U}, 2)$ is the second Chern class $\mathrm{ch}_2(\{f^C, g\})$ of the Steinberg symbol $\{f^C, g\}$ which is given by

$$\left(d \log f^C, \frac{1}{p} \log f_0^C \right) \cup \left(d \log g, \frac{1}{p} \log g_0 \right) = (0, \theta_0(f^C, g)) \tag{2.23}$$

with

$$\theta_0(f^C, g) = \frac{1}{p^2} \log f_0^C d \log \varphi^* g - \frac{1}{p} \log g_0 d \log f^C.$$

Under the isomorphism $H^2_{\mathrm{ms}}(\mathcal{U}, 2) \cong H^1_{\mathrm{rig}}(\mathcal{U}_s)$ the image of $\mathrm{ch}_2(\{f^C, g\})$ in $H^1_{\mathrm{rig}}(\mathcal{U}_s)$ is given by the class of any form $\theta(f^C, g) \in \Omega^1_{A^\dagger/\mathbb{Q}_p}$ satisfying

$$\left(1 - \frac{\varphi^*}{p^2}\right) \theta(f^C, g) = \theta_0(f^C, g) + d(?). \tag{2.24}$$ Q7

Let

$$P(t) = 1 - \frac{t}{p}, \quad Q(s) = 1 - \frac{s}{p}.$$

There exist polynomials $a(t, s), b(t, s)$ with

$$P * Q(ts) := \left(1 - \frac{ts}{p^2}\right) = a(t, s) P(t) + b(t, s) Q(s).$$

By choosing

$$a(t, s) = \frac{s}{p}, \quad b(t, s) = 1,$$

we get the representation

$$1 - \frac{ts}{p^2} = \left(1 - \frac{t}{p}\right) \frac{s}{p} + \left(1 - \frac{s}{p}\right).$$

Let the two variable polynomials act on $A_{\mathrm{Col},1}(Y) \otimes \Omega^1_{\mathrm{Col},1}(Y)$ by letting t act as $\varphi^* \otimes \mathrm{id}$ and s as $\mathrm{id} \otimes \varphi^*$. Then (2.23) is equivalent to

$$\begin{aligned} P * Q(\varphi^*)\theta(f^C, g) &= a(t,s)P(\varphi^*)\log f^C \otimes d\log g \\ &\quad + b(t,s) d\log f^C \otimes Q(\varphi^*)\log g + dh \\ &= \frac{1}{p^2}\log f_0^C d\log \varphi^* g - \frac{1}{p}\log g_0 d\log f^C + dh \\ &= \theta_0(f^C, g) + dh. \end{aligned} \tag{2.25}$$

Consider the bilinear pairing introduced by Besser–de Jeu [6]

$$\langle\langle\ \rangle\rangle : A_{\mathrm{Col},1}(Y) \otimes \Omega^1_{\mathrm{Col},1}(Y) \longrightarrow \mathbb{Q}_p$$

between Coleman forms and Coleman functions on the basic wide open Y, given by

$$\langle\langle F, GdH\rangle\rangle = \langle F, G, H\rangle_{\mathrm{gl}}. \tag{2.26}$$

By Besser [5, Proposition 2.14] we have, for $\theta \in \Omega^1_{A^\dagger/\mathbb{Q}_p} = \Omega^1(Y)$,

$$\langle\langle F, \theta\rangle\rangle = \langle F, F_\theta\rangle_{\mathrm{gl}}, \tag{2.27}$$

where $\langle F, F_\theta\rangle_{\mathrm{gl}}$ is Besser's global double index on Y. Using Serre's cup product formula and the definition of the double index, we have in our situation that

$$\theta(f^C, g) \cup \eta' = \langle F_\eta, F_{\theta(f^C,g)}\rangle_{\mathrm{gl}}, \tag{2.28}$$

here η' is the image of $\eta \in H^0(E, \Omega^1)$ in $H^1_{\mathrm{rig},c}(U_s/\mathbb{Q}_p)$ given by $\{\eta, (F_\eta)_e\}$, where F_η and $F_{\theta(f^C,g)}$ are the Coleman integrals of η and $\theta(f^C, g)$, respectively. Then (2.27) implies

$$\langle F_\eta, F_{\theta(f^C,g)}\rangle_{\mathrm{gl}} = \langle\langle F_\eta, \theta(f^C, g)\rangle\rangle. \tag{2.29}$$

Hence, we need to show that

$$\langle\langle F_\eta, \theta(f^C, g)\rangle\rangle = \langle\langle F_\eta, \log f^C d\log g\rangle\rangle. \tag{2.30}$$

We are in the same situation as in the proof of [5, Proposition 5.3]: consider both sides in (2.30) as functions of η; these are functionals on the cohomology $H^1_{dR}(U/\mathbb{Q}_p)$. We see that (2.30) follows, by applying [5, (2.15)], from the formula

$$\langle\langle F_\eta, P * Q(\varphi^*)\theta_{(f^C,g)}\rangle\rangle = \langle\langle F_\eta, P * Q(\varphi^*)\log f^C d\log g\rangle\rangle. \tag{2.31}$$

To prove (2.31), one then follows the proof of [5, Proposition 5.3]; the last lemma [5, Lemma 5.4] can be applied as well: by the triple identity it remains to show that

$$\langle(\varphi^*)^n Q(\varphi^*)\log g, (\varphi^*)^m \log f^C, F_\eta\rangle_{\mathrm{gl}} = 0. \tag{2.32}$$

This is true because for a function $h \in \mathcal{O}(Y)^*$ we have, by [13, Lemma 2.5.1], that $\log(h^p/\varphi^*(h))$ is in $\mathcal{O}(Y)$. One then applies [6, Lemma 7.4]. This completes the proof of Proposition 2.21. □

Combining (2.18), (2.22) and Theorem 1.2, we obtain the following proposition.

Proposition 2.33. *Let the assumptions be as in Theorem 1.2. Then the syntomic regulator $r_{\mathrm{syn}}(z'')$ of the element $z'' \in H^1_{\mathrm{zar}}(\mathcal{U} \times \mathcal{E}, \mathcal{K}_2)$ satisfies*

$$r_{\mathrm{syn}}(z'') \cup \pi_1^*(\mathrm{reg}(g)) \cup \pi_1^*\eta = c_{f,g} \cdot \Omega_p \cdot L_p(E, 0).$$

3. *A regulator-decomposable element*

Consider the smooth proper model $\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p}$ of $E\times E$ over $\mathbb{Z}_p$, where p is a good ordinary reduction prime. Its Néron–Severi group $\mathrm{NS}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p})$ has rank 4, generated by the cycles $\{\mathcal{E}_{\mathbb{Z}_p}\times\{*\}\}$, $\{\{*\}\times\mathcal{E}_{\mathbb{Z}_p}\}$, Δ and the CM-cycle, that is, the graph of the complex multiplication $\mathcal{E}_{\mathbb{Z}_p}\xrightarrow[\sqrt{-d}]{}\mathcal{E}_{\mathbb{Z}_p}$, if $K=\mathbb{Q}(\sqrt{-d})$.

We consider the image of the composite map

$$\begin{aligned}\mathrm{Pic}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p})\otimes\mathbb{Z}_p^* &\xrightarrow{\cup} H^1(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},\mathcal{K}_2)\\ &\longrightarrow H^3_{\mathrm{ms}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},2).\end{aligned}$$

It follows from diagram (3.6) below that this image is generated by the elements $r_{\mathrm{syn}}(\gamma_i)$, $i=1,2,3,4$, with

$$\begin{aligned}\gamma_1&=r_{\mathrm{syn}}(\mathcal{E}\times\{0\},c),\\ \gamma_2&=r_{\mathrm{syn}}(\{0\}\times\mathcal{E},c),\\ \gamma_3&=r_{\mathrm{syn}}(\Delta,c),\\ \gamma_4&=r_{\mathrm{syn}}(\text{CM-cycle},c),\end{aligned}$$

where c is a topological generator of the subgroup $\mathbb{Z}_p\subset\mathbb{Z}_p^*$.

THEOREM 3.1. *Under the above assumptions we have the following properties.* Q12

(i) *The syntomic regulator*

$$r_{\mathrm{syn}}: H^1(\widehat{\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p}},\mathcal{K}_2)\otimes\mathbb{Q}_p\longrightarrow H^3_{\mathrm{syn}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},S_{\mathbb{Q}_p}(2))$$

is an injection and the dimension of $\mathrm{coker}(r_{\mathrm{syn}})$ *is at most* 1.

(ii) *Let z be the element defined in* (1.8). *Then z is regulator-decomposable, that is,* $r_{\mathrm{syn}}(z)\in H^1_f(\mathbb{Q}_p,\mathrm{NS}(\bar{X})\otimes\mathbb{Q}_p(1))$.

REMARK. The p-adic points conjecture implies that $H^3_{\mathrm{syn}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},S_{\mathbb{Q}_p}(2))\cong H^1_f(\mathbb{Q}_p,V)$, so (ii) means that $r_{\mathrm{syn}}(z)$ is in the subspace corresponding to $H^1_f(\mathbb{Q}_p,\mathrm{NS}(\bar{X})\otimes\mathbb{Q}_p(1))$ under this isomorphism. (Recall that $\bar{X}=(E\times E_{\mathbb{Q}_p})\times\bar{\mathbb{Q}}_p$, $V=H^2_{\mathrm{et}}(\bar{X},\mathbb{Q}_p(2))$.)

Proof of (i). Let B_{dR} be Fontaine's ring of p-adic periods and $DR(V)=(B_{dR}\otimes V)^{G_{\mathbb{Q}_p}}$ be defined as in [8]. There is a natural filtration on B_{dR} that induces a filtration on $DR(V)$. The Bloch–Kato-exponential map [8]

$$\exp: DR(V)\longrightarrow H^1(G_{\mathbb{Q}_p},V)$$

induces an isomorphism

$$DR(V)/DR^0(V)\xrightarrow{\sim}H^1_f(G_{\mathbb{Q}_p},V), \tag{3.2}$$

and via the B_{dR}-comparison isomorphism we have an isomorphism

$$DR(V)/DR^0(V)\xrightarrow{\sim}H^2_{dR}(X)/\mathrm{Fil}^2.$$

Hence, $\dim H^3_{\mathrm{syn}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},S_{\mathbb{Q}_p}(2))=\dim H^1_f(\mathbb{Q}_p,V)=5$.

We have a commutative diagram

$$\begin{array}{ccc}\mathrm{Pic}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p})\otimes\mathbb{Z}_p^* & \xrightarrow{\cup} & H^1(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},\mathcal{K}_2)\\ \downarrow & & \downarrow r_{\mathrm{syn}}\\ H^2_{\mathrm{syn}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},S_{\mathbb{Q}_p}(1))\cup H^1_{\mathrm{syn}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},S_{\mathbb{Q}_p}(1)) & \xrightarrow{\cup} & H^3_{\mathrm{syn}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},S_{\mathbb{Q}_p}(2)).\end{array} \tag{3.3}$$

We show that the image of $\mathrm{Pic}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p})\otimes\mathbb{Z}_p^*$ in $H^3_{\mathrm{syn}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p}, S_{\mathbb{Q}_p}(2))$ generates a 4-dimensional subspace. This is seen as follows.

Let $\mathrm{Pic}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p})\supset N = N_0\oplus N_1\oplus N_2\oplus N_3$ be the subgroup generated by $N_0=\mathbb{Z}\cdot\Delta$, $N_1=\mathbb{Z}\cdot[\mathcal{E}_{\mathbb{Z}_p}\times 0]$, $N_2=\mathbb{Z}\cdot[0\times\mathcal{E}_{\mathbb{Z}_p}]$, $N_3=\mathbb{Z}\cdot[\text{CM-cycle}]$.

Let $M=M_0\oplus M_1\oplus M_2\oplus M_3\subset H^2(\bar{E}\times\bar{E},\mathbb{Q}_p(1))$ be the subgroup defined by

$$\begin{aligned}
M_0 &= (\wedge^2 H^1(\bar{E},\mathbb{Q}_p))(1) = H^2(\bar{E},\mathbb{Q}_p(1))\cong\mathbb{Q}_p,\\
M_1 &= H^0(\bar{E},\mathbb{Q}_p)\otimes H^2(\bar{E},\mathbb{Q}_p(1))\cong\mathbb{Q}_p,\\
M_2 &= H^2(\bar{E},\mathbb{Q}_p(1))\otimes H^0(\bar{E},\mathbb{Q}_p)\cong\mathbb{Q}_p,\\
M_3 &= \langle c_{\mathrm{et}}(\text{CM-cycle})\rangle\cong\mathbb{Q}_p\subset \mathrm{Sym}^2 H^1(\bar{E},\mathbb{Q}_p)(1),\\
&\quad\subset H^2(\bar{E}\times\bar{E},\mathbb{Q}_p(1)).
\end{aligned}$$

One has an isomorphism for $i=0,1,2,3$:

$$\begin{aligned}
N_i\otimes_{\mathbb{Z}} H^1_f(\mathbb{Q}_p,\mathbb{Q}_p(1)) &\cong M_i\otimes_{\mathbb{Q}_p} H^1_f(\mathbb{Q}_p,\mathbb{Q}_p(1))\\
&\cong H^1_f(\mathbb{Q}_p,\mathbb{Q}_p(1)),
\end{aligned}\tag{3.4}$$

which induces an isomorphism

$$\begin{array}{ccc}
N\otimes H^1_f(\mathbb{Q}_p,\mathbb{Q}_p(1)) & \xrightarrow[\cong]{\alpha} & M\otimes H^1_f(\mathbb{Q}_p,\mathbb{Q}_p(1))\\
 & & \| \\
 & & H^1_f(\mathbb{Q}_p, M\otimes\mathbb{Q}_p(1))\\
 & & \downarrow\!\!\hookrightarrow \\
 & & H^1_f(\mathbb{Q}_p, H^2(\bar{E}\times\bar{E},\mathbb{Q}_p(2)).
\end{array}\tag{3.5}$$

Now consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{Pic}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p})\otimes\mathbb{Z}_p^* & \longrightarrow & N\otimes\mathbb{Z}_p^*\\
\downarrow\cup & & \downarrow \mathrm{id}\otimes\log_p\\
 & & N\otimes H^1_{\mathrm{syn}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p}, S_{\mathbb{Q}_p}(1))\\
H^1(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},\mathcal{K}_2) & & \cong\downarrow\\
 & & M\otimes H^1_f(\mathbb{Q}_p,\mathbb{Q}_p(1))\\
\downarrow & & \downarrow\!\!\hookrightarrow\\
H^3_{\mathrm{syn}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p}, S_{\mathbb{Q}_p}(2)) & \cong & H^1_f(\mathbb{Q}_p, H^2(\bar{E}\times\bar{E},\mathbb{Q}_p(2))).
\end{array}\tag{3.6}$$

The isomorphisms follow from the p-adic points conjecture as proved in [16, §6]. The commutativity of (3.6) follows from the commutative diagram

$$\begin{array}{ccc}
\mathrm{Pic}(E\times E_{\mathbb{Q}_p})\otimes\mathbb{Q}_p^* & \longrightarrow & N\otimes H^1(\mathbb{Q}_p,\mathbb{Q}_p(1))\\
 & & \downarrow=\\
\downarrow\cup & & H^1(\mathbb{Q}_p, N\otimes\mathbb{Q}_p(1))\\
 & & \downarrow\!\!\hookrightarrow\\
H^1(E\times E_{\mathbb{Q}_p},\mathcal{K}_2) & \longrightarrow & H^1(\mathbb{Q}_p, H^2(\bar{E}\times\bar{E},\mathbb{Q}_p(2)))
\end{array}\tag{3.7}$$

where the upper and lower horizontal arrows arise as boundary maps of Kummer sequences. (See [14, Lemma 2.8].)

The diagram shows that the image of $\mathrm{Pic}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p})\otimes\mathbb{Z}_p^*$ in $H^3_{\mathrm{syn}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p}, S_{\mathbb{Q}_p}(2))$ generates a 4-dimensional subspace.

By Bloch–Ogus theory, one has an exact sequence

$$0\to H^1(E\times E_{\mathbb{Q}_p},\mathcal{K}_2)/p^n\longrightarrow NH^3_{\mathrm{et}}(E\times E_{\mathbb{Q}_p},\mathbb{Z}/p^n(2))\longrightarrow \mathrm{Ch}^2(E\times E_{\mathbb{Q}_p})_{p^n}\longrightarrow 0,\tag{3.8}$$

where

$$NH^3_{\mathrm{et}}(E\times E_{\mathbb{Q}_p},\mathbb{Z}/p^n(2)) = \ker(H^3_{\mathrm{et}}(E\times E_{\mathbb{Q}_p},\mathbb{Z}/p^n(2)) \longrightarrow H^3_{\mathrm{et}}(k(E\times E_{\mathbb{Q}_p}),\mathbb{Z}/p^n(2))$$

is the first step in the coniveau filtration.

Taking inverse limits and using that $H^3_{\mathrm{et}}(E\times E_{\mathbb{Q}_p},\mathbb{Q}_p(2)) \cong H^1(\mathbb{Q}_p,V)$, we obtain an injection $H^1(\widehat{E\times E_{\mathbb{Q}_p}},\mathcal{K}_2)\otimes\mathbb{Q}_p \hookrightarrow H^1(\mathbb{Q}_p,V)$.

As the kernel of $H^1(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},\mathcal{K}_2)\to H^1(E\times E_{\mathbb{Q}_p},\mathcal{K}_2)$ is a finite torsion group, we also obtain an injection

$$H^1(\widehat{\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p}},\mathcal{K}_2)\otimes\mathbb{Q}_p \hookrightarrow H^1(\widehat{E\times E_{\mathbb{Q}_p}},\mathcal{K}_2)\otimes\mathbb{Q}_p \hookrightarrow H^1(\mathbb{Q}_p,V),$$

the image of which is contained in $H^1_f(\mathbb{Q}_p,V)\cong H^3_{\mathrm{syn}}(\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p},S_{\mathbb{Q}_p}(2))$, hence part (i) of Theorem 3.1 follows. □

Proof of (ii). Consider the subrepresentation W of V generated by algebraic cycles in the Néron–Severi group, so $W=N\otimes\mathbb{Q}_p(1)$. As we have seen above all four generators of N are already defined over $\mathbb{Q}_p$. The exponential map respects subrepresentations, hence we obtain an isomorphism

$$\exp: DR(W)=N\otimes DR(\mathbb{Q}_p(1)) \xrightarrow{\sim} N\otimes H^1_f(\mathbb{Q}_p,\mathbb{Q}_p(1)). \tag{3.9}$$

The map is given by

$$DR(\mathbb{Q}_p(1))\cong\mathbb{Q}_p \xrightarrow{\exp} \mathbb{Z}_p^*\otimes\mathbb{Q}_p\cong H^1_f(\mathbb{Q}_p,\mathbb{Q}_p(1)),$$

where the last isomorphism is given by the boundary map of the Kummer sequence.

Note that $DR^0(W)=0$ and $DR(\mathbb{Q}_p(1))=DR^{-1}(\mathbb{Q}_p(1))$, hence

$$DR(W)\subseteq DR^{-1}(V)\cong \mathrm{Fil}^1 H^2_{dR}(X).$$

Poincaré-duality on $H^2_{dR}(X)$ induces an isomorphism

$$H^2_{dR}(X)/\mathrm{Fil}^2 H^2_{dR}(X)\cong \mathrm{Hom}(\mathrm{Fil}^1 H^2_{dR}(X)\longrightarrow\mathbb{Q}_p).$$

The restriction of this isomorphism to Fil^1 induces a non-degenerate pairing on $\mathrm{Fil}^1/\mathrm{Fil}^2$ that coincides with the intersection pairing on N.

In order to show that $r_{\mathrm{syn}}(z)$ is contained in $H^1_f(\mathbb{Q}_p,N\otimes\mathbb{Q}_p(1))$, we need to show that $r_{\mathrm{syn}}(z)$, considered as a linear form on $\mathrm{Fil}^1 H^2_{dR}(X)$ via the above isomorphisms, vanishes on $\mathrm{Fil}^2 H^2_{dR}(X)=H^0(X,\Omega^2)$. Let ω be an invariant, hence nowhere vanishing, holomorphic 1-form on E, so $\omega\in H^0(E,\Omega^1)$. We consider the pullbacks $\omega_1=\pi_1^*\omega$, $\omega_2=\pi_2^*\omega$ on $E\times E$ via the canonical projections π_i. Then we need to compute $r_{\mathrm{syn}}(z)(\omega_1\cup\omega_2)$. Now we apply Besser's triple index formula [**5**, Theorem 1.1].

Let $\mathcal{Y}$ be the open surface obtained from $\mathcal{X}=\mathcal{E}\times\mathcal{E}_{\mathbb{Z}_p}$ by deleting the points $[x_i,x_i]$, $[x_i,0]$, $i=1,\dots,M-1$ and $[0,0]$. Let $\mathcal{Y}_0=(\mathcal{E}\times\{0\})_{\mathcal{X}}\times\mathcal{Y}$, $\mathcal{Y}_i=(\{x_i\}\times\mathcal{E})_{\mathcal{X}}\times\mathcal{Y}$. Let $\mathcal{V}$ be the complement of $\mathrm{supp}(\mathrm{div}(f))$, embedded diagonally in $\mathcal{Y}$, so we have finite maps

$$\lambda_i:\mathcal{Y}_i\longrightarrow\mathcal{Y} \quad\text{and}\quad \Delta:\mathcal{V}\longrightarrow\mathcal{Y}.$$

Then Besser's formula, applied to the element z in (1.8), yields

$$\begin{aligned} r_{\mathrm{syn}}(z)(\omega_1\cup\omega_2) &= \sum_{i=1}^{M-1}\langle\lambda_i^*F_{\omega_1},\log h_i,\lambda_i^*F_{\omega_2}\rangle_{\mathrm{gl},\hat{\mathcal{Y}}_i} \\ &\quad+\sum_{i=1}^{M-1}\langle\lambda_0^*F_{\omega_1},\log h_i,\lambda_0^*F_{\omega_2}\rangle_{\mathrm{gl},\hat{\mathcal{Y}}_0} \\ &\quad+\langle\Delta^*F_{\omega_1},\log f^M,\Delta^*F_{\omega_2}\rangle_{\mathrm{gl},\hat{\mathcal{V}}}. \end{aligned} \tag{3.10}$$

The sum runs over global triple indices on Coleman integrals defined on the wide open $\hat{\mathcal{Y}}_i$ and $\hat{\mathcal{V}}$ that we can associate to the open curves $\mathcal{Y}_i$ and $\mathcal{V}$.

The choice of the global Coleman integrals F_{ω_1} and F_{ω_2} on $E \times E_{\mathbb{Q}_p}$ is as follows.

Let F_ω be the unique Coleman integral of ω, which satisfies $F_\omega(0) = 0$. Then define $F_{\omega_1} := \pi_1^* F_\omega$ and $F_{\omega_2} := \pi_2^* F_\omega$.

As $\lambda_0^* F_{\omega_2}$ and $\lambda_i^* F_{\omega_1}$ vanish for all i, we get

$$\begin{aligned} r_{\mathrm{syn}}(z)(\omega_1 \cup \omega_2) &= \langle \Delta^* F_{\omega_1}, \log f^M, \Delta^* F_{\omega_2} \rangle_{\mathrm{gl}, \hat{\mathcal{V}}} \\ &= \langle F_\omega, \log f^M, F_\omega \rangle_{\mathrm{gl}, \hat{\mathcal{V}}}. \end{aligned} \tag{3.11}$$

The Coleman integral F_ω, which satisfies $F_\omega(0) = 0$, also vanishes at all torsion points on E by [12, Proposition 3.1].

By the triple identity we get

$$\begin{aligned} \langle F_\omega, \log f^M, F_\omega \rangle_{\mathrm{gl}, \hat{\mathcal{V}}} &= -\frac{1}{2} \langle F_\omega, F_\omega, \log f^M \rangle_{\mathrm{gl}, \hat{\mathcal{V}}} \\ &= -\frac{1}{2} \sum_e \mathrm{Res}_e F_\omega^2 d \log f^M. \end{aligned} \tag{3.12}$$

The residues are taken at annuli ends e associated to the points in $\mathrm{supp}((f))$. As the zeros of F_ω kill the poles of $d \log f^M$, all residues vanish and hence

$$r_{\mathrm{syn}}(z)(\omega_1 \cup \omega_2) = 0.$$

This completes the proof of Theorem 3.1. □

Acknowledgement. We thank M. Asakura and A. Besser for very helpful discussions.

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